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CERTAIN EXPANSION FORMULAE INVOLVING INCOMPLETE H AND $\overline{H}$-FUNCTIONS

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Abstract. The object of this paper is to derive the expansion formulae for incomplete H-function and incomplete $\overline{H}$-function. Further, their special cases are also point out in terms of different type of special functions (Meijer's ${}^{(\Gamma)}G$-function, incomplete Fox-Wright ${}_p\Psi_q^{(\Gamma)}$-function and incomplete hypergeometric function) which are general in nature and very useful for further investigation.

1. Introduction and Preliminaries

Special functions are important in the study of differential equation solutions and therefore are associated with a wide range of problems in a number of areas of science and engineering [10]. In fact, H-function and its implementations have also been described with important in numerous sub-fields of relevant mathematical analysis, see [1, 2, 3, 11, 12, 24]. In the domain of heat conduction and astrophysics, it has already been recognised that there are certain complications where even the most basic groups of special functions are not sufficient to respond to those situations. During this instance, the concept and study of incomplete Gamma functions and their generalisations were included in the overview. As a consequence, numerous articles [4, 5, 6, 9, 13, 14, 18, 19, 20] on incomplete special functions have previously been studied by academicians together with similar higher transcendental special functions. So keeping this in mind, we derive expansion formulae of incomplete H-function and incomplete $\overline{H}$-function to make these more useful which are general in nature and helpful for further studies.

In the study of incomplete generalized hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ Srivastava et al. [23] define the following pair of Mellin-Barnes type contour integral representation

$${}_p\gamma_q\left[\begin{array}{c}(c_1,x),c_2,\cdots,c_p;\\ d_1,d_2,\cdots,d_q;\end{array} z\right]=\frac{\prod_{j=1}^q\Gamma(d_j)}{\prod_{j=1}^p\Gamma(c_j)}\sum_{k=0}^{\infty}\frac{\gamma(c_1+k,x)\prod_{j=2}^p\Gamma(c_j+k)}{\prod_{j=1}^q\Gamma(d_j+k)}\frac{z^k}{k!}$$

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$$= \frac{1}{2\pi i} \frac{\prod_{j=1}^{q}\Gamma(d_j)}{\prod_{j=1}^{p}\Gamma(c_j)} \int_L \frac{\gamma(c_1+s,x)\prod_{j=2}^{p}\Gamma(c_j+s)}{\prod_{j=1}^{q}\Gamma(d_j+s)}\Gamma(-s)(-z)^s ds, \quad (|\arg(-z)|<\pi), \tag{1}$$

and

$${}_p\Gamma_q\left[\begin{array}{c}(c_1,x),c_2,\cdots,c_p;\\ d_1,d_2,\cdots,d_q;\end{array} z\right] = \frac{\prod_{j=1}^{q}\Gamma(d_j)}{\prod_{j=1}^{p}\Gamma(c_j)} \sum_{k=0}^{\infty} \frac{\Gamma(c_1+k,x)\prod_{j=2}^{p}\Gamma(c_j+k)}{\prod_{j=1}^{q}\Gamma(d_j+k)}\frac{z^k}{k!}$$

$$= \frac{1}{2\pi i} \frac{\prod_{j=1}^{q}\Gamma(d_j)}{\prod_{j=1}^{p}\Gamma(c_j)} \int_L \frac{\Gamma(c_1+s,x)\prod_{j=2}^{p}\Gamma(c_j+s)}{\prod_{j=1}^{q}\Gamma(d_j+s)}\Gamma(-s)(-z)^s ds, \quad (|\arg(-z)|<\pi), \tag{2}$$

where $L = L_{(i\tau;\infty)}$ is the Mellin-Barnes type contour which starts from $\tau - i\infty$ and terminate at $\tau + i\infty$ ($\tau \in \Re$) with indentations that each set poles are separate to each other in the integrand in each case.

