A NOTE ON FRACTIONAL ASKEY–WILSON INTEGRALS

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ABSTRACT. In this paper, we generalize fractional q-integrals by the method of q-difference equation. In addition, we deduce fractional Askey–Wilson integral, reversal type fractional Askey–Wilson integral and Ramanujan type fractional Askey–Wilson integral.

1. INTRODUCTION

The fractional q-calculus theories have been applied successfully in many fields, which is a very suitable tool in describing and solving a lot of problems in numerous sciences [23, 18], such as high-energy physics, system control, biomedical engineering and economics. Its theoretical and applied research has become a hot spot in the world. The treatment from the point view of the q-calculus can open new perspectives as it did, for example, in optimal control problems [7, 8, 15, 27]. For more information, see details in [10, 11, 2, 3, 21, 29, 30, 31, 32, 33, 24, 25, 26].

In this paper, we follow the notations and terminology in [16] and suppose that $0 < q < 1$. We first show a list of various definitions and notations in q-calculus which are useful to understand the subject of this paper. The basic hypergeometric series $_r\phi_s$ [16]

$$_r\phi_s\left[\begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \qquad (1)$$

converges absolutely for all z if $r \leq s$ and for $|z| < 1$ if $r = s+1$ and for terminating. The compact factorials of $_r\phi_s$ are defined respectively by

$$(a;q)_0 = 1, \quad [a]_q := \frac{1-q^a}{1-q}, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \quad (2)$$

and $(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$, where $m \in \mathbb{N} := \{1, 2, 3, \cdots\}$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The q-gamma function is defined by [16]

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x \in \mathbb{R}\backslash\{0, -1, -2, \ldots\}. \qquad (3)$$

The Thomae–Jackson q-integral is defined by [16, 17, 28]

$$\int_a^b f(x)\,\mathrm{d}_q\,x = (1-q)\sum_{n=0}^{\infty}\Big[bf(bq^n) - af(aq^n)\Big]q^n. \tag{4}$$

The Riemann–Liouville fractional q-integral operator is introduced in [1]

$$\Big(I_q^{\alpha}f\Big)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x \big(qt/x;q\big)_{\alpha-1}f(t)\,\mathrm{d}_q\,t. \tag{5}$$

The generalized Riemann–Liouville fractional q-integral operator is given by [25]

$$\Big(I_{q,a}^{\alpha}f\Big)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_a^x \big(qt/x;q\big)_{\alpha-1}f(t)\,\mathrm{d}_q\,t, \quad \alpha \in \mathbb{R}^+. \tag{6}$$

In fact, we rewrite fractional q-integral (6) equivalently as follows by (4)

$$\Big(I_{q,a}^{\alpha}f\Big)(x) = \frac{x^{\alpha-1}(1-q)}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty}\Big[x\big(q^{n+1};q\big)_{\alpha-1}f\big(xq^n\big) - a\big(aq^{n+1}/x;q\big)_{\alpha-1}f\big(aq^n\big)\Big]q^n.$$

In paper [13], author built the relations between the following fractional q-integrals and certain generating functions for q-polynomials.

Proposition 1 ([13, Theorem 3]). *For $\alpha \in \mathbb{R}^+$ and $0 < a < x < 1$, if $\max\{|at|,|az|\} < 1$, we have*

$$I_{q,a}^{\alpha}\left\{\frac{(bxz,xt;q)_{\infty}}{(xs,xz;q)_{\infty}}\right\}$$
$$= \frac{(1-q)^{\alpha}(abz,at;q)_{\infty}}{(as,az;q)_{\infty}}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}\big(a/x;q\big)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\,_3\phi_2\left[\begin{array}{c} q^{-k},as,az \\ at,abz \end{array};q,q\right]. \tag{7}$$

In this paper, we generalize fractional q-integrals and give applications of fractional Askey–Wilson integrals as follows.

Theorem 2. *For $\alpha \in \mathbb{R}^+$ and $0 < a < x < 1$, if $\max\{|at|,|az|,|aru|\} < 1$, we have*

$$I_{q,a}^{\alpha}\left\{\frac{(bxz,xt,xru;q)_{\infty}}{(xs,xz,xu;q)_{\infty}}\right\}$$
$$= \frac{(1-q)^{\alpha}(abz,at,aru;q)_{\infty}}{(as,az,au;q)_{\infty}}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}\big(a/x;q\big)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\,_4\phi_3\left[\begin{array}{c} q^{-k},as,az,au \\ abz,at,aru \end{array};q,q\right]. \tag{8}$$

Remark 3. *For $u = 0$ in Theorem 2, equation (8) reduces to (7).*

Before the proof of Theorem 2, the following lemmas are necessary.

