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CONVEXITY OF CERTAIN INTEGRAL OPERATOR DEFINED BY MITTAG-LEFFLER FUNCTIONS

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Abstract. Recently, there has been a vivid interest on special functions from the point of view of geometric function theory. Geometric properties of special functions like univalency, starlikeness and convexity appear in works of many mathematicians. In this paper, firstly, we obtain a new family of integral operators involving normalized Mittag-Leffler functions. Then, we give various sufficient conditions for convexity of this integral operator in the open unit disk. Several consequences of the main results are also shown.

1. Introduction and Preliminaries

Let A be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

and satisfy the usual normalization condition

$$f(0) = f'(0) - 1 = 0.$$

We denote by S the subclass of A consisting of the functions which are also univalent in U.

A function $f \in A$ is said to be starlike of order α $(0 \le \alpha < 1)$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U. \tag{2}$$

We denote by $S^*(\alpha)$ the class of all such functions. Also, we note that $S^* = S^*(0)$ is the usual class of starlike functions in U.

A function $f \in A$ is said to be starlike of order α $(0 \le \alpha < 1)$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in U. \tag{3}$$

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We denote by $K(\alpha)$ the class of all such functions. Also, we note that $K = K(0)$ is the usual class of convex functions in U.

The theory of fractional calculus has been applied in the theory of analytic functions. The classical concepts of a fractional differential operator and a fractional integral operator and their generalizations have fruitfully been employed in finding, for example, the characterization properties, coefficients estimate [7], distortion inequalities [24] and convolution properties for difference subclasses of analytic functions.

The classical Mittag-Leffler functions have already proved their efficiency as solutions of fractional-order differential and integral equations and thus have become important elements of the fractional calculus' theory and applications. The Mittag-Leffler function arises especially in the investigations of fractional generalization of kinetic equation, random walks, Levy flights, super-diffusive transport and in the study of complex systems. The most essential properties of these entire functions, investigated by many mathematicians, can be found in [11], [12].

The function $E_\lambda(z)$ defined by the following infinite series

$$E_\lambda(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + 1)},\ \ \mathrm{Re}(\lambda) > 0,\ \ \lambda, z \in \mathbb{C}, \tag{4}$$

was introduced by Mittag-Leffler [17] and is, therefore, known as the Mittag-Leffler function. It is an entire function of z with order $[\mathrm{Re}(\beta)]^{-1}$. A more general function $E_{\lambda,\mu}(z)$, generalizing $E_\lambda(z)$, is defined by

$$E_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + \mu)},\ \ \mathrm{Re}(\lambda) > 0,\ \ \lambda, \mu, z \in \mathbb{C}. \tag{5}$$

Note that Mittag-Leffler function $E_{\lambda,\mu}(z)$, defined by (5) does not belong to the class A. Thus, it is natural to consider the following normalization of the Mittag-Leffler function:

$$\tilde{E}_{\lambda,\mu}(z) := \Gamma(\mu) z E_{\lambda,\mu}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\mu) z^n}{\Gamma(\lambda(n-1) + \mu)}, \tag{6}$$

$$\left(\mathrm{Re}(\lambda) > 0,\ \ \lambda, \mu, z \in \mathbb{C},\ \ \mu \in \mathbb{Z}_0^-\right)$$

where $\mathbb{Z}_0^- = \{0, -1, -2, ... - n, ...\}$. Whilst formula (6) holds for complex-valued λ, μ and $z \in \mathbb{C}$, however in this paper, we shall restrict our attention to the case of real-valued λ, μ and $z \in U$. Observe that the function $\tilde{E}_{\lambda,\mu}(z)$ contains many well-known functions as its special case, for example,

$$\begin{cases} \tilde{E}_{0,1}(z) = \frac{z}{1-z}; \\ \tilde{E}_{1,1}(z) = ze^z; \\ \tilde{E}_{2,1}(z) = z\cosh(\sqrt{z}); \\ \tilde{E}_{2,2}(z) = \sqrt{z}\sinh(\sqrt{z}). \end{cases} \tag{7}$$

Geometric properties including starlikeness, convexity and close-to-convexity for the Mittag-Leffler function $\tilde{E}_{\lambda,\mu}(z)$ were recently investigated by Bansal and Prajapat [1]. Raducanu [21], investigated the ratio of the normalized Mittag-Leffler function. Srivastava et al. [23] found sufficient conditions for univalence of integral operators involving the normalized Mittag-Leffler function.

