Journal of Fractional Calculus and Applications
Vol. 11(2) July 2020, pp. 270-280.
ISSN: 2090-5858.
http://math-frac.oreg/Journals/JFCA/

ON A GENERAL THEOREM CONNECTING LAPLACE TRANSFORM AND GENERALIZED WEYL FRACTIONAL INTEGRAL OPERATOR INVOLVING FOX'S H-FUNCTION AND A GENERAL CLASS OF FUNCTIONS

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ABSTRACT. The aim of the present paper is to establish a general theorem which asserts an interesting relationship between the Laplace transform and generalized Weyl fractional integral operator involving Fox's H- function and a general class of functions. The general theorem involves a multidimensional series with essentially arbitrary sequence of complex numbers. By suitably assigning different values to the sequence, one can easily evaluate generalized Weyl fractional integral of special functions of several variables. An illustration for (Srivastava- Daoust) generalized Lauricella function is mentioned. On account of general nature of generalized Lauricella function, Fox's H-function and general class of functions a number of results involving special functions can be obtained merely by specializing the parameters. For the sake of illustration I have given generalized Weyl fractional integral of elementary special functions including G-function due to Lorenzo-Hartley, which is a generalization of a number of functions of practical utility in fractional calculus.

1. Introduction

Definition 1. Fox's H-function occurring in this paper is defined and represented by means of the following Mellin-Barnes type contour integral [14]:

$$H[z] = H_{P,Q}^{M,N}\left[z\left|\begin{matrix}(k_J,\tau_J)_{1,P}\\(\rho_J,\omega_J)_{1,Q}\end{matrix}\right.\right] = \frac{1}{2\pi\omega}\int_L \phi(\xi)\, z^{\xi}\, d\xi, \tag{1}$$

where $\omega = \sqrt{-1}, L$ is a contour which goes from $c - i\infty$ to $c + i\infty$ and

$$\phi(\xi) = \frac{\prod_{J=1}^{M}\Gamma(\rho_J - \omega_J\xi)\prod_{J=1}^{N}\Gamma(1 - k_J + \tau_J\xi)}{\prod_{J=M+1}^{Q}\Gamma(1 - \rho_J + \omega_J\xi)\prod_{J=N+1}^{P}\Gamma(k_J - \tau_J\xi)}. \tag{2}$$

2010 *Mathematics Subject Classification.* 26A33,33C65,44A10.

Key words and phrases. Weyl fractional integral operator, Laplace transform, generalized Lauricella function, Fox's H-function, general class of functions.

Submitted Mach 23, 2019. Revised April 20, 2019.

For the convergence, existence conditions and other details of the above Fox's H -function, one may refer to the treatise by Srivastava, Gupta and Goyal [14].

Definition 2. Parseval-Goldstein theorem [6] for Laplace transform is defined as follows:

If

$$L\left\{f_1\left(x\right);\, s\right\} = \phi_1\left(s\right)$$

and

$$L\left\{f_2\left(x\right);\, s\right\} = \phi_2\left(s\right)$$

then

$$\int_0^\infty f_1\left(x\right)\, \phi_2\left(x\right) dx = \int_0^\infty f_2\left(x\right)\, \phi_1\left(x\right)\, dx, \tag{3}$$

provided that both the integrals in (3) are absolutely convergent.

Definition 3. The generalized Lauricella function introduced by Srivastava and Daoust [15, p.454] is defined as follows:

$$\begin{aligned} F\left[z_1, ..., z_v\right] &= F^{b:\, p_1\,;...;\, p_v}_{l:\, q_1\,;...;\, q_v} \begin{bmatrix} z_1 \\ \vdots \\ z_v \end{bmatrix} \\ &= F^{b:\, p_1\,;...;\, p_v}_{l:\, q_1\,;...;\, q_v} \left[\begin{matrix} \left(A_J; \alpha'_J\,, ..., \alpha_J^{(v)}\right)_{1,\, b} : \left(c'_J\,, \gamma'_J\right)_{1, p_1}\,;...;\left(c_J^{(v)}, \gamma_J^{(v)}\right)_{1,\, p_v}; \\ \left(B_J; \beta'_J\,, ..., \beta_J^{(v)}\right)_{1,\, l} : \left(d'_J\,, \delta'_J\right)_{1,\, q_1}\,;...;\left(d_J^{(v)}, \delta_J^{(v)}\right)_{1,\, q_v}; \end{matrix} z_1\,, ...,\, z_v \right] \\ &= \sum_{r_1\,, ...,\, r_v = 0}^{\infty} \Omega\left(r_1\,, ...,\, r_v\right) \frac{(z_1)^{r_1}}{r_1!} \cdots \frac{(z_v)^{r_v}}{r_v!}, \end{aligned} \tag{4}$$

where

$$\Omega\left(r_1\,, ...,\, r_v\right) = \frac{\prod\limits_{J=1}^{b} \left(A_J\right)_{\sum\limits_{i=1}^{v} r_i \alpha_J^{(i)}} \prod\limits_{J=1}^{p_1} \left(c'_J\right)_{r_1 \gamma'_J} \cdots \prod\limits_{J=1}^{p_v} \left(c_J^{(v)}\right)_{r_v \gamma_J^{(v)}}}{\prod\limits_{J=1}^{l} \left(B_J\right)_{\sum\limits_{J=1}^{v} r_i \beta_J^{(i)}} \prod\limits_{J=1}^{q_1} \left(d'_J\right)_{r_1 \delta'_J} \cdots \prod\limits_{J=1}^{q_v} \left(d_J^{(v)}\right)_{r_v \delta_J^{(v)}}}. \tag{5}$$