In this sequel, by the inspiration of equation (1) and (2), Srivastava et al. [26] investigate the corresponding pairs of incomplete H-functions and also define their several properties. The incomplete H-functions $\gamma_{p,q}^{m,n}$and $\Gamma_{p,q}^{m,n}$ are defined as follows:

$$\begin{aligned}\gamma_{p,q}^{m,n}(z) &= \gamma_{p,q}^{m,n}\left[z\left|\begin{array}{c}(c_1,C_1,x),(c_j,C_j)_{2,p}\\ (d_j,D_j)_{1,q}\end{array}\right.\right]\\ &= \gamma_{p,q}^{m,n}\left[z\left|\begin{array}{c}(c_1,C_1,x),(c_2,C_2),\cdots,(c_p,C_p)\\ (d_1,D_1),(d_2,D_2),\cdots,(d_q,D_q)\end{array}\right.\right]\\ &= \frac{1}{2\pi i}\int_L g(s,x)z^{-s}ds,\end{aligned} \tag{3}$$

where

$$g(s,x) = \frac{\gamma(1-c_1-C_1s,x)\prod_{j=1}^{m}\Gamma(d_j+D_js)\prod_{j=2}^{n}\Gamma(1-c_j-C_js)}{\prod_{j=m+1}^{q}\Gamma(1-d_j-D_js)\prod_{j=n+1}^{p}\Gamma(c_j+C_js)}, \tag{4}$$

and

$$\begin{aligned}\Gamma_{p,q}^{m,n}(z) &= \Gamma_{p,q}^{m,n}\left[z\left|\begin{array}{c}(c_1,C_1,x),(c_j,C_j)_{2,p}\\ (d_j,D_j)_{1,q}\end{array}\right.\right]\\ &= \Gamma_{p,q}^{m,n}\left[z\left|\begin{array}{c}(c_1,C_1,x),(c_2,C_2),\cdots,(c_p,C_p)\\ (d_1,D_1),(d_2,D_2),\cdots,(d_q,D_q)\end{array}\right.\right]\\ &= \frac{1}{2\pi i}\int_L G(s,x)z^{-s}ds,\end{aligned} \tag{5}$$

where

$$G(s,x) = \frac{\Gamma(1-c_1-C_1s,x)\prod_{j=1}^{m}\Gamma(d_j+D_js)\prod_{j=2}^{n}\Gamma(1-c_j-C_js)}{\prod_{j=m+1}^{q}\Gamma(1-d_j-D_js)\prod_{j=n+1}^{p}\Gamma(c_j+C_js)}. \tag{6}$$

These incomplete H-functions are exists for all $x \geq 0$ under the same conditions (see, for details [10, 11, 12, 24]).

These incomplete H-functions are symmetric in the set of pair of parameters. Also the following decomposition formula holds.

$$\gamma_{p,q}^{m,n}(z) + \Gamma_{p,q}^{m,n}(z) = H_{p,q}^{m,n}(z).$$

Buschman and Srivastava [7] defined $\overline{H}$-function in the form Mellin-Barnes type contour integral of type L_0. Since for the wide applications of $\overline{H}$- function for solving the fractional- order differential equation given by Srivastava et al. [25], the following set of incomplete $\overline{H}$-functions $\overline{\gamma}_{p,q}^{m,n}$ and $\overline{\Gamma}_{p,q}^{m,n}$ defined by Srivastava et al. [26] as follows:

$$\overline{\gamma}_{p,q}^{m,n}(z) = \overline{\gamma}_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (c_1, C_1; \alpha_1 : x), (c_j, C_j; \alpha_j)_{2,n}, (c_j, C_j)_{n+1,p} \\ (d_j, D_j)_{1,m}, (d_j, D_j; \beta_j)_{m+1,q} \end{matrix}\right]$$

$$= \overline{\gamma}_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (c_1, C_1; \alpha_1 : x), (c_2, C_2; \alpha_2), \cdots, (c_n, C_n; \alpha_n), (c_{n+1}, C_{n+1}), \cdots, (c_p, C_p) \\ (d_1, D_1), \cdots, (d_m, D_m), (d_{m+1}, D_{m+1}; \beta_{m+1}), \cdots, (d_q, D_q; \beta_q) \end{matrix}\right]$$