Lemma 4 ([22, Proposition 1.2]). *Let $f(a,b,c)$ be a three-variable analytic function in a neighborhood of $(a,b,c) = (0,0,0) \in C^3$. If $f(a,b,c)$ satisfies the difference equation*

$$(c-b)f(a,b,c) = abf(a,bq,cq) - bf(a,b,cq) + (c-ab)f(a,bq,c), \tag{9}$$

then we have

$$f(a,b,c) = T(a,bD_c)\{f(a,0,c)\}, \tag{10}$$

where

$$T(a,bD_c) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}(bD_c)^n, \quad D_c\{f(c)\} = \frac{f(c)-f(cq)}{c}. \tag{11}$$

Lemma 5. *For* $\max\{|as|, |az|, |au|\} < 1$, *we have*

$$
(s - u)\frac{(abz, at, aru; q)_\infty}{(as, az, au)_\infty}
$$
$$
= ur\frac{(abz, at, aruq; q)_\infty}{(asq, az, auq)_\infty} - u\frac{(abz, at, aru; q)_\infty}{(asq, az, au)_\infty} + (s - ur)\frac{(abz, at, aruq; q)_\infty}{(as, az, auq)_\infty}. \quad (12)
$$

Proof of Lemma 5. The right-hand side (RHS) of equation (12) equals

$$
ur\frac{(1 - xs)(1 - xu)}{1 - xur}\frac{(abz, at, aru; q)_\infty}{(as, az, au)_\infty} - u(1 - xs)\frac{(abz, at, aru; q)_\infty}{(as, az, au)_\infty}
$$
$$
+ (s - ur)\frac{1 - xu}{1 - xur}\frac{(abz, at, aru; q)_\infty}{(as, az, au)_\infty} = (s - u)\frac{(abz, at, aru; q)_\infty}{(as, az, au)_\infty}, \quad (13)
$$

which is equals to the left-hand side (LHS) of the equation (12). The proof is complete.

Lemma 6 ([14, Eq. (2.3)]). *For* $\max\{|bt|, |ct|\} < 1$, *we have*

$$
T(a, bD_c)\left\{\frac{1}{(ct; q)_\infty}\right\} = \frac{(abt; q)_\infty}{(bt, ct; q)_\infty}. \quad (14)
$$

Proof of Theorem 2. Denoting the RHS of the equation (8) by $f(r, u, s)$, and rewriting $f(r, u, s)$ equivalently by

$$
f(r, u, s) = \sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x; q)_{\alpha+k}}{a^k(q; q)_{\alpha+k}}\sum_{j=0}^{k}\frac{(q^{-k}; q)_j q^j}{(q; q)_j} \cdot \frac{(1 - q)^\alpha(abzq^j, atq^j, aruq^j; q)_\infty}{(asq^j, azq^j, auq^j; q)_\infty},
$$

we check that $f(r, u, s)$ satisfies the equation (9) by Lemma 5, then we have

$$
f(r, u, s) = T(r, uD_s)\left\{f(r, 0, s)\right\}
$$
$$
= T(r, uD_s)\left\{\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x; q)_{\alpha+k}}{a^k(q; q)_{\alpha+k}}\sum_{j=0}^{k}\frac{(q^{-k}; q)_j q^j}{(q; q)_j} \cdot \frac{(1 - q)^\alpha(abzq^j, atq^j; q)_\infty}{(asq^j, azq^j; q)_\infty}\right\}
$$
$$
= T(r, uD_s)\left\{I_{q,a}^\alpha\left\{\frac{(xbz, xt; q)_\infty}{(xs, xz; q)_\infty}\right\}\right\}
$$
$$
= I_{q,a}^\alpha\left\{T(r, uD_s)\left\{\frac{(xbz, xt; q)_\infty}{(xs, xz; q)_\infty}\right\}\right\}
$$
$$
= I_{q,a}^\alpha\left\{\frac{(xbz, xt; q)_\infty}{(xz; q)_\infty} \cdot T(r, uD_s)\left\{\frac{1}{(xs; q)_\infty}\right\}\right\},
$$

which equals the LHS of the equation (8) by Lemma 6. The proof is complete.

The rest of the paper is organized as follows. In Section 2, we give the fractional Askey–Wilson integral. In Section 3, we obtain the reversal type fractional Askey–Wilson integral. In Section 4, we get the Ramanujan type fractional Askey–Wilson integral.