The multiple series (4) converges absolutely (see[16]) for all $z_1\,, ...,\, z_v$, when $Q_i > 0$ or for $Q_i = 0$ and $|z_i| < \rho_i$, $(i = 1,\, 2,\, ...,\, v)$, where [16, p.157]

$$Q_i = 1 + \sum_{J=1}^{l} \beta_J^{(i)} + \sum_{J=1}^{q_i} \delta_J^{(i)} - \sum_{J=1}^{b} \alpha_J^{(i)} - \sum_{J=1}^{p_i} \gamma_J^{i}, \quad i = 1, ..., v, \tag{6}$$

$$\rho_i = \min_{r_1\,, ...,\, r_v > 0} \left\{E_i\right\}, \quad i = 1\,, ...,\, v \tag{7}$$

and

$$E_i = r_i^{\left(1 + \sum\limits_{J=1}^{q_i} \delta_J^{(i)} - \sum\limits_{J=1}^{p_i} \gamma_J^{(i)}\right)} \frac{\prod\limits_{J=1}^{l} \left(\sum\limits_{i=1}^{v} r_i \beta_J^{(i)}\right)^{\beta_J^{(i)}} \prod\limits_{J=1}^{q_i} \left(\delta_J^{(i)}\right)^{\delta_J^{(i)}}}{\prod\limits_{J=1}^{b} \left(\sum\limits_{i=1}^{v} r_i \alpha_J^{(i)}\right)^{\alpha_J^{(i)}} \prod\limits_{J=1}^{p_i} \left(\gamma_J^{(i)}\right)^{\gamma_J^{(i)}}}. \tag{8}$$

Definition 4. The author [8] introduced a general class of functions called V-function defined in the following form (see also [10]):

$$V_n(x) = V_n^{h_m,\, d,\, g_j}\left[p,\, \tau,\, k,\, w,\, q,\, k_m,\, a_j,\, b_r,\, \alpha,\, \beta,\, \delta;\, x\right]$$

$$= \lambda \sum_{n=0}^{\infty} \frac{(-p)^n \prod_{m=1}^{t} \left[(h_m)_{n+k_m}\right] (d+\alpha n+\beta)^{-\tau} (x/2)^{nk+dw+q}}{\prod_{j=1}^{s} \left[(g_j)_{n+a_j}\right] \prod_{r=1}^{u} \left[(d)_{\alpha n\delta+b_r}\right]}. \tag{9}$$

where
(i) $p,\, k,\, w,\, q,\, \beta,\, \delta,\, k_m,\, a_j,\, b_r$ $(m = 1, ..., t;\, j = 1, ..., s;\, r = 1, ..., u)$ are real numbers.
(ii) $t,\, s,\, u$ are natural numbers.
(iii) $h_m,\, g_j \geq 1$ $(m = 1, ...t;\, j = 1, ..., s)$.
(iv) $\alpha > 0$, $\mathrm{Re}(\tau) > 0$, $\mathrm{Re}(d) > 0$, x is a variable and λ is an arbitrary constant.
(v) The series on the R.H.S of (9) converges absolutely if $t < s$ or $t = s$ with $\left|p(x/2)^k\right| \leq 1$.

For details of convergence conditions of the series on the R.H.S of (9) one may refer to the paper [9].

The general class of functions defined by (9) is very important and general in nature as it unifies and extends a number of useful functions such as unified Riemann-zeta function [5], generalized hypergeometric function [1], Bessel function [2], generalized Bessel function [2], Struve's function [2], Lommel's function [2], generalized Mittag-Leffler function [7, 11], exponential function, sine function, cosine function and MacRobert's E-function [1] etc.(see, e.g.[8, 10]).

Definition 5. The well known Laplace transform is defined as follows:

$$L\{f(x);\, s\} = \int_0^{\infty} e^{-sx} f(x)\, dx = F(s), \qquad \mathrm{Re}(s) > 0, \tag{10}$$

Definition 6. The generalized Weyl fractional integral operator occurring in this paper is defined as follows:

$$W^{\mu}\left[g(x;\, z_1, \ldots, z_v)\, V\{y(x-a)^{\rho}\}\, H\left[z(x-a)^{-\sigma}\right];\, a,\, y,\, z\right]$$

$$= \frac{1}{\Gamma(\mu)} \int_a^{\infty} (x-a)^{\mu-1}\, V\{y(x-a)^{\rho}\}\, H_{P,\,Q}^{M,\,N}\left[z(x-a)^{-\sigma} \left| \begin{matrix} (k_J, \tau_J)_{1,\,P} \\ (\rho_J, \omega_J)_{1,\,Q} \end{matrix} \right.\right] \tag{11}$$

$$\times g(x;\, z_1, ..., z_v)\, dx,$$

provided that the integral on R.H.S of (11) converges absolutely.