$$= \frac{1}{2\pi i}\int_L \overline{g}(s, x) z^{-s} ds, \tag{7}$$

where

$$\overline{g}(s, x) = \frac{[\gamma(1 - c_1 - C_1 s, x)]^{\alpha_1} \prod_{j=1}^{m} \Gamma(d_j + D_j s) \prod_{j=2}^{n} [\Gamma(1 - c_j - C_j s)]^{\alpha_j}}{\prod_{j=m+1}^{q} [\Gamma(1 - d_j - D_j s)]^{\beta_j} \prod_{j=n+1}^{p} \Gamma(c_j + C_j s)}, \tag{8}$$

and

$$\overline{\Gamma}_{p,q}^{m,n}(z) = \overline{\Gamma}_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (c_1, C_1; \alpha_1 : x), (c_j, C_j; \alpha_j)_{2,n}, (c_j, C_j)_{n+1,p} \\ (d_j, D_j)_{1,m}, (d_j, D_j; \beta_j)_{m+1,q} \end{matrix}\right]$$

$$= \overline{\Gamma}_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (c_1, C_1; \alpha_1 : x), (c_2, C_2; \alpha_2), \cdots, (c_n, C_n; \alpha_n), (c_{n+1}, C_{n+1}), \cdots, (c_p, C_p) \\ (d_1, D_1), \cdots, (d_m, D_m), (d_{m+1}, D_{m+1}; \beta_{m+1}), \cdots, (d_q, D_q; \beta_q) \end{matrix}\right]$$

$$= \frac{1}{2\pi i}\int_L \overline{G}(s, x) z^{-s} ds, \tag{9}$$

where

$$\overline{G}(s, x) = \frac{[\Gamma(1 - c_1 - C_1 s, x)]^{\alpha_1} \prod_{j=1}^{m} \Gamma(d_j + D_j s) \prod_{j=2}^{n} [\Gamma(1 - c_j - C_j s)]^{\alpha_j}}{\prod_{j=m+1}^{q} [\Gamma(1 - d_j - D_j s)]^{\beta_j} \prod_{j=n+1}^{p} \Gamma(c_j + C_j s)}. \tag{10}$$

In this paper, we derive certain expansions of incomplete H-function and $\overline{H}$-function by using the generalized Taylor's series formula given by Osler [17] as follows:

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{\rho D_z^{\rho n+\eta} f(z)|_{z=w} (z - w)^{\rho n+\eta}}{\Gamma(\rho n + \eta + 1)}, \tag{11}$$

where η is the arbitrary complex number so the order of the derivative is arbitrary and $0 < \rho \leq 1$ and n is the integer over the summation.

2. Main Results

In this section, we establish certain expansion formulae of incomplete H-function by using the Taylor's series formula defined in Osler [17] and our results are presented in Theorem 2.1 and Theorem 2.2 as below.

Theorem 2.1 Let $h > 0, m - 1 \leq \Re(\rho n + \eta) \leq m, \eta \in \mathbb{C}, 0 < \rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then

$$\Gamma_{p,q}^{m,n}\left[az^h \,\middle|\, \begin{matrix} (c_1, C_1, x), (c_j, C_j)_{2,p} \\ (d_j, D_j)_{1,q} \end{matrix}\right]$$

$$= \sum_{n=-\infty}^{\infty} \frac{\rho(z - w)^{\rho n+\eta}}{\Gamma(\rho n + \eta + 1)} z^{-\rho n-\eta}\, \Gamma_{p+1,q+1}^{m,n+1}\left[az^h \,\middle|\, \begin{matrix} (c_1, C_1, x), (0, h), (c_j, C_j)_{2,p} \\ (d_j, D_j)_{1,q}, (\rho n + \eta, h) \end{matrix}\right].$$