In this paper, we give various sufficient conditions for convexity of integral operator

$$G^{\beta}_{\lambda,\mu}(z)=\left\{\beta\int_0^z t^{\beta-2}\tilde{E}_{\lambda,\mu}(t)dt\right\}^{1/\beta},\quad z\in U,\ \beta\in\mathbb{C}/\left\{0\right\} \tag{8}$$

which involve the normalized form of the Mittag-Leffler function $\tilde{E}_{\lambda,\mu}(z)$.

In recent years, the problem of geometric properties (such as univalence, starlikeness and convexity) of some integral operators discussed by many authors (see [2]-[6], [8]-[10], [13]-[16], [18]-[20], [22]).

In our investigation, we shall need the following results.

Lemma 1 Let $\lambda\geq 1$ and $\mu>\mu_0$ where $\mu_0\cong 1.618$ is the root of the equation

$$\mu^2-\mu-1=0. \tag{9}$$

Then the following inequality holds for all $z\in U$

$$\left|\frac{z\left(\tilde{E}_{\lambda,\mu}(z)\right)'}{\tilde{E}_{\lambda,\mu}(z)}-1\right|\leq\frac{2\mu+1}{\mu^2-\mu-1}. \tag{10}$$

Proof. By using the definition of the normalized Mittag-Leffler function $\tilde{E}_{\lambda,\mu}(z)$ for all $z\in U$, we obtain

$$\begin{aligned}\left|\frac{z\left(\tilde{E}_{\lambda,\mu}(z)\right)'}{\tilde{E}_{\lambda,\mu}(z)}-1\right| &= \left|\frac{z\left(\tilde{E}_{\lambda,\mu}(z)\right)'-\tilde{E}_{\lambda,\mu}(z)}{\tilde{E}_{\lambda,\mu}(z)}\right|\\ &= \left|\frac{\sum_{n=2}^{\infty}\frac{(n-1)\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}z^n}{z+\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}z^n}\right|\\ &\leq \frac{\sum_{n=2}^{\infty}\frac{(n-1)\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}}{1-\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}}.\end{aligned} \tag{11}$$

Under hypothesis $\lambda\geq 1$, the inequality $\Gamma(n-1+\mu)\leq\Gamma(\lambda(n-1)+\mu),\ n\in\mathbb{N}$ holds, which is equivalent to

$$\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}\leq\frac{1}{(\mu)_{n-1}},\ n\in\mathbb{N} \tag{12}$$

where $(\mu)_n=\Gamma(n+\mu)/\Gamma(\mu)=\mu(\mu+1)\cdots(\mu+n-1),\ (\mu)_0=1$ is Pochhammer (or Appell) symbol, defined in terms of Euler gamma function.

Using (12) we obtain

$$\sum_{n=2}^{\infty}\frac{(n-1)\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}\leq\sum_{n=2}^{\infty}\frac{n-1}{(\mu)_{n-1}}=\sum_{n=1}^{\infty}\frac{n}{(\mu)_n}=\frac{1}{\mu}+\sum_{n=2}^{\infty}\frac{n}{(\mu)_n}. \tag{13}$$

Further, for all $n\in\mathbb{N}/\left\{1\right\}$ and $\mu\geq 1$ the inequality

$$\frac{n}{(\mu)_n}\leq\frac{1}{\mu(\mu+1)^{n-2}} \tag{14}$$

holds true. If we use (14) in (13), we get

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{(n-1)\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} &\leq \frac{1}{\mu}+\sum_{n=2}^{\infty}\frac{1}{\mu(\mu+1)^{n-2}}\\ &= \frac{1}{\mu}+\frac{1}{\mu}\sum_{n=0}^{\infty}\frac{1}{(\mu+1)^{n}}\\ &= \frac{2\mu+1}{\mu^2}. \qquad (15)\end{aligned}$$