In this paper we establish a general theorem which interconnects the Laplace transform and generalized Weyl FIO defined by (10) and (11) respectively. This theorem is then applied to evaluate generalized Weyl FIO of generalized Srivastava-Daoust Lauricella function defined by (4). Several (known and new) special cases of our results are mentioned briefly. Our findings may be useful in handling the problems involving Weyl FIO, as our results involve certain special functions which are highly useful in fractional calculus.

2. Elementary result

In this section an elementary result is established. This result will be used to prove the general theorem in the next section.

Lemma 7. If $\mathrm{Re}\,(c) > 0$, $\rho \in R$, $\sigma > 0$, $\mathrm{Re}\,(\mu + \rho nk + \rho dw + \rho q) > 0$ and the conditions pertaining to the Fox's H-function and V-function are satisfied, then

$$L\left\{x^{\mu-1}\, V_n^{h_m,\,d,\,g_j}\,[p,\,\tau,\,k,\,w,\,q,\,k_m,\,a_j,\,b_r,\,\alpha,\,\beta,\,\delta;\,yx^\rho]\right.$$
$$\left.\times\; H_{P,Q}^{M,N}\left[zx^{-\sigma}\left|\begin{matrix}(k_J,\tau_J)_{1,\,P}\\(\rho_J,\omega_J)_{1,\,Q}\end{matrix}\right.\right];\,c\right\}$$
$$=\frac{\lambda}{c^\mu}\sum_{n=0}^{\infty}\frac{(-p)^n\prod_{m=1}^{t}\left[(h_m)_{n+k_m}\right](d+\alpha n+\beta)^{-\tau}\,\{y/(2c^\rho)\}^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]} \tag{12}$$
$$\times H_{P,\,Q+1}^{M+1,\,N}\left[zc^\sigma\left|\begin{matrix}(k_J,\tau_J)_{1,\,P}\\(\mu+\rho nk+\rho dw+\rho q,\sigma),(\rho_J,\omega_J)_{1,\,Q}\end{matrix}\right.\right]=\psi\,(c)$$

Proof. We first express the V-function occurring on the left hand side of (12) in series form and the Fox's H-function involved therein in terms of Mellin-Barnes type contour integral. Then, changing the order of summation and integrations, we find that left-hand side of (12) is

$$=\lambda\sum_{n=0}^{\infty}\frac{(-p)^n\prod_{m=1}^{t}\left[(h_m)_{n+k_m}\right](d+\alpha n+\beta)^{-\tau}(y/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n+a_j}\right]\prod_{r=1}^{u}(d)_{\alpha n\delta+b_r}} \tag{13}$$
$$\times\frac{1}{2\pi\omega}\int_L\phi\,(\xi)\;z^\xi\left[\int_0^\infty e^{-cx}\,x^{\mu+\rho nk+\rho dw+\rho q-\sigma\xi-1}\,dx\right]d\xi.$$

Now, evaluating the inner integral in (13) with the help of the following known result [3, p.137, Eq.(1)]:

$$\int_0^\infty e^{-cx}\,x^{\lambda-1}\,dx=\frac{\Gamma\,(\lambda)}{c^\lambda},\qquad[\mathrm{Re}\,(c)>0,\,\mathrm{Re}\,(\lambda)>0] \tag{14}$$

we get that left-hand side of (12) is

$$=\lambda\sum_{n=0}^{\infty}\frac{(-p)^n\prod_{m=1}^{t}\left[(h_m)_{n+k_m}\right](d+\alpha n+\beta)^{-\tau}\,(y/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]} \tag{15}$$
$$\times\frac{1}{2\pi\omega}\int_L\phi\,(\xi)\;z^\xi\,\frac{\Gamma(\mu+\rho nk+\rho dw+\rho q-\sigma\xi)}{c^{\mu+\rho nk+\rho dw+\rho q-\sigma\xi}}\,d\xi,$$

provided that $\mathrm{Re}\,(c)>0, \mathrm{Re}\,(\mu+\rho nk+\rho dw+\rho q-\sigma\xi)>0.$

On interpreting the resulting Mellin-Barnes type contour integral in terms of H-function, we arrive at the desired result (12).