2. A generalization of Askey–Wilson integrals

In 1985, Askey and Wilson gave the famous Askey–Wilson integral, which greatly promoted the research and development of orthogonal polynomials. Chen and Gu [14]??Liu [19] and Cao [9] et al have promoted Askey–Wilson integral by different methods. For more information, see details in [4, 9, 14, 19].

In this section, we use the fractional q-integrals to generalize Askey–Wilson integrals.

Proposition 7 ([4, Theorem 2.1]). *If $\max\{|a|,|b|,|c|,|d|\} < 1$, we have*

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b, c, d)} d\theta = \frac{2\pi(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}, \tag{15}$$

where

$$h(\cos \theta; a) = (ae^{i\theta}, ae^{-i\theta}; q)_\infty,$$

$$h(\cos \theta; a_1, a_2, \cdots, a_m) = h(\cos \theta; a_1) h(\cos \theta; a_2) \cdots h(\cos \theta; a_m).$$

Theorem 8. *For $\alpha \in \mathbb{R}^+$, if $\max\{|a|,|b|,|c|,|d|\} < 1$, we have*

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b, c, d)} \sum_{k=0}^\infty \frac{x^{\alpha+k}(a/x; q)_{\alpha+k}}{a^k (q; q)_{\alpha+k}} {}_4\phi_3 \left[\begin{array}{c} q^{-k}, abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{array} ; q, q \right] d\theta$$

$$= \frac{2\pi(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty} \frac{x^\alpha (a/x; q)_\alpha}{(q; q)_\alpha}. \tag{16}$$

Corollary 9. *For $\alpha \in \mathbb{R}^+$, if $\max\{|a|,|b|,|c|\} < 1$, we have*

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b, c)} \sum_{k=0}^\infty \frac{x^{\alpha+k}(a/x; q)_{\alpha+k}}{a^k (q; q)_{\alpha+k}} {}_3\phi_2 \left[\begin{array}{c} q^{-k}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac \end{array} ; q, q \right] d\theta$$

$$= \frac{2\pi}{(q, ab, ac, bc; q)_\infty} \frac{x^\alpha (a/x; q)_\alpha}{(q; q)_\alpha}. \tag{17}$$

Remark 10. *For $d = 0$ in Theorem 8, equation (16) reduces to (17).*

Proof of Theorem 8. The equation (16) can be rewrite equivalently by

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; b, c, d)} \frac{(ab, ac, ad; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, abcd; q)_\infty} \sum_{k=0}^\infty \frac{x^{\alpha+k}(a/x; q)_{\alpha+k}}{a^k (q; q)_{\alpha+k}}$$

$$\times {}_4\phi_3 \left[\begin{array}{c} q^{-k}, abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{array} ; q, q \right] d\theta = \frac{2\pi}{(q, bc, bd, cd; q)_\infty} \frac{x^\alpha (a/x; q)_\alpha}{(q; q)_\alpha}. \tag{18}$$

By using Theorem 2, the LHS of the equation (18) is equivalent to

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; b, c, d)} I_{q,a}^\alpha \left\{ \frac{(xb, xc, xd; q)_\infty}{(xe^{i\theta}, xe^{-i\theta}, xbcd; q)_\infty} \right\} d\theta$$

$$= I_{q,a}^\alpha \left\{ \int_0^\pi \frac{h(\cos 2\theta; 1)(xb, xc, xd; q)_\infty}{h(\cos \theta; x, b, c, d)(xbcd; q)_\infty} d\theta \right\}$$

$$= I_{q,a}^\alpha \left\{ \frac{2\pi}{(q, bc, bd, cd; q)_\infty} \right\}$$

$$= \frac{2\pi}{(q, bc, bd, cd; q)_\infty} \frac{x^\alpha (a/x; q)_\alpha}{(q; q)_\alpha},$$

which is the RHS of the equation (18). The proof is complete.

3. A generalization of reversal type Askey–Wilson integrals

In this section, we use the fractional q-integrals to expand reversal Askey–Wilson integrals.