Proof. We start from the L.H.S, by using the generalized Taylor's series (11). For this in our investigation, we consider

$$f(z) = \Gamma_{p,q}^{m,n}\left[az^h \,\middle|\, \begin{array}{c} (c_1, C_1, x), (c_j, C_j)_{2,p} \\ (d_j, D_j)_{1,q} \end{array}\right],$$

then we obtain

$$\Gamma_{p,q}^{m,n}\left[az^h \,\middle|\, \begin{array}{c} (c_1, C_1, x), (c_j, C_j)_{2,p} \\ (d_j, D_j)_{1,q} \end{array}\right]$$
$$= \sum_{n=-\infty}^{\infty} \frac{\rho(z-w)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)} D_z^{\rho n+\eta}\ \Gamma_{p,q}^{m,n}\left[az^h \,\middle|\, \begin{array}{c} (c_1, C_1, x), (c_j, C_j)_{2,p} \\ (d_j, D_j)_{1,q} \end{array}\right], \tag{12}$$

for solving the above fractional derivative, we use the fractional derivative formula which is given by Meena et al. ([15], Eq. 21) as

$$D^{\mu}\left[z^{\lambda-1}\Gamma_{p,q}^{m,n}(az^{\sigma})\right] = z^{\lambda-\mu-1}\Gamma_{p+1,q+1}^{m,n+1}\left[az^{\sigma} \,\middle|\, \begin{array}{c} (c_1, C_1, x), (1-\lambda, \sigma), (c_j, C_j)_{2,p} \\ (d_j, D_j)_{1,q}, (1-\lambda+\mu, \sigma) \end{array}\right],$$

then we get

$$D_z^{\rho n+\eta}\left[\Gamma_{p,q}^{m,n}(az^h)\right] = z^{-\rho n-\eta}\Gamma_{p+1,q+1}^{m,n+1}\left[az^h \,\middle|\, \begin{array}{c} (c_1, C_1, x), (0, h), (c_j, C_j)_{2,p} \\ (d_j, D_j)_{1,q}, (\rho n+\eta, h) \end{array}\right], \tag{13}$$

by using equation (13) in equation (1), we get the required result. □

In the similar way, we can derive Theorem 2.2 for $\gamma_{p,q}^{m,n}$.

Theorem 2.2 Let $h > 0, m-1 \leq \Re(\rho n+\eta) \leq m, \eta \in \mathbb{C}, 0 < \rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then

$$\gamma_{p,q}^{m,n}\left[az^h \,\middle|\, \begin{array}{c} (c_1, C_1, x), (c_j, C_j)_{2,p} \\ (d_j, D_j)_{1,q} \end{array}\right]$$
$$= \sum_{n=-\infty}^{\infty} \frac{\rho(z-w)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)} z^{-\rho n-\eta}\gamma_{p+1,q+1}^{m,n+1}\left[az^h \,\middle|\, \begin{array}{c} (c_1, C_1, x), (0, h), (c_j, C_j)_{2,p} \\ (d_j, D_j)_{1,q}, (\rho n+\eta, h) \end{array}\right].$$

Further, by using the Taylor's series formula defined in Osler [17], we are going to investigate certain expansion formula of incomplete $\overline{H}$-function and our results are presented in Theorem 2.3 and Theorem 2.4 as below.

Theorem 2.3 Let $h > 0, m-1 \leq \Re(\rho n+\eta) \leq m, \eta \in \mathbb{C}, 0 < \rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then

$$\overline{\Gamma}_{p,q}^{m,n}\left[az^h \,\middle|\, \begin{array}{c} (c_1, C_1; \alpha_1 : x), (c_j, C_j; \alpha_j)_{2,n}, (c_j, C_j)_{n+1,p} \\ (d_j, D_j)_{1,m}, (d_j, D_j; \beta_j)_{m+1,q} \end{array}\right]$$
$$= \sum_{n=-\infty}^{\infty} \frac{\rho(z-w)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)} z^{-\rho n-\eta}$$
$$\times\, \overline{\Gamma}_{p+1,q+1}^{m,n+1}\left[az^h \,\middle|\, \begin{array}{c} (c_1, C_1; \alpha_1 : x), (0, h; 1), ..., (c_n, C_n; \alpha_n), (c_j, C_j)_{n+1,p} \\ (d_j, D_j)_{1,m}, (d_j, D_j; \beta_j)_{m+1,q}, (\rho n+\eta, h; 1) \end{array}\right].$$