3. General theorem

Statement 8. Let $\Omega(r_1, ..., r_v)$ be a sequence of arbitrary complex numbers, $\mathrm{Re}(c) > 0$, $\mathrm{Re}(\mu) > 0$, $a > 0$, $u_i \in R$, $i \in \{1, ..., v\}$,

$$\phi(x;\, z_1, ..., z_v) = x^{\eta-1} \sum_{r_1,...,r_v=0}^{\infty} \Omega(r_1, ..., r_v) \prod_{i=1}^{v} \left\{ \frac{(z_i)^{r_i}}{r_i!} x^{u_i r_i} \right\} \tag{16}$$

and

$$g(c;\, z_1, ..., z_v) = L\{\phi(x;\, z_1, ..., z_v);\, c\} \tag{17}$$

then

$$W^{\mu}\left[g(x;\, z_1, ..., z_v)\, V\{y(x-a)^{\rho}\}\, H\left[z(x-a)^{-\sigma}\right];\, a,\, y,\, z\right]$$

$$= \lambda \frac{a^{\mu-\eta}}{\Gamma(\mu)} \sum_{n,\, r_1,...,r_v=0}^{\infty} \frac{(-p)^n \prod_{m=1}^{t}\left[(h_m)_{n+k}\right](d+\alpha n+\beta)^{-\tau}\,(a^{\rho}y/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]}$$

$$\times \Omega(r_1, ..., r_v) \prod_{i=1}^{v}\left\{\frac{(z_i)^{r_i}}{r_i!}\, a^{-u_i r_i}\right\} \tag{18}$$

$$\times H_{P+1,\,Q+1}^{M+1,\,N+1}\left[za^{-\sigma} \left| \begin{matrix} \left(1+\mu+\rho nk+\rho dw+\rho q-\eta-\sum_{i=1}^{v} u_i r_i,\, \sigma\right), (k_J, \tau_J)_{1,\,P} \\ (\mu+\rho nk+\rho dw+\rho q,\, \sigma), (\rho_J, \omega_J)_{1,\,Q} \end{matrix} \right. \right],$$

provided that

$$\sigma > 0,\ \mathrm{Re}(\mu) - \sigma \max_{1\le J\le N} \mathrm{Re}\left(\frac{k_J - 1}{\tau_J}\right) > 0,\ A > 0,\ |\arg z| < \frac{1}{2} A\pi,\ \rho \in R,$$

where

$$A = \sum_{J=1}^{N} \tau_J - \sum_{J=N+1}^{P} \tau_J + \sum_{J=1}^{M} \omega_J - \sum_{J=M+1}^{Q} \omega_J \tag{19}$$

and the multiple series on the R. H. S. of (16) converges absolutely.

Proof. On substituting the value of $\phi(x;\, z_1, ..., z_v)$ from (16) in (17) and evaluating the Laplace transform with the help of (14), we find that

$$g(c;\, z_1, ..., z_v) = c^{-\eta} \sum_{r_1,...,r_v=0}^{\infty} \Omega(r_1, ..., r_v) \prod_{i=1}^{v}\left\{\frac{(z_i)^{r_i}}{r_i!} c^{-u_i r_i}\right\} \Gamma\left(\eta + \sum_{i=1}^{v} u_i r_i\right), \tag{20}$$

where $\mathrm{Re}(c) > 0$, $\mathrm{Re}(\eta) > 0$, $u_i \in R$, $i \in \{1, ..., v\}$ and multiple series on R.H.S of (20) is absolutely convergent.

Replacing x by $(x-a)$ and using the second shifting theorem for Laplace transform in (12), we get

$$L\left\{(x-a)^{\mu-1}H\,(x-a)\,V_n^{h_m,\,d,\,g_j}\,[p,\,\tau,\,k,\,w,\,q,\,k_m,\,a_j,\,b_r,\,\alpha,\,\beta,\,\delta;\,y(x-a)^{\rho}]\right.$$
$$\left.\times\;H_{P,\,Q}^{M,\,N}\left[z(x-a)^{-\sigma}\left|\begin{smallmatrix}(k_J,\,\tau_J)_{1,\,p}\\(\rho_J,\,\omega_J)_{1,\,Q}\end{smallmatrix}\right.\right]\,;\,c\right\}=e^{-ac}\,\psi\,(c) \tag{21}$$

where $H(x-a)$ is Heaviside Unit function.

Now applying well known Parseval-Goldstein theorem for the Laplace transform for the pairs (17) and (21), we obtain

$$\int_0^{\infty}(x-a)^{\mu-1}H\,(x-a)\,V_n^{h_m,\,d,\,g_j}\,[p,\,\tau,\,k,\,w,\,q,\,k_m,\,a_j,\,b_r,\,\alpha,\,\beta,\,\delta;\,y(x-a)^{\rho}]$$
$$\times\,H_{P,\,Q}^{M,\,N}\left[z(x-a)^{-\sigma}\left|\begin{smallmatrix}(k_J,\,\tau_J)_{1,\,P}\\(\rho_J,\,\omega_J)_{1,\,Q}\end{smallmatrix}\right.\right]g\,(x;\,z_1,...,\,z_v)\;dx$$
$$=\int_0^{\infty}e^{-ax}\,\psi\,(x)\;\phi\,(x;\,z_1,...,\,z_v)\;dx. \tag{22}$$

Expressing L.H.S. of (22) in terms of generalized Weyl fractional integral operator, substituting the series form of $\phi\,(x;\,z_1,...,\,z_v)$ from (16), $\psi\,(x)$ from (12) in R.H.S of (22) and interchanging the order of integration and summation therein, and then evaluating the x-integral thus obtained by using the Laplace transform, we get required result (18).