Proposition 11 (Reversal Askey–Wilson integral [5])**.** *For $|qabcd| < 1$, there holds*

$$\int_{-\infty}^{\infty} \frac{h(i\sinh x; qa, qb, qc, qd)}{h(\cosh 2x; -q)} dx = \frac{(q, qab, qac, qad, qbc, qbd, qcd; q)_{\infty}}{(qabcd; q)_{\infty}} \log(q^{-1}),$$

$$(19)$$

where

$$h(i\sinh \alpha x; t) = \prod_{k=0}^{\infty}(1 - 2iq^k t \sinh \alpha x + q^{2k}t^2) = (ite^{\alpha x}, -ite^{-\alpha x}; q)_{\infty}.$$

Theorem 12. *For $\alpha \in \mathbb{R}^+$ and $|qabcd| < 1$, we have*

$$\int_{-\infty}^{\infty} \frac{h(i\sinh t; qa, qb, qc, qd)}{h(\cosh 2t; -q)} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x; q)_{\alpha+k}}{a^k(q; q)_{\alpha+k}} {}_4\phi_3 \left[\begin{array}{c} q^{-k}, qab, qac, qad \\ iaqe^t, -iaqe^{-t}, qabcd \end{array} ; q, q \right] dt$$

$$= \frac{(q, qab, qac, qad, qbc, qbd, qcd; q)_{\infty}}{(qabcd; q)_{\infty}} \frac{x^{\alpha}(a/x; q)_{\alpha}}{(q; q)_{\alpha}} \log(q^{-1}). \qquad (20)$$

Corollary 13. *For $\alpha \in \mathbb{R}^+$, we have*

$$\int_{-\infty}^{\infty} \frac{h(i\sinh t; qa, qb, qc)}{h(\cosh 2t; -q)} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x; q)_{\alpha+k}}{a^k(q; q)_{\alpha+k}} {}_3\phi_2 \left[\begin{array}{c} q^{-k}, qab, qac \\ iaqe^t, -iaqe^{-t} \end{array} ; q, q \right] dt$$

$$= \frac{(q, qab, qac, qbc; q)_{\infty} x^{\alpha}(a/x; q)_{\alpha}}{(q; q)_{\alpha}} \log(q^{-1}). \quad (21)$$

Remark 14. *For $d = 0$ in Theorem 12, equation (20) reduces to (19).*

Proof of Theorem 12. The equation (20) can be rewrite

$$\int_{-\infty}^{\infty} \frac{h(i\sinh t; qb, qc, qd)}{h(\cosh 2t; -q)} \frac{(iaqe^t, -iaqe^{-t}, qabcd; q)_{\infty}}{(qab, qac, qad; q)_{\infty}} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x; q)_{\alpha+k}}{a^k(q; q)_{\alpha+k}}$$

$$\times {}_4\phi_3 \left[\begin{array}{c} q^{-k}, qab, qac, qad \\ iaqe^t, -iaqe^{-t}, qabcd \end{array} ; q, q \right] dt$$

$$= (q, qbc, qbd, qcd; q)_{\infty} \frac{x^{\alpha}(a/x; q)_{\alpha}}{(q; q)_{\alpha}} \log(q^{-1}), \quad (22)$$

By using Theorem 2, the LHS of the equation (22) equals

$$\int_{-\infty}^{\infty} \frac{h(i\sinh t; qb, qc, qd)}{h(\cosh 2t; -q)} I_{q,a}^{\alpha} \left\{ \frac{(ixqe^t, -ixqe^{-t}, qxbcd; q)_{\infty}}{(qxb, qxc, qxd; q)_{\infty}} \right\} dt$$

$$= I_{q,a}^{\alpha} \left\{ \int_{-\infty}^{\infty} \frac{h(i\sinh t; qx, qb, qc, qd)(qxbcd; q)_{\infty}}{h(\cosh 2t; -q)(qxb, qxc, qxd; q)_{\infty}} dt \right\}$$

$$= I_{q,a}^{\alpha} \left\{ (q, qbc, qbd, qcd; q)_{\infty} \log(q^{-1}) \right\}$$

$$= (q, qbc, qbd, qcd; q)_{\infty} \frac{x^{\alpha}(a/x; q)_{\alpha}}{(q; q)_{\alpha}} \log(q^{-1}).$$

which is the RHS of the equation (22). The proof is complete.

4. A generalization of Ramanujan type Askey–Wilson integrals

In 1994, Atakishiyev discovered the Atakishiyev integral by Ramanujan's method. After that, Wang [33] and Liu [20] et al conducted different generalization and proving lines for this integral.

In this part, we get the generating functions of Atakishiyev integral by fractional q-integral.

