Journal of Fractional Calculus and Applications
Vol. 11(2) July 2020, pp. 208-223.
ISSN: 2090-5858.
http://math-frac.oreg/Journals/JFCA/

EXISTENCE AND UNIQUENESS OF MILD INTEGRABLE SOLUTIONS TO SOME QUASILINEAR CAUCHY PROBLEMS FOR NONLOCAL FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS

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Abstract. The purpose of this paper is to discuss the existence and uniqueness of mild L^1-solutions to some quasilinear Cauchy problems for Caputo fractional integrodifferential equations with nonlocal conditions. The nonlinear term of the considered problem contains a fractional derivative or fractional integral. Illustrative examples will be given.

1. Introduction

In this paper, we discuss the existence and uniqueness of mild integrable solutions to the quasilinear Cauchy problems

$$^cD^\alpha u(t) = A(t,u)u(t) + f(t, u(t), I^\alpha u(t)),\ a.e.,\ t \in J \tag{1}$$

and

$$^cD^\alpha u(t) = A(t,u)u(t) + f(t, u(t), {}^cD^\alpha u(t)),\ a.e.,\ t \in J \tag{2}$$

each together with the nonlocal condition

$$\sum_{k=1}^{m} a_k\ u(t_k) = u_0 \tag{3}$$

where $u_0 \in D(A)$, $\sum_{k=1}^m a_k \neq 0$ and $J = [0,T],\ T < \infty$. $^cD^\alpha, I^\alpha$ denote the Caputo derivative and fractional integral of order $\alpha \in (0,1)$, respectively. $A(t,u)$ is a bounded linear operator. t_k satisfy $0 < t_1 < t_2 < .. < t_m < T,\ k = 1, 2, .., m$. Our results are based upon the contraction mapping principle and Krasnoselskii's fixed-point theorem.

In fact, papers on integrable solutions for fractional-order integrodifferential equations are limited, see for instance: El-Sayed and Abd El-Salam [10, 11], Benchohra and Souid [2, 3, 4, 5], Gaafer [15] and Souid [30]. Integrodifferential equations of fractional-order have affirmed to be valuable tools in modelling of many

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2010 *Mathematics Subject Classification.* 26A33, 34A08, 45N05 .
Key words and phrases. Caputo derivative, Krasnoselskii's fixed-point theorem, Kolmogorov compactness criterion.
Submitted Aug. 25, 2019.

phenomena in various fields of science and engineering. For the history, applications and significant results on fractional derivatives and integrals, we refer to [1, 17, 19, 23, 25, 27, 31]. Many author's are interested in investigating the existence and uniqueness of solutions to quasilinear fractional Cauchy problems in Banach spaces, see [26, 28, 29]. A lot of papers contain a fractional derivative or integral in the nonlinear term of the considered Cauchy problem, see [4, 18, 22]. The existence of solutions for abstract Cauchy differential equations with nonlocal condition in a Banach space has been considered first by Byszewski [7]. Deng [9] indicated that the nonlocal condition, as a generalization of the classical condition, gives more precise measurements, accurate results and better effect for describing natural phenomena. For different forms of nonlocal conditions, see [12, 16].

This paper is organized as follows: In section 2, Some notations, main definitions and theorems, which are used through out the paper, will be given. In section 3, we will study the existence and uniqueness of mild L^1-solutions to the quasilinear problem (1) with the nonlocal condition (3). A clarifying example will be given. In section 4, we will investigate the existence and uniqueness of mild L^1-solutions to the nonlocal quasilinear problem (2)-(3) with giving an illustrative example.

2. Preliminaries

Here, we introduce some notations, main definitions and theorems which are crucial in what follows.