Proof. We start from the L.H.S, by using the generalized Taylor's series (11), for this in our investigation, we consider

$$f(z) = \overline{\Gamma}_{p,q}^{m,n}\left[az^h \,\middle|\, \begin{array}{c} (c_1, C_1; \alpha_1 : x), (c_j, C_j; \alpha_j)_{2,n}, (c_j, C_j)_{n+1,p} \\ (d_j, D_j)_{1,m}, (d_j, D_j; \beta_j)_{m+1,q} \end{array}\right],$$

and we have

$$\overline{\Gamma}^{m,n}_{p,q}\left[az^h \middle| \begin{array}{c}(c_1,C_1;\alpha_1:x),(c_j,C_j;\alpha_j)_{2,n},(c_j,C_j)_{n+1,p}\\(d_j,D_j)_{1,m},(d_j,D_j;\beta_j)_{m+1,q}\end{array}\right]$$

$$=\sum_{n=-\infty}^{\infty}\frac{\rho(z-w)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)}$$

$$\times\ D_z^{\rho n+\eta}\ \overline{\Gamma}^{m,n}_{p,q}\left[az^h \middle| \begin{array}{c}(c_1,C_1;\alpha_1:x),(c_j,C_j;\alpha_j)_{2,n},(c_j,C_j)_{n+1,p}\\(d_j,D_j)_{1,m},(d_j,D_j;\beta_j)_{m+1,q}\end{array}\right], \tag{14}$$

for solving the above fractional derivative, we use the fractional derivative formula given in ([15], Eq. 21) as

$$D^{\mu}\left[z^{\lambda-1}\ \Gamma^{m,n}_{p,q}(az^{\sigma})\right]=z^{\lambda-\mu-1}\Gamma^{m,n+1}_{p+1,q+1}\left[az^{\sigma} \middle| \begin{array}{c}(c_1,C_1,x),(1-\lambda,\sigma),(c_j,C_j)_{2,p}\\(d_j,D_j)_{1,q},(1-\lambda+\mu,\sigma)\end{array}\right],$$

then we get

$$D_z^{\rho n+\eta}\left[\overline{\Gamma}^{m,n}_{p,q}(az^h)\right]=z^{-\rho n-\eta}$$

$$\times\ \overline{\Gamma}^{m,n+1}_{p+1,q+1}\left[az^h \middle| \begin{array}{c}(c_1,C_1;\alpha_1:x),(0,h;1),...(c_n,C_n;\alpha_n),(c_j,C_j)_{n+1,p}\\(d_j,D_j)_{1,m},(d_j,D_j;\beta_j)_{m+1,q},(\rho n+\eta,h;1)\end{array}\right], \tag{15}$$

by using equation (15) in equation (14), we get the required result. □

In the similar way we can derive Theorem 2.4 for $\overline{\gamma}^{m,n}_{p,q}$.

Theorem 2.4 Let $h>0, m-1\leq\Re(\rho n+\eta)\leq m, \eta\in\mathbb{C}, 0<\rho\leq 1$, where a is the arbitrary constant and n is the integer over the summation, then

$$\overline{\gamma}^{m,n}_{p,q}\left[az^h \middle| \begin{array}{c}(c_1,C_1;\alpha_1:x),(c_j,C_j;\alpha_j)_{2,n},(c_j,C_j)_{n+1,p}\\(d_j,D_j)_{1,m},(d_j,D_j;\beta_j)_{m+1,q}\end{array}\right]$$

$$=\sum_{n=-\infty}^{\infty}\frac{\rho(z-w)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)}z^{-\rho n-\eta}$$

$$\times\ \overline{\gamma}^{m,n+1}_{p+1,q+1}\left[az^h \middle| \begin{array}{c}(c_1,C_1;\alpha_1:x),(0,h;1),\cdots,(c_n,C_n;\alpha_n),(c_j,C_j)_{n+1,p}\\(d_j,D_j)_{1,m},(d_j,D_j;\beta_j)_{m+1,q},(\rho n+\eta,h;1)\end{array}\right].$$

Remark It is easy to observe that the Theorems 2.3 and 2.4 are generalizations of the results provided in Theorems 2.1 and 2.2.