Similarly, we have

$$\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \leq \sum_{n=2}^{\infty}\frac{1}{(\mu)_{n-1}}.$$

Further, the inequality

$$(\mu)_{n-1}=\mu(\mu+1)\cdots(\mu+n-2)\geq \mu(\mu+1)^{n-2},\ n\in\mathbb{N} \qquad (16)$$

is true, which is equivalent to $1/(\mu)_{n-1}\leq 1/\mu(\mu+1)^{n-2}$, $n\in\mathbb{N}$. Using (16), we get

$$\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \leq \sum_{n=2}^{\infty}\frac{1}{\mu(\mu+1)^{n-2}}=\frac{1}{\mu}\sum_{n=0}^{\infty}\frac{1}{(\mu+1)^{n}}=\frac{\mu+1}{\mu^2}. \qquad (17)$$

From (17), (15) and (11), we obtain

$$\left|\frac{z\left(\tilde{E}_{\lambda,\mu}(z)\right)'}{\tilde{E}_{\lambda,\mu}(z)}-1\right| \leq \frac{2\mu+1}{\mu^2-\mu-1}.$$

Thus, the proof of Lemma 1 is completed.

2. Main Result

On the convexity of the function $G^{\beta}_{\lambda,\mu}(z)$, we give the following theorem.

Theorem 1 Let $\lambda\geq 1$, β be a complex number such that $\beta\notin\mathbb{Z}_0^-=\{0,-1,-2,...,-n,...\}$ and $\mu>\mu_0$ where $\mu_0\cong 1.618$ is the root of the equation (9). Then, the function $G^{\beta}_{\lambda,\mu}: U\to\mathbb{C}$ defined by (8) belongs to the class $K(\alpha)$ if the following condition is satisfied:

$$(1-\alpha)\mu^2-(3+|\beta-1|-\alpha)\mu-(2+|\beta-1|-\alpha)\geq 0. \qquad (18)$$

Proof. Since $\tilde{E}_{\lambda,\mu}\in A$, clearly $G^{\beta}_{\lambda,\mu}\in A$, $G^{\beta}_{\lambda,\mu}(0)={G^{\beta}_{\lambda,\mu}}'(0)-1=0$. From the definition, a function $G^{\beta}_{\lambda,\mu}\in A$ belongs to the class $K(\alpha)$ if and only if

$$\operatorname{Re}\left(1+\frac{z\left(G^{\beta}_{\lambda,\mu}\right)''(z)}{\left(G^{\beta}_{\lambda,\mu}\right)'(z)}\right)>\alpha,\ z\in U. \qquad (19)$$

It suffices to show that

$$\left|\frac{z\left(G^{\beta}_{\lambda,\mu}\right)''(z)}{\left(G^{\beta}_{\lambda,\mu}\right)'(z)}\right| \le 1-\alpha,\ z\in U. \tag{20}$$

On the other hand, it is easy to see that

$$\frac{\left(G^{\beta}_{\lambda,\mu}\right)''(z)}{\left(G^{\beta}_{\lambda,\mu}\right)'(z)} = \left(\frac{1-\beta}{\beta}\right)\frac{z^{\beta-2}\tilde{E}_{\lambda,\mu}(z)}{\int_0^z t^{\beta-2}\tilde{E}_{\lambda,\mu}(t)dt} + \frac{\left(\tilde{E}_{\lambda,\mu}(t)\right)'}{\tilde{E}_{\lambda,\mu}(z)} + \frac{\beta-2}{z}$$

and

$$\begin{aligned}\frac{z\left(G^{\beta}_{\lambda,\mu}\right)''(z)}{\left(G^{\beta}_{\lambda,\mu}\right)'(z)} &= \left(\frac{1-\beta}{\beta}\right)\frac{z^{\beta-1}\tilde{E}_{\lambda,\mu}(z)}{\int_0^z t^{\beta-2}\tilde{E}_{\lambda,\mu}(t)dt} + \frac{z\left(\tilde{E}_{\lambda,\mu}(t)\right)'}{\tilde{E}_{\lambda,\mu}(z)} + (\beta-2)\\ &= \left(\frac{1-\beta}{\beta}\right)\frac{z^{\beta-1}\tilde{E}_{\lambda,\mu}(z)}{\int_0^z t^{\beta-2}\tilde{E}_{\lambda,\mu}(t)dt} + (\beta-1) + \frac{z\left(\tilde{E}_{\lambda,\mu}(t)\right)' - \tilde{E}_{\lambda,\mu}(z)}{\tilde{E}_{\lambda,\mu}(z)}.\end{aligned}$$