As usual, let $\mathbb{R}$ be the set of real numbers. $AC(J,\mathbb{R})$ be the space of functions which are absolutely continuous on J, $L^1(J,\mathbb{R})$ be the class of Lebesgue integrable functions $v : J \to \mathbb{R}$ with the norm $\|v\|_{L^1}= \int_J |v(t)|dt$, and $B(L^1(J,\mathbb{R}))$ be the set of all bounded linear operators from $L^1(J,\mathbb{R})$ into itself with the norm $\|A\|_B = \sup_{\|u\|=1}\{\|Au(t)\|, u \in L^1(J,\mathbb{R})\}$.

Definition 1 [19, 24] The fractional integral of order $\alpha \in \mathbb{R}^+$ with the lower limit 0 of a function $u \in L^1(J,\mathbb{R})$ is defined by

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds,$$

where $\Gamma(.)$ is the Euler gamma function.

If $v \in L^1(J,\mathbb{R})$ and $\alpha > 0$, the integral $I^\alpha v(t)$ exists for almost every $t \in J$. Moreover, the function $I^\alpha v$ itself is also an element of $L^1(J,\mathbb{R})$.

Definition 2 [19, 24] The fractional derivative of order α where $0 < \alpha < 1$ with the lower limit 0 of a function $u \in AC(J,\mathbb{R})$ is defined by

$$^cD^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u'(s)ds$$

where the prime sign denotes the usual first derivative.

For the Caputo fractional derivative and the fractional integral, we have

$$^cD^\alpha I^\alpha u(t) = u(t) \text{ and } I^\alpha\ {}^cD^\alpha u(t) = u(t) - u(0),\ \alpha \in (0,1).$$

The Kolmogorov compactness criterion [4, 8, 20] gives the necessary and sufficient conditions in order to the set ψ of functions in $L^p(J,\mathbb{R})$ be relatively compact.

Theorem 1 Let $\psi \subseteq L^p(J,\mathbb{R}),\ 1 \leq p \leq \infty$. If

(a) ψ is bounded in $L^p(J,\mathbb{R})$, and

(b) $y_h := \frac{1}{h}\int_t^{t+h} y(s)ds \to y$ as $h \to 0$ uniformly with respect to $y \in \psi$,

then ψ is relatively compact in $L^p(J,\mathbb{R})$.

The results of Schauder's fixed point theorem and the contraction mapping principle are combined into the following result by M.A. Krasnoselskii [13, 21, 32].

Theorem 2 Let Q be a nonempty, closed and convex subset of a Banach space X. Suppose that $A : Q \to X$ and $B : Q \to X$ satisfy the following properties:

(a) $Ax + By \in Q$ for all $x, y \in Q$;

(b) A is continuous and compact;

(c) B is a contraction mapping.

Then, there exists $q \in Q$ such that $Aq + Bq = q$.

3. Nonlocal quasilinear fractional integrodifferential equation

This section deals with investigating the existence and uniqueness of mild L^1-solutions to the quasilinear problem (1) with the nonlocal condition (3).

We introduce the following assumptions:

(H_1) $A(t,u)$ is a bounded linear operator on $L^1(J,\mathbb{R})$, for each $t \in J$ and $u \in L^1(J,\mathbb{R})$, and there exist constants $a, b > 0$ such that for all $t \in J$ and $u, v \in L^1(J,\mathbb{R})$

$$\|A(.,u) - A(.,v)\| \leq a\|u - v\|_{L_1} \text{ and } b = \max_{t\in J} \|A(t,0)\|.$$

(H_2) $f : J \times \mathbb{R}^2 \to \mathbb{R}$ is measurable in $t \in J$, for any $(u,v) \in \mathbb{R}^2$ and continuous in $(u,v) \in \mathbb{R}^2$, for almost all $t \in J$;

(H_3) There exists a constant $q > 0$ such that:

$$|f(t,u_2,v_2) - f(t,u_1,v_1)| \ \leq\ q\,(\ |u_2 - u_1| + |v_2 - v_1|\ ),$$

where $(t,u_i,v_i) \in J \times \mathbb{R}^2,\ i = 1,2$ and there exists a positive function $\omega(t) \in L^1(J,\mathbb{R})$ such that for all $t \in J$,

$$|f(t,0,0)| \leq \omega(t).$$

Consider the nonempty, convex, bounded and closed set B_r such that

$$B_r = \left\{u \in L^1(J,\mathbb{R}) : \|u\|_{L^1} \leq r,\ r > 0\right\}. \tag{4}$$