3. Special cases and concluding remarks

In this section, we present certain expansion formulae for other well known incomplete functions as special cases of the main results, in the form of Meijer's ${}^{(\Gamma)}G$-function, incomplete Fox-Wright ${}_p\Psi_q^{(\Gamma)}$-function and incomplete hypergeometric function (see details [8, 16, 22]), by substituting particular value to the parameters in the below results.

If we put $C_j=1(j=1,2,\cdots,p)$, $D_j=1(j=1,2,\cdots,q)$, $h=1$ and using the relation

$$\Gamma^{m,n}_{p,q}\left[z \middle| \begin{array}{c}(c_1,1,x),(c_j,1)_{2,p}\\(d_j,1)_{1,q}\end{array}\right]={}^{(\Gamma)}G^{m,n}_{p,q}\left[z \middle| \begin{array}{c}(c_1,x),(c_j)_{2,p}\\(d_j)_{1,q}\end{array}\right], \tag{16}$$

in Theorem 2.1, we obtain the following Corollary.

Corollary 3.1 Let $h > 0, m-1 \leq \Re(\rho n+\eta) \leq m, \eta \in \mathbb{C}, 0 < \rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then

$$ {}^{(\Gamma)}G^{m,n}_{p,q}\left[az\,\middle|\,\begin{matrix}(c_1,x),(c_j)_{2,p}\\(d_j)_{1,q}\end{matrix}\right] $$

$$ = \sum_{n=-\infty}^{\infty}\frac{\rho(z-w)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)}z^{-\rho n-\eta}\,{}^{(\Gamma)}G^{m,n+1}_{p+1,q+1}\left[az\,\middle|\,\begin{matrix}(c_1,x),0,(c_j)_{2,p}\\(d_j)_{1,q},\rho n+\eta\end{matrix}\right], $$

which is in terms of incomplete Meijer's ${}^{(\Gamma)}G$-function.

Now if we put $C_j = 1(j = 1, 2, \cdots, p)$, $D_j = 1(j = 1, 2, \cdots, q)$, $h = 1$, $\alpha_1 = 1$, $\alpha_j = 1, \beta_j = 1$ and using the above relation (16), then Theorem 2.3 becomes as below.

Corollary 3.2 Let $h > 0, m-1 \leq \Re(\rho n+\eta) \leq m, \eta \in \mathbb{C}, 0 < \rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then

$$ {}^{(\Gamma)}G^{m,n}_{p,q}\left[az\,\middle|\,\begin{matrix}(c_1,x),(c_j)_{2,p}\\(d_j)_{1,q}\end{matrix}\right] $$

$$ = \sum_{n=-\infty}^{\infty}\frac{\rho(z-w)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)}z^{-\rho n-\eta}\,{}^{(\Gamma)}G^{m,n+1}_{p+1,q+1}\left[az\,\middle|\,\begin{matrix}(c_1,x),(0;1),(c_j)_{2,p}\\(d_j)_{1,q},(\rho n+\eta;1)\end{matrix}\right], $$

which is in terms of incomplete Meijer's ${}^{(\Gamma)}G$-function.

Now, if we substitute $a = -a, m = 1, n = p, q = q+1, c_j \to (1-c_j)(j = 1, \cdots, p)$, and $d_j \to (1-d_j)(j = 1, \cdots, q)$ in Theorem 2.1 and $a = -a, m = 1, n = p, q = q+1$, $\alpha_1 = 1, \alpha_j = 1$, $\beta_j = 1$, $c_j \to (1-c_j)(j = 1, \cdots, p), d_j \to (1-d_j)(j = 1, \cdots, q)$ in Theorem 2.3 and using the following relation (see [26])

$$ \Gamma^{1,p}_{p,q+1}\left[-z\,\middle|\,\begin{matrix}(1-c_1,C_1,x),(1-c_j,C_j)_{2,p}\\(0,1),(1-d_j,D_j)_{1,q}\end{matrix}\right] = {}_p\Psi^{(\Gamma)}_q\left[\begin{matrix}(c_1,C_1,x),(c_j,C_j)_{2,p};\\(d_j,D_j)_{1,q};\end{matrix}\; z\right], \tag{17} $$

then, we have the following Corollaries 3.3 and 3.4 in term of incomplete Fox-Wright ${}_p\Psi^{(\Gamma)}_q$-function.