4. Special cases

(i)If we take $n=0$, $d=1, m=1, j=1, r=1, \tau=1, w=0, q=0, k_1=0, a_1=0, b_1=0, \beta=0, \delta=0$ and $\lambda=1$ in (18), the V-function reduces to unity and we obtain the results due to Goyal and Goyal [4].

(ii)If we take $\rho=1, r=1, d=1, t=p', s=q', p=-2, \tau=1, k=1, w=0, q=0, k_m=0, a_j=0, b_1=-1, \alpha=1, \beta=-1, \delta=1, \lambda=1, M=1, N=P, \sigma=1$ and $Q=Q+1$ in (18), the V-function reduces to generalized hypergeometric function [8] and using the following known relationship [14, p.18, Eq.(2.6.3)] therein:

$$H_{p,\,q+1}^{1,\,p}\left[x\left|\begin{smallmatrix}(1-a_j,\,1)_{1,\,p}\\(0,\,1),\,(1-b_j,\,1)_{1,\,q}\end{smallmatrix}\right.\right]=\prod_{j=1}^{p}\Gamma\,(a_j)\left\{\prod_{j=1}^{q}\Gamma\,(b_j)\right\}^{-1}{}_pF_q\,[(a_p)\,;\,(b_q)\,;\,-x]\,, \tag{23}$$

we have after a little simplification

Corollary 9.

$$W^{\mu}\left[g\left(x;\, z_1,...,\, z_v\right)\, {}_{p'}F_{q'}\left[\left(h_{p'}\right);\, \left(g_{q'}\right);\, y\left(x-a\right)\right]\right.$$

$$\left.\times\ {}_PF_Q\left[\left(A_P\right);\, \left(B_Q\right);\, \tfrac{-z}{x-a}\right]\ ;\, a,\, y,\, z\right]$$

$$= \tfrac{1}{\Gamma(\mu)}\int\limits_a^{\infty}\left(x-a\right)^{\mu-1}{}_{p'}F_{q'}\left[\left(h_{p'}\right);\, \left(g_{q'}\right);\, y\left(x-a\right)\right]$$

$$\times\, {}_PF_Q\left[\left(A_P\right);\, \left(B_Q\right);\, \tfrac{-z}{x-a}\right]\, g\left(x;\, z_1,...,\, z_v\right)dx$$

$$= \tfrac{a^{\mu-\eta}}{\Gamma(\mu)}\prod_{J=1}^{Q}\Gamma\left(B_J\right)\left\{\prod_{J=1}^{P}\Gamma\left(A_J\right)\right\}^{-1}\sum_{n,\, r_1,...,\, r_v=0}^{\infty}\Omega\left(r_1,...,\, r_v\right)\prod_{i=1}^{v}\left\{\tfrac{(z_i)^{r_i}}{r_i!}a^{-u_ir_i}\right\}$$

$$\times\,\frac{\prod\limits_{m=1}^{p'}(h_m)_n}{\prod\limits_{j=1}^{q'}(g_j)_n}(ay)^n\, G^{2,\, P+1}_{P+1,\, Q+2}\left[\tfrac{z}{a}\left|\begin{matrix}\left(1+\mu+n-\eta-\sum\limits_{i=1}^{v}u_ir_i\right),(1-A_P)\\ 0,(\mu+n),(1-B_Q)\end{matrix}\right.\right], \tag{24}$$

where $g\left(x;\, z_1,...,\, z_v\right)$ is given by (20). The conditions of validity of (24) are
$(i)\, a>0,\ \mathrm{Re}\left(\mu\right)>0,\ u_i\in R,\ i\in\{1,\, 2,...,\, v\}$.
$(ii)\ p'\le q'$ or $p'=q'+1$ with $|y|<1$ and $P\le Q$ or $P=Q+1$ with $|z|<1$.
(iii) Series on R.H.S of (24) are absolutely convergent.

(iii)If we take $\rho=1,\ m=1,\ j=1,\ r=1,\ h_1=1,\ g_1=1,\ p=-2,\ \tau=1,\ k=1,\ w=0,\ q=0,\ k_1=0,\ a_1=0,\ b_1=-1,\ \beta=-1,\ \delta=1,\ \lambda=1/\Gamma\left(d\right),\ \sigma=1,\ M=2,\ N=0,\ P=1,\ Q=2,\ k_1=a'-\lambda'+1,\ \tau_1=1,\ \rho_1=a'+\mu'+1/2,\ \omega_1=1,\ \rho_2=a'-\mu'+1/2$ and $\omega_2=1$ in (18), the V-function reduces to the generalized Mittag-Leffler function [7,11] and using the following known relationship [13, p.11, Eq.(1.7.6)] therein:

$$H^{1,\, p}_{p,\, q+1}\left[x\left|\begin{matrix}(a-\lambda+1,1)\\ (a+\mu+1/2,1),(a-\mu+1/2,1)\end{matrix}\right.\right]=x^a\, e^{-\frac{x}{2}}\, W_{\lambda,\, \mu}\left(x\right), \tag{25}$$

we have after a little simplification

Corollary 10.