To facilitate the next discussion, let

$$\gamma_1 := \frac{T^\alpha}{\Gamma(\alpha+1)},\ \gamma_2 := \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{2\alpha}}{\Gamma(2\alpha+1)} \text{ and } \rho := \frac{1}{\sum_{k=1}^m |a_k|}.$$

Now, we give some assistant calculations.

From (H_1), we get

$$\begin{aligned}\|A(.,u)\| &\leq \|A(.,u) - A(.,0)\| + \|A(.,0)\| \\ &\leq a\|u\|_{L^1} + b. \end{aligned} \tag{5}$$

Moreover,

$$\begin{aligned}\|A(.,u)u - A(.,v)v\|_{L^1} &\leq \|A(.,u)u - A(.,u)v\|_{L^1} + \|A(.,u)v - A(.,v)v\|_{L^1}\\ &\leq \|A(.,u)\|\|u-v\|_{L^1} + \|A(.,u) - A(.,v)\|\|v\|_{L^1}\\ &\leq (a\|u\|_{L^1} + b)\|u-v\|_{L^1} + a\|u-v\|_{L^1}\|v\|_{L^1}\\ &\leq \left[a(\|u\|_{L^1} + \|v\|_{L^1}) + b\right]\|u-v\|_{L^1}. \end{aligned} \tag{6}$$

From (H_3), we obtain

$$\begin{aligned}|f(t,u,v)| &\leq |f(t,u(t),v(t)) - f(t,0,0)| + |f(t,0,0)|\\ &\leq q\,(\ |u(t)| + |v(t)|\ ) + \omega(t),\end{aligned} \tag{7}$$

then

$$\begin{aligned}\|f(.,u,v)\|_{L^1} &\leq \int_0^T [\ q\,(\ |u(t)| + |v(t)|\ ) + \omega(t)\ ]\,dt\\ &\leq\ q\,(\|u\|_{L^1} + \|v\|_{L^1}) + \|\omega\|_{L^1}.\end{aligned} \tag{8}$$

Moreover,

$$\begin{aligned}\|f(.,u_2,v_2) - f(.,u_1,v_1)\|_{L^1} &= \int_0^T |f(t,u_2(t),v_2(t) - f(t,u_1(t),v_1(t)|\ dt\\ &\leq \int_0^T q\,(\ |u_2(t) - u_1(t)| + |v_2(t) - v_1(t)|\ )\ dt\\ &\leq\ q\,(\ \|u_2 - u_1\|_{L^1} + \|v_2 - v_1\|_{L^1})\,.\end{aligned} \tag{9}$$

Consider the integral

$$\begin{aligned}\int_0^t s^\alpha (t-s)^{\alpha-1} ds &= t^{\alpha-1}\int_0^t s^\alpha \left(1 - st^{-1}\right)^{\alpha-1} ds\\ &= t^{2\alpha}\int_0^1 (z)^\alpha\,(1-z)^{\alpha-1}\,dz\\ &= \frac{\Gamma(\alpha+1)\Gamma(\alpha)}{\Gamma(2\alpha+1)}\ t^{2\alpha}.\end{aligned} \tag{10}$$

Using (7), (10) and Young's convolution inequality [6], we obtain

$$\begin{aligned}&\|I^\alpha f(.,u(.),I^\alpha u(.))\|_{L^1}\\ &\quad = \int_0^T |I^\alpha f(t,u(t),I^\alpha u(t))|\,dt\\ &\quad \leq \int_0^T\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}u(\tau)\ d\tau\ )|\ ds\ dt\\ &\quad \leq \int_0^T\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[q\left(|u(s)| + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|u(\tau)|\ d\tau\right) + |w(s)|\right] ds\ dt\\ &\quad \leq\ q\gamma_2\ \|u\|_{L_1} + \gamma_1\ \|\omega\|_{L_1}.\end{aligned} \tag{11}$$