Corollary 3.3 Let $h > 0, m-1 \leq \Re(\rho n+\eta) \leq m, \eta \in \mathbb{C}, 0 < \rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then

$$ {}_p\Psi^{(\Gamma)}_q\left[\begin{matrix}(c_1,C_1,x),(c_j,C_j)_{2,p};\\(d_j,D_j)_{1,q};\end{matrix}\; az^h\right] = \sum_{n=-\infty}^{\infty}\frac{\rho(z-w)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)}z^{-\rho n-\eta} $$

$$ \times{}_{p+1}\Psi^{(\Gamma)}_{q+1}\left[\begin{matrix}(c_1,C_1,x),(0,h),(c_j,C_j)_{2,p};\\(d_j,D_j)_{1,q},(\rho n+\eta,h);\end{matrix}\; az^h\right]. $$

Corollary 3.4 Let $h > 0, m-1 \leq \Re(\rho n+\eta) \leq m, \eta \in \mathbb{C}, 0 < \rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then

$$ {}_p\Psi^{(\Gamma)}_q\left[\begin{matrix}(c_1,C_1,x),(c_j,C_j)_{2,p};\\(d_j,D_j)_{1,q};\end{matrix}\; az^h\right] = \sum_{n=-\infty}^{\infty}\frac{\rho(z-w)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)}z^{-\rho n-\eta} $$

$$ \times{}_{p+1}\Psi^{(\Gamma)}_{q+1}\left[\begin{matrix}(c_1,C_1,x),(0,h;1),(c_j,C_j)_{2,p};\\(d_j,D_j)_{1,q},(\rho n+\eta,h;1);\end{matrix}\; az^h\right]. $$

Again, if we substitute $h = 1, C_j = 1 (j = 1, \cdots, p), D_j = 1 (j = 1, \cdots, q)$ in Corollaries 3.3 and 3.4, and use the below relations (17) and (see [26])

$$\Gamma^{1,p}_{p,q+1}\left[-z\left|\begin{array}{c}(1-c_1,1,x),(1-c_j,1)_{2,p}\\(0,1),(1-d_j,1)_{1,q}\end{array}\right.\right] = c^p_{q}{}_p\Gamma_q\left[\begin{array}{c}(c_1,x),c_2,\cdots,c_p;\\d_1,\cdots,d_q;\end{array}\;z\right], \quad (18)$$

where, $c^p_q = \frac{\prod_{j=1}^{p}\Gamma(c_j)}{\prod_{j=1}^{q}\Gamma(d_j)}$, then we have the following Corollaries:

Corollary 3.5 Let $h > 0, m-1 \leq \Re(\rho n+\eta) \leq m, \eta \in \mathbb{C}, 0 < \rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then

$$c^p_{q}{}_p\Gamma_q\left[\begin{array}{c}(c_1,x),c_2,\cdots,c_p;\\d_1,\cdots,d_q;\end{array}\;az\right] = \sum_{n=-\infty}^{\infty}\frac{\rho(z-w)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)\Gamma(\rho n+\eta)}z^{-\rho n-\eta}$$
$$\times{}_{p+1}\Gamma_{q+1}\left[az\left|\begin{array}{c}(c_1,x),0,c_2,\cdots,c_p;\\d_1,\cdots,d_q,\rho n+\eta;\end{array}\;az\right.\right].$$