$$W^{\mu}\left[g\left(x;\, z_1,...,\, z_v\right)E_{\alpha,\, d}\left\{y\left(x-a\right)\right\}W_{\lambda',\, \mu'}\left(\tfrac{z}{x-a}\right);\, a,\, y,\, z\right]$$

$$= \tfrac{1}{\Gamma(\mu)}\int\limits_a^{\infty}\left(x-a\right)^{\mu-a'-1}E_{\alpha,\, d}\left\{y\left(x-a\right)\right\}e^{-\frac{z}{2(x-a)}}\, W_{\lambda',\, \mu'}\left(\tfrac{z}{x-a}\right)$$

$$\times\, g\left(x;\, z_1,...,\, z_v\right)\, dx$$

$$= \tfrac{z^{-a'}a^{\mu-\eta}}{\Gamma(\mu)}\sum_{n,\, r_1,...,\, r_v=0}^{\infty}\Omega\left(r_1,...,\, r_v\right)\prod_{i=1}^{v}\left\{\tfrac{(z_i)^{r_i}}{r_i!}a^{-u_ir_i}\right\}\tfrac{(ay)^n}{\Gamma(\alpha n+d)} \tag{26}$$

$$\times\, H^{3,\, 1}_{2,\, 3}\left[\tfrac{z}{a}\left|\begin{matrix}\left(1+\mu+n-\eta-\sum\limits_{i=1}^{v}u_ir_i,1\right),\left(a'-\lambda'+1,1\right)\\ (\mu+n,1),(a'+\mu'+1/2,1),(a'-\mu'+1/2,1)\end{matrix}\right.\right],$$

where $W_{\lambda',\,\mu'}\left(\frac{z}{x-a}\right)$ is the Whittaker function [1].

The conditions of validity of (26) are easily obtainable from those given with (24).

5. Applications

Example 11. If we choose $\Omega\,(r_1, ..., r_v)$ as given by (5), then from (16) and (20) we have

$$\phi\,(x;\, z_1, ..., z_v)$$

$$= x^{\eta-1} F^{b:\, p_1; ...; \,p_v}_{l:\, q_1;\, ...;\, q_v}\left[\begin{matrix}\left(A_J; \alpha'_J, ..., \alpha_J^{(v)}\right)_{1,\,b} : \left(c'_J, \gamma'_J\right)_{1,\,p_1}; ...; \left(c_J^{(v)}, \gamma_J^{(v)}\right)_{1,\,p_v};\\ \left(B_J; \beta'_J, ..., \beta_J^{(v)}\right)_{1,\,l} : \left(d'_J, \delta'_J\right)_{1,\,q_1}; ...; \left(d_J^{(v)}, \delta_J^{(v)}\right)_{1,\,q_v};\end{matrix} z_1 x^{u_1}, ..., z_v x^{u_v}\right] \tag{27}$$

and

$$g\,(c;\, z_1, ..., z_v)$$

$$= c^{-\eta}\,\Gamma\,(\eta)\, F^{b+1:\, p_1; ...; \,p_v}_{l:\, q_1; ...; \,q_v}\left[\begin{matrix}(\eta; z_1; ...; z_v), \left(A_J; \alpha'_J, ..., \alpha_J^{(v)}\right)_{1,b} :\\ \left(B_J;\, \beta'_J, ..., \beta_J^{(v)}\right)_{1,\,l} \quad :\end{matrix}\right. \tag{28}$$

$$\left.\begin{matrix}\left(c'_J, \gamma'_J\right)_{1,\,p_1}; ...; \left(c_J^{(v)}, \gamma_J^{(v)}\right)_{1,\,p_v};\\ \left(d'_J, \delta'_J\right)_{1,\,q_1}; ...; \left(d_J^{(v)}, \delta_J^{(v)}\right)_{1,\,q_v};\end{matrix}\; z_1 c^{-u_1}, ..., z_v c^{-u_v}\right]$$

$$= c^{-\eta}\,\Gamma\,(\eta)\, F\left[z_1 c^{-u_1}, ..., z_v c^{-u_v}\right], \tag{29}$$

or

$$g\,(x;\, z_1, ..., z_v) = x^{-\eta}\,\Gamma\,(\eta)\, F\left[z_1 x^{-u_1}, ..., z_v x^{-u_v}\right], \tag{30}$$

provided that $\mathrm{Re}\,(c) > 0$, $\mathrm{Re}\,(\eta) > 0$, $u_i \in R^+$, $i \in \{1, ..., v\}$ and the conditions mentioned with (4) are satisfied.