Applying (H_3), (10) and Young's convolution inequality, we get

$$\begin{aligned}
&\|I^\alpha\left[f(.,u(.),I^\alpha u(.))-f(.,v(.),I^\alpha v(.))\right]\|_{L^1}\\
&=\int_0^T |\ I^\alpha\left[f(t,u(t),I^\alpha u(t))-f(t,v(t),I^\alpha v(t))\right]|\,dt\\
&\leq \int_0^T\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s,u(s),\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}u(\tau)\ d\tau\ )\right.\\
&\qquad\qquad \left.-f(t,v(t),\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}v(\tau)\ d\tau\ )\right| dsdt\\
&\leq \int_0^T\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ q\left(|u(s)-v(s)|+\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|u(\tau)-v(\tau)|\ d\tau\right) ds\ dt\\
&\leq\ q\gamma_2\ \|u-v\|_{L_1}. \qquad (12)
\end{aligned}$$

From (5),

$$\|I^\alpha A(.,u)u(.)|\|_{L^1}\leq\ \gamma_1\,(a\|u\|_{L_1}+b)\,\|u\|_{L_1}. \qquad (13)$$

From (6),

$$\|I^\alpha\left[A(.,u)u(.)-A(.,v)v(.)\right]\|_{L_1}\leq\ \gamma_1\ \left[a(\|u\|_{L^1}+\|v\|_{L^1})+b\right]\|u-v\|_{L^1}. \qquad (14)$$

We need the following lemma to give the main results.

Lemma 1 The solution of the quasilinear problem (1) with the nonlocal condition (3) can be expressed by the integral equation

$$\begin{aligned}
u(t)\ &=\ \frac{u_0}{\sum_{k=1}^m a_k}-\frac{1}{\sum_{k=1}^m a_k}\sum_{k=1}^m a_k\ I^\alpha A(t,u)u(t)|_{t=t_k}\\
&-\frac{1}{\sum_{k=1}^m a_k}\sum_{k=1}^m a_k\ I^\alpha f(t,u(t),I^\alpha u(t))|_{t=t_k}\\
&+\ I^\alpha A(t,u)u(t)+I^\alpha f(t,u(t),I^\alpha u(t)). \qquad (15)
\end{aligned}$$

Proof

Let $u(t)$ be a solution of problem (1) together with condition (3). Operating I^α on (1), we have

$$u(t)\ =\ u(0)+I^\alpha A(t,u)u(t)+I^\alpha f(t,u(t),I^\alpha u(t)). \qquad (16)$$

Putting $t=t_k$ in (16) and using (3), we have

$$\begin{aligned}
u_0&=\sum_{k=1}^m a_k\ u(t_k)\\
&=\sum_{k=1}^m a_k\ u(0)+\sum_{k=1}^m a_k\ I^\alpha A(t,u)u(t)|_{t=t_k}+\sum_{k=1}^m a_k\ I^\alpha f(t,u(t),I^\alpha u(t))|_{t=t_k},
\end{aligned}$$

then

$$\begin{aligned}
u(0)&=\frac{u_0}{\sum_{k=1}^m a_k}-\frac{1}{\sum_{k=1}^m a_k}\sum_{k=1}^m a_k\ I^\alpha A(t,u)u(t)|_{t=t_k}\\
&-\frac{1}{\sum_{k=1}^m a_k}\sum_{k=1}^m a_k\ I^\alpha f(t,u(t),I^\alpha u(t))|_{t=t_k}. \qquad (17)
\end{aligned}$$

Substituting (17) into (16), we get the required.