From (6), we write

$$\frac{z\left(G^{\beta}_{\lambda,\mu}\right)''(z)}{\left(G^{\beta}_{\lambda,\mu}\right)'(z)} = \frac{\sum_{n=2}^{\infty}\left(\frac{\beta}{n+\beta-1}-1\right)\frac{(\beta-1)\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}z^{n+\beta-1}}{z^{\beta}+\sum_{n=2}^{\infty}\frac{\beta}{n+\beta-1}\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}z^{n+\beta-1}} + \frac{\sum_{n=2}^{\infty}\frac{(n-1)\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}z^{n}}{z+\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}z^{n}}.$$

Hence,

$$\left|\frac{z\left(G^{\beta}_{\lambda,\mu}\right)''(z)}{\left(G^{\beta}_{\lambda,\mu}\right)'(z)}\right| \le \frac{\sum_{n=2}^{\infty}\left(1-\left|\frac{\beta}{n+\beta-1}\right|\right)\frac{|\beta-1|\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}}{1-\sum_{n=2}^{\infty}\left|\frac{\beta}{n+\beta-1}\right|\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}} + \frac{\sum_{n=2}^{\infty}\frac{(n-1)\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}}{1-\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}}.$$

Also, we can write

$$\begin{aligned}\left|\frac{z\left(G^{\beta}_{\lambda,\mu}\right)''(z)}{\left(G^{\beta}_{\lambda,\mu}\right)'(z)}\right| &\le \frac{\sum_{n=2}^{\infty}\left((n-1)+|\beta-1|\right)\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}}{1-\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)(n-1)!}}\\ &= \frac{\sum_{n=2}^{\infty}\frac{(n-1)\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}+\sum_{n=2}^{\infty}\frac{|\beta-1|\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}}{1-\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}}\end{aligned}$$

and from (15) and (17)

$$\left|\frac{z\left(G^{\beta}_{\lambda,\mu}\right)''(z)}{\left(G^{\beta}_{\lambda,\mu}\right)'(z)}\right| \le \frac{\frac{2\mu+1}{\mu^2}+|\beta-1|\frac{\mu+1}{\mu^2}}{1-\frac{\mu+1}{\mu^2}} = \frac{2\mu+1+|\beta-1|(\mu+1)}{\mu^2-\mu-1}.$$

Thus, from the last inequality we see that the inequality (20) is true if the last expression is bounded by $(1-\alpha)$, which is equivalent to

$$(1-\alpha)\left(\mu^2-\mu-1\right)-|\beta-1|(\mu+1)-(2\mu+1)\ge 0.$$

With this, the proof of Theorem 1 is completed.

By setting $\alpha = 0$ in Theorem 1, we arrive at the following corollary.

Corollary 1 Let $\lambda \geq 1$, β be a complex number such that $\beta \notin \mathbb{Z}_0^- = \{0, -1, -2, ..., -n, ...\}$ and $\mu > \mu_0$ where $\mu_0 \cong 1.618$ is the root of the equation (9). Then, the function $G_{\lambda,\mu}^{\beta} : U \to \mathbb{C}$ defined by (8) belongs to the class K if the following condition is satisfied:

$$\mu^2 - (3 + |\beta - 1|)\mu - (2 + |\beta - 1|) \geq 0.$$

For $\beta = 0$ in Corollary 1, we have the following corollary.

Corollary 2 Let $\lambda \geq 1$ and $\mu > \mu_0$ where $\mu_0 \cong 1.618$ is the root of the equation (9). Then, the function $G_{\lambda,\mu}^{1} : U \to \mathbb{C}$ defined by (8) belongs to the class K if the following condition is satisfied:

$$\mu^2 - 3\mu - 2 \geq 0.$$

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