Substituting the value of $\Omega\,(r_1, ..., r_v)$and $g\,(x;\, z_1, ..., z_v)$ in (18) of general theorem, then replacing H-function in the expression so obtained in contour integral form with the help of (1), changing the order of integration and summation therein and reinterpreting the series involved in terms of F-function, we obtain

$$W^{\mu}\left[x^{-\eta} F\left[z_1 x^{-u_1}, ..., z_v x^{-u_v}\right] V\left\{y(x-a)^{\rho}\right\} H\left[z(x-a)^{-\sigma}\right]; a,\, y,\, z\right]$$

$$= \lambda \sum_{n=0}^{\infty} \frac{(-p)^n \prod\limits_{m=1}^{t}\left[(h_m)_{n+k_m}\right](d+\alpha n+\beta)^{-\tau}(a^{\rho}y/2)^{nk+dw+q}(a)^{\mu-\eta}}{\prod\limits_{j=1}^{s}\left[(g_j)_{n+a_j}\right]\prod\limits_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]\Gamma(\eta)\Gamma(\mu)}$$

$$\times \frac{1}{2\pi\omega}\int\limits_{L} \phi\,(\xi)\,(za^{-\sigma})^{\xi}\,\Gamma\,(\mu+\rho nk+\rho dw+\rho q-\sigma\xi)$$

$$\times\Gamma\,(\eta-\mu-\rho nk-\rho dw-\rho q+\sigma\xi)\, F^{b+1:\, p_1; ...; \,p_v}_{l:\, q_1; ...; \,q_v}\left[\begin{matrix}(\eta-\mu-\rho nk-\rho dw-\rho q+\sigma\xi,\, u_1, ..., u_v),\\ --------\end{matrix}\right.$$

$$\left.\begin{matrix}\left(A_J; \alpha'_J, ..., \alpha_J^{(v)}\right)_{1,\,b} : \left(c'_J, \gamma'_J\right)_{1,\,p_1}; ...; \left(c_J^{(v)}, \gamma_J^{(v)}\right)_{1,\,p_v};\\ \left(B_J; \beta'_J, ..., \beta_J^{(v)}\right)_{1,\,l} : \left(d'_J, \delta'_J\right)_{1,\,q_1}; ...; \left(d_J^{(v)}, \delta_J^{(v)}\right)_{1,\,q_v};\end{matrix}\; z_1 a^{-u_1}, ..., z_v a^{-u_v}\right] d\xi. \tag{31}$$

Now, expressing F-function in terms of Mellin-Barnes type contour integral form [14, p.88, Eq. (6.4.2)] and reinterpreting the expression so obtained in terms of H-function of $(v+1)$ variables [14, p. 253-254, Eq. (C.9], we finally obtain

$$W^{\mu}\left[x^{-\eta}F\left[z_1x^{-u_1},...,\, z_vx^{-u_v}\right]V\left\{y(x-a)^{\rho}\right\}H\left[z(x-a)^{-\sigma}\right];\, a,\, y,\, z\right]$$

$$= \lambda\frac{a^{\mu-\eta}}{\Gamma(\mu)\Gamma(\eta)}\sum_{n=0}^{\infty}\frac{(-p)^n\prod_{m=1}^{t}\left[(h_m)_{n+k_m}\right](d+\alpha n+\beta)^{-\tau}(a^{\rho}y/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]}$$

$$\times\prod_{J=1}^{l}\Gamma\left(B_J\right)\left\{\prod_{J=1}^{b}\Gamma\left(A_J\right)\right\}^{-1}\prod_{i=1}^{v}\left[\prod_{J=1}^{q_i}\Gamma\left(d_J^i\right)\left\{\prod_{J=1}^{p_i}\Gamma\left(c_J^i\right)\right\}^{-1}\right]$$

$$\times H^{0,\,b+1:\,M+1,\,N;\,1,\,P_1;...;\,1,\,P_v}_{b+1,\,l:\,P,\,Q+1;\,P_1,\,Q_1+1;...;\,P_v,\,Q_v+1}\left[\begin{matrix} za^{-\sigma}\\ -z_1a^{-u_1}\\ \vdots\\ -z_va^{-u_v}\end{matrix}\,\middle|\,\begin{matrix}(1-\eta+\mu+\rho nk+\rho dw+\rho q;\sigma,u_1,...,u_v),\\ \left(1-B_J;0,\beta'_J,...,\beta_J^{(v)}\right)_{1,\,l}:\end{matrix}\right.$$

$$\left.\begin{matrix}\left(1-A_J;0,\alpha'_J,...,\alpha_J^{(v)}\right)_{1,\,b}:(k_J,\tau_J)_{1,\,P};\left(1-c'_J,\gamma'_J\right)_{1,\,p_1};...;\left(1-c_J^{(v)},\gamma_J^{(v)}\right)_{1,\,p_v}\\ (\mu+\rho nk+\rho dw+\rho q;\sigma),(\rho_J,\omega_J)_{1,\,Q};(0,1),(1-d'_J,\delta'_J)_{1,\,q_1};...;(0,1),\left(1-d_J^{(v)},\delta_J^{(v)}\right)_{1,\,q_v}\end{matrix}\right] \tag{32}$$

The conditions of validity of (32) are

$(i)\, a>0,\, \sigma>0,\, u_i\in R^+,\, i\in\{1,\,2,...,\,v\}$, $\mathrm{Re}\,(\eta)+\sigma\min_{1\le J\le M}\mathrm{Re}\left(\frac{\rho_J}{\omega_J}\right)>0$, $\mathrm{Re}\,(\mu)-\sigma\max_{1\le J\le N}\mathrm{Re}\left(\frac{k_J-1}{\tau_J}\right)>0$.