By a mild integrable solution of the quasilinear problem (1) with the nonlocal condition (3), we mean a function $u \in L^1(J,\mathbb{R})$ such that $\sum_{k=1}^m a_k\ u(t_k) = u_0,\ \sum_{k=1}^m a_k \neq 0$ and $u(t)$ satisfies (15).

The following theorem gives the uniqueness result.

Theorem 3 Let the assumptions (H_1)-(H_3) be satisfied. Then, the quasilinear problem (1) with the nonlocal condition (3) has a unique mild solution $u \in L^1(J,\mathbb{R})$ if for $\lambda_1 := b\gamma_1 + q\gamma_2 \in (0, \frac{1}{2})$, we have

$$a\gamma_1[\ \rho\|u_0\|_{L_1} + 2\gamma_1\|\omega\|_{L_1}\ ] \leq \frac{1}{8}(1-2\lambda_1)^2 \text{ and } r < \frac{1-2\lambda_1}{4a\gamma_1}$$

where $r > 0$ is the solution of the quadratic equation

$$2a\gamma_1\ r^2 + [2\,(b\gamma_1 + q\gamma_2) - 1]\ r + \rho\|u_0\|_{L^1} + 2\gamma_1\|\omega\|_{L^1} = 0. \tag{18}$$

Proof

Suppose that the operator $K : L^1(J,\mathbb{R}) \to L^1(J,\mathbb{R})$ is defined by

$$\begin{aligned} Ku(t) = &\ \frac{u_0}{\sum_{k=1}^m a_k} - \frac{1}{\sum_{k=1}^m a_k}\sum_{k=1}^m a_k\ I^\alpha A(t,u)u(t)|_{t=t_k} \\ &- \frac{1}{\sum_{k=1}^m a_k}\sum_{k=1}^m a_k\ I^\alpha f(t,u(t),I^\alpha u(t))|_{t=t_k} +\ I^\alpha A(t,u)u(t) \\ &+ I^\alpha f(t,u(t),I^\alpha u(t)). \end{aligned} \tag{19}$$

The proof will be given in two steps.

Step 1. $(KB_r \subset B_r)$

Let u be an arbitrary element in B_r.

Using (19) with applying (11) and (13), we have

$$\begin{aligned} \|Ku\|_{L^1} = &\int_0^T \|Ku(t)\|dt \\ &\leq \rho\int_0^T \|u_0\|dt + \rho\sum_{k=1}^m |a_k|\int_0^T \|I^\alpha A(t,u)u(t)|_{t=t_k}\|\,dt \\ &\quad + \rho\sum_{k=1}^m |a_k|\int_0^T \|I^\alpha f(t,u(t),I^\alpha u(t))|_{t=t_k}\|\,dt \\ &\quad + \int_0^T \|I^\alpha A(t,u)u(t)\|\,dt + \int_0^T \|I^\alpha f(t,u(t),I^\alpha u(t))\|\,dt \\ &\leq\ \rho\|u_0\|_{L^1} + \rho\sum_{k=1}^m |a_k|\ \|I^\alpha A(t,u)u(t)|_{t=t_k}\|_{L^1} \\ &\quad + \rho\sum_{k=1}^m |a_k|\ \|I^\alpha f(t,u(t),I^\alpha u(t))|_{t=t_k}\|_{L^1} \\ &\quad +\ \|I^\alpha A(.,u)u(.)|\|_{L^1} + \|I^\alpha f(.,u(.),I^\alpha u(.))\|_{L^1}, \end{aligned}$$

$$\|Ku\|_{L^1}$$
$$\leq \rho\|u_0\|_{L^1} + \rho\gamma_1(a\|u\|_{L^1}+b)\|u\|_{L^1}\sum_{k=1}^{m}|a_k| + \rho\left[q\gamma_2\|u\|_{L^1}+\gamma_1\|\omega\|_{L^1}\right]\sum_{k=1}^{m}|a_k|$$
$$+\gamma_1(a\|u\|_{L^1}+b)\|u\|_{L^1} + q\gamma_2\|u\|_{L^1}+\gamma_1\|\omega\|_{L^1},$$

then

$$\|Ku\|_{L^1} \leq r$$

where r satisfies the quadratic equation (18). Therefore, K maps B_r into itself.