Proposition 15 (Atakishiyev integral [6]). *If α is a real number and $q = exp(-2\alpha^2)$, then we have*

$$\int_{-\infty}^{\infty} h(i\sinh\alpha t; a, b, c, d)e^{-t^2}\cosh\alpha t\,dt$$

$$= \sqrt{\pi}q^{-\frac{1}{8}}\frac{(ab/q, ac/q, ad/q, bc/q, bd/q, cd/q; q)_\infty}{(abcd/q^3; q)_\infty}. \quad (23)$$

Theorem 16. *For $\alpha \in \mathbb{R}^+$ and $\left|abcd/q^3\right| < 1$, if α is a real number and $q = exp(-2\alpha^2)$, then we have*

$$\int_{-\infty}^{\infty} h(i\sinh\alpha t; a, b, c, d)e^{-x^2}\cosh\alpha t\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x; q)_{\alpha+k}}{a^k(q; q)_{\alpha+k}}$$

$$\times {}_4\phi_3\left[\begin{array}{c} q^{-k}, ab/q, ac/q, ad/q \\ iae^{\alpha t}, -iae^{-\alpha t}, abcd/q^3 \end{array}; q, q\right]dt$$

$$= \sqrt{\pi}q^{-\frac{1}{8}}\frac{(ab/q, ac/q, ad/q, bc/q, bd/q, cd/q; q)_\infty}{(abcd/q^3; q)_\infty}\frac{x^\alpha(a/x; q)_\alpha}{(q; q)_\alpha}. \quad (24)$$

Corollary 17. *For $\alpha \in \mathbb{R}^+$, if α is a real number and $q = exp(-2\alpha^2)$, then we have*

$$\int_{-\infty}^{\infty} h(i\sinh\alpha t; a, b, c)e^{-x^2}\cosh\alpha t\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x; q)_{\alpha+k}}{a^k(q; q)_{\alpha+k}}{}_3\phi_2\left[\begin{array}{c} q^{-k}, ab/q, ac/q \\ iae^{\alpha t}, -iae^{-\alpha t} \end{array}; q, q\right]dt$$

$$= \sqrt{\pi}q^{-\frac{1}{8}}(ab/q, ac/q, bc/q; q)_\infty\frac{x^\alpha(a/x; q)_\alpha}{(q; q)_\alpha}. \quad (25)$$

Remark 18. *For $d = 0$ in Theorem 16, equation (24) reduces to (25).*

Proof of Theorem 16. The equation (24) can be rewrited

$$\int_{-\infty}^{\infty} h(i\sinh\alpha t; b, c, d)e^{-x^2}\cosh\alpha t\frac{(iae^{\alpha t}, -iae^{-\alpha t}, abcd/q^3; q)_\infty}{(ab/q, ac/q, ad/q; q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x; q)_{\alpha+k}}{a^k(q; q)_{\alpha+k}}$$

$$\times {}_4\phi_3\left[\begin{array}{c} q^{-k}, ab/q, ac/q, ad/q \\ iae^{\alpha t}, -iae^{-\alpha t}, abcd/q^3 \end{array}; q, q\right]dt$$

$$= \sqrt{\pi}q^{-\frac{1}{8}}(bc/q, bd/q, cd/q; q)_\infty\frac{x^\alpha(a/x; q)_\alpha}{(q; q)_\alpha}. \quad (26)$$

By using Theorem 2, the LHS of the equation (26) is equal to

$$\int_{-\infty}^{\infty} h(i\sinh\alpha t; b, c, d)e^{-x^2}\cosh\alpha t \cdot I_{q,a}^{\alpha}\left\{\frac{(ixe^{\alpha t}, -ixe^{-\alpha t}, xbcd/q^3; q)_{\infty}}{(xb/q, xc/q, xd/q; q)_{\infty}}\right\}dt$$

$$= I_{q,a}^{\alpha}\left\{\int_{-\infty}^{\infty} h(i\sinh\alpha t; x, b, c, d)e^{-x^2}\cosh\alpha t \cdot \frac{(xbcd/q^3; q)_{\infty}}{(xb/q, xc/q, xd/q; q)_{\infty}}dt\right\}$$

$$= I_{q,a}^{\alpha}\left\{\sqrt{\pi}q^{-\frac{1}{8}}(bc/q, bd/q, cd/q; q)_{\infty}\right\}$$

$$= \sqrt{\pi}q^{-\frac{1}{8}}(bc/q, bd/q, cd/q; q)_{\infty}\frac{x^{\alpha}(a/x; q)_{\alpha}}{(q; q)_{\alpha}},$$

which is the RHS of the equation (26). The proof is complete.

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