$(ii)\, A>0, |\arg\, z|<\frac{1}{2}A\pi,$ where A is defined by (19).

(iii) The conditions (modified appropriately) which are given just below (4).

In (32) and elsewhere, $H\left[z_1,...,\, z_v\right]$ stands for the multivariable H-function introduced by Srivastava and Panda through a series of research papers. For the convergence, existence conditions and other details of the above multivariable H-function, one may refer to the treatise by Srivastava, Gupta and Goyal [14].

6. Special cases of (32)

(i)If we take $n=0$, $d=1,\ m=1,\ j=1,\ r=1,\ \tau=1,\ w=0,\ q=0,\ k_1=0,\ a_1=0,\ b_1=0,\ \beta=0,\ \delta=0$ and $\lambda=1$ in (32), the V-function reduces to unity and we obtain the result due to Goyal and Goyal [4].

(ii)If we take $\rho=1,\ m=1,\ j=2,\ r=1,\ h_1=1,\ g_1=1,\ g_2=1,\ p=1,\ \tau=1,\ k=2,\ w=1,\ q=0,\ k_1=0,\ a_1=0,\ a_2=0,\ b_1=0,\ \alpha=1,\ \beta=0,\ \delta=1,\ \lambda=1/\Gamma\left(d\right),\ u_j\to 0,\ J\in\{2,...,\,v\}$, $u_1\,(=u')\neq 0$ and $t=l=0$ in (32), the V-function reduces to the Bessel function and using the following known relationship [14 , p.19, Eq (2.6.11)] therein:

$$_p\psi_q\left[\begin{matrix}(a_j,\alpha_j)_{1,\,p}:\\ (b_j,\beta_j)_{1,\,q}:\end{matrix}x\right]=\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_jr\right)}{\prod_{j=1}^{q}\Gamma\left(b_j+\beta_jr\right)}\frac{x^r}{r!}, \tag{33}$$

we have after a little simplification

Corollary 12.

$$W^{\mu}\left\{x^{-\eta}{}_{p'+1}\psi_{q'}\left[\begin{matrix}(\eta,u'),(c_J,\gamma_J)_{1,\,p'}: \\ (d_J,\delta_J)_{1,\,q'} \quad :\end{matrix}\, z'x^{-\mu'}\right] J_d\left\{y\left(x-a\right)\right\} H\left[z(x-a)^{-\sigma}\right]; a,\, y,\, z\right\}$$

$$= \frac{a^{\mu-\eta}}{\Gamma(\mu)\Gamma(\eta)} \sum_{n=0}^{\infty} \frac{(-1)^n (ay/2)^{2n+d}}{\Gamma(1+d+n)\, n!}$$

$$\times H^{0,\,1:\,M+1,\,N;\,1,\,p_1}_{1,0:\,P,\,Q+1;\,p_1,\,q_1+1}\left[\begin{matrix} za^{-\sigma} \\ -z'a^{-\mu'}\end{matrix}\left|\begin{matrix}\left(1-\eta+\mu+2n+d;\sigma,u'\right):(k_J,\tau_J)_{1,\,P};(1-c_J,\gamma_J)_{1,\,p_1} \\ (\mu+2n+d;\sigma),(\rho_J,\omega_J)_{1,\,Q};(0,1),(1-d_J,\delta_J)_{1,\,q_1}\end{matrix}\right.\right], \tag{34}$$

where the functions occurring on the L.H.S and R.H.S of (34) are Wright's generalized hypergeometric function and H-function of two variables respectively (see e.g [14]).

(iii) Next putting $p_1 = 0$, $q_1 = 1$, $u' = 1$, $d_1 = \eta q' - v$ and $\delta_1 = q'$ in (34), we get

$$W^{\mu}\left\{x^{-\eta}G_{q',\,v,\,\eta}\left[z',\, x^{-1/q'}\right] J_d\left\{y\left(x-a\right)\right\} H\left[z(x-a)^{-\sigma}\right]; a,\, y,\, z\right\}$$

$$= \frac{a^{\mu-\eta}}{\Gamma(\mu)\Gamma(\eta)} \sum_{n=0}^{\infty} \frac{(-1)^n (ay/2)^{2n+d}}{\Gamma(1+d+n)\ n!}$$

$$\times H^{0,\,1:\,M+1,\,N;\,1,\,0}_{1,\,0:\,P,\,Q+1;\,0,\,2}\left[\begin{matrix} za^{-\sigma} \\ -z'/a\end{matrix}\left|\begin{matrix}(1-\eta+\mu+2n+d;\sigma,1):(k_J,\tau_J)_{1,\,P} \quad ; \\ - \quad :(\mu+2n+d;\sigma),(\rho_J,\omega_J)_{1,\,Q};(0,1),(1-\eta q'+v,q')\end{matrix}\right.\right], \tag{35}$$

where $G_{q',\,v,\,\eta}\left[z',\, t\right]$ is Lorenzo-Hartley G-function [12, p.15, Eq.(101)].

Acknowledgements.

The author is extremely thankful to the worthy referee for suggestions given for the improvement of the paper.

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