Step 2. (K is a contraction mapping)

Using (19) with applying (12) and (14), we have

$$\begin{aligned}\|Ku(t)-Kv(t)\| \leq\ & \rho\sum_{k=1}^{m}|a_k|\ \|I^\alpha\left[A(t,u)u(t)-A(t,v)v(t)\right]|_{t=t_k}\| \\ &+\rho\sum_{k=1}^{m}|a_k|\ \|I^\alpha\left[f(t,u(t),I^\alpha u(t))-f(t,v(t),I^\alpha v(t))\right]|_{t=t_k}\| \\ &+\|I^\alpha\left[A(.,u)u(.)-A(.,v)v(.)\right]\| \\ &+\|I^\alpha\left[f(t,u(t),I^\alpha u(t))-f(t,v(t),I^\alpha v(t))\right]\|,\end{aligned}$$

then

$$\|Ku-Kv\|_{L_1}$$
$$\leq \rho\gamma_1\left[a(\|u\|_{L^1}+\|v\|_{L^1})+b\right]\|u-v\|_{L^1}\sum_{k=1}^{m}|a_k| + \rho q\gamma_2\ \|u-v\|_{L^1}\sum_{k=1}^{m}|a_k|$$
$$+\gamma_1\left[a(\|u\|_{L^1}+\|v\|_{L^1})+b\right]\ \|u-v\|_{L^1}+q\gamma_2\ \|u-v\|_{L^1}$$

For $u,v\in B_r$, we get

$$\|Ku-Kv\|_{L^1} \leq 2(2a\gamma_1 r+b\gamma_1+q\gamma_2)\ \|u-v\|_{L^1} \tag{20}$$

Since $2(2a\gamma_1 r+b\gamma_1+q\gamma_2)<1$, K is a contraction mapping [13, 14] and it has a unique fixed point which is the unique solution of the integral equation (15). Therefore, by lemma 1, the quasilinear problem (1) with the nonlocal condition (3) has a unique mild solution $u\in B_r\subset L^1(J,\mathbb{R})$. This completes the proof.

For the existence result, we give the following theorem.

Theorem 4 Let the assumptions (H_1)-(H_3) are satisfied. The quasilinear problem (1) with the nonlocal condition (3) has at least one mild integrable solution $u\in L^1(J,\mathbb{R})$ if for $\lambda_1\in(0,\frac{1}{2})$, we have

$$a\gamma_1[\ \rho\|u_0\|_{L_1}+2\gamma_1\|\omega\|_{L_1}\ ] \leq \frac{1}{8}(1-2\lambda_1)^2 \text{ and } r<\frac{1+2(q\gamma_2-\lambda_1)}{4a\gamma_1}$$

where $r>0$ is the solution of the quadratic equation (18).

Proof

Suppose that the operator K is defined by $Ku(t)=K_1u(t)+K_2u(t)$, where

$$K_1u(t) = I^\alpha f(t,u(t),I^\alpha u(t)) - \frac{1}{\sum_{k=1}^{m}a_k}\sum_{k=1}^{m}a_k\ I^\alpha f(t,u(t),I^\alpha u(t))|_{t=t_k}, \tag{21}$$

and

$$K_2u(t) = \frac{u_0}{\sum_{k=1}^m a_k} - \frac{1}{\sum_{k=1}^m a_k}\sum_{k=1}^m a_k\ I^\alpha(A(t,u)u(t))|_{t=t_k} + I^\alpha(A(t,u)u(t)). \tag{22}$$

The proof will be given in four steps.