Corollary 3.6 Let $h > 0, m-1 \leq \Re(\rho n+\eta) \leq m, \eta \in \mathbb{C}, 0 < \rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then

$$c^p_{q}{}_p\Gamma_q\left[\begin{array}{c}(c_1,x),c_2,\cdots,c_p;\\d_1,\cdots,d_q;\end{array}\;az\right] = \sum_{n=-\infty}^{\infty}\frac{\rho(z-w)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)}z^{-\rho n-\eta}$$
$$\times{}_{p+1}\Gamma_{q+1}\left[az\left|\begin{array}{c}(c_1,x),(0;1),c_2,\cdots,c_p;\\d_1,\cdots,d_q,(\rho n+\eta;1);\end{array}\;az\right.\right].$$

Finally, if we make the particular substitutions $p = 2, q = 1$ and $p = q = 1$ in the above Corollaries 3.5 and 3.6, we get the results for incomplete Gauss's and Kummer's hypergeometric functions as below:

Corollary 3.7 Let $h > 0, m-1 \leq \Re(\rho n+\eta) \leq m, \eta \in \mathbb{C}, 0 < \rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then

$$c^2_{1}{}_2\Gamma_1\left[\begin{array}{c}(c_1,x),c_2;\\d_1;\end{array}\;az\right] = \sum_{n=-\infty}^{\infty}\frac{\rho(z-w)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)\Gamma(\rho n+\eta)}z^{-\rho n-\eta}$$
$$\times\;{}_3\Gamma_2\left[az\left|\begin{array}{c}(c_1,x),0,c_2;\\d_1,\rho n+\eta;\end{array}\;az\right.\right].$$

Corollary 3.8 Let $h > 0, m-1 \leq \Re(\rho n+\eta) \leq m, \eta \in \mathbb{C}, 0 < \rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then

$$c^2_{1}{}_2\Gamma_1\left[\begin{array}{c}(c_1,x),c_2;\\d_1;\end{array}\;az\right] = \sum_{n=-\infty}^{\infty}\frac{\rho(z-w)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)}z^{-\rho n-\eta}$$
$$\times\;{}_3\Gamma_2\left[az\left|\begin{array}{c}(c_1,x),(0;1),c_2;\\d_1,(\rho n+\eta;1);\end{array}\;az\right.\right].$$

Corollary 3.9 Let $h > 0, m-1 \leq \Re(\rho n+\eta) \leq m, \eta \in \mathbb{C}, 0 < \rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then

$$c^1_{1}{}_1\Gamma_1\left[\begin{array}{c}(c_1,x);\\d_1;\end{array}\;az\right] = \sum_{n=-\infty}^{\infty}\frac{\rho(z-w)^{\rho n+\eta}\,z^{-\rho n-\eta}}{\Gamma(\rho n+\eta+1)\Gamma(\rho n+\eta)}{}_2\Gamma_2\left[az\left|\begin{array}{c}(c_1,x),0;\\d_1,\rho n+\eta;\end{array}\;az\right.\right].$$

Corollary 3.10 Let $h > 0, m-1 \leq \Re(\rho n+\eta) \leq m, \eta \in \mathbb{C}, 0 < \rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then

$$c^1_{12}\Gamma_1\left[\begin{array}{c}(c_1,x);\\ d_1;\end{array} az\right] = \sum_{n=-\infty}^{\infty}\frac{\rho(z-w)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)}z^{-\rho n-\eta}{}_2\Gamma_2\left[az\left|\begin{array}{c}(c_1,x),(0;1);\\ d_1,(\rho n+\eta;1);\end{array} az\right.\right].$$

Remark. In the same sequel, special cases for the Theorem 2.2 and Theorem 2.4 may be derived.

Remark. By taking into account the decomposition formula of incomplete H-functions or by setting $x = 0$, the Theorem 2.1 lead to the known result provided earlier by Raina [21].

Moreover, it is important to note that the particular cases of the results obtained in this paper for $x = 0$ would give the corresponding results for classical $\overline{H}$-function and other special functions. Therefore, we conclude with the remark that, by specializing the parameter in the main results, one can derive number of expansion formulas for variety of special functions.

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