Step 1. ($K_1u + K_2v \in B_r$ whenever $u, v \in B_r$)

Using (21) and (22) with applying (11) and (13), we obtain

$$\begin{aligned}\|K_1u(t) + K_2v(t)\| &\\ \leq\ &\|I^\alpha f(t,u(t),I^\alpha u(t))\| + \rho\sum_{k=1}^m|a_k|\ \|I^\alpha f(t,u(t),I^\alpha u(t))|_{t=t_k}\|\\ &+\rho\|u_0\|\ +\rho\sum_{k=1}^m|a_k|\ \|I^\alpha A(t,v)v(t)|_{t=t_k}\| +\ \|I^\alpha A(.,v)v(.)|\|,\end{aligned}$$

then

$$\begin{aligned}\|K_1u + K_2v\|_{L^1} &\\ \leq\ &q\gamma_2\|u\|_{L^1} + \gamma_1\|\omega\|_{L^1} +\ \rho[q\gamma_2\|u\|_{L^1} + \gamma_1\|\omega\|_{L^1}]\ \sum_{k=1}^m|a_k|\\ &+\rho\|u_0\|_{L^1} + \rho\gamma_1(a\|v\|_{L^1}+b)\|v\|_{L^1}\sum_{k=1}^m|a_k| + \gamma_1(a\|v\|_{L^1}+b)\|v\|_{L^1}.\end{aligned}$$

For $u, v \in B_r$, we get

$$\|K_1u + K_2v\|_{L^1}\ \leq\ r,$$

where r is the solution of the quadratic equation (18).

Therefore, $K_1u + K_2v \in B_r$ whenever $u, v \in B_r$.

Step 2. (K_1 is continuous)

Let $\{u_n\}_{n=1}^\infty$ be a sequence in $L^1(J,\mathbb{R})$ such that $u_n \to u \in L^1(J,\mathbb{R})$ as $n \to \infty$ for all $t \in J$.

By using (21) and applying (12), we have

$$\begin{aligned}\|K_1u_n(t) - K_1u(t)\| &\\ \leq\ &\|I^\alpha\left(f(t,u_n(t),I^\alpha u_n(t)) - f(t,u(t),I^\alpha u(t))\right)\|\\ &+\rho\sum_{k=1}^m|a_k|\ \|I^\alpha\left(f(t,u_n(t),I^\alpha u_n(t)) - f(t,u(t),I^\alpha u(t))\right)|_{t=t_k}\|,\end{aligned}$$

then

$$\begin{aligned}\|K_1u_n - K_1u\|_{L_1} &\leq\ q\gamma_2\ \|u_n-u\|_{L^1}\ +\ \rho q\gamma_2\ \|u_n-u\|_{L^1}\sum_{k=1}^m|a_k|\\ &\leq\ 2q\gamma_2\ \|u_n-u\|_{L^1}.\end{aligned} \tag{23}$$

Letting $n \to \infty$, the right hand side of (23) tends to zero. Therefore, K_1 is continuous.

Step 3. (K_1 is a compact operator)

Clearly that K_1B_r is bounded in $L^1(J,\mathbb{R})$ which is the first condition of Kolmogorov compactness criterion (Theorem 1).

Firstly, we prove the continuity of $K_1 u$ for all $u \in B_r$. For $0 \leq t_1 < t_2 \leq T$, we have

$$\|K_1u(t_2) - K_1u(t_1)\| \leq \int_0^{t_1} \left(\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \|f(s,u(s),I^\alpha u(s))\| \, ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(s,u(s),I^\alpha u(s))\| \, ds,$$

then

$$\begin{aligned} \|K_1u(t_2) - K_1u(t_1)\|_{L^1} \leq \; & (q\gamma_2 \, r + \|\omega\|_{L^1}) \int_0^{t_1} \left(\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \right) ds \\ & + (q\gamma_2 \, r + \|\omega\|_{L^1}) \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \, ds \\ \leq \; & \frac{q\gamma_2 \, r + \|\omega\|_{L^1}}{\Gamma(\alpha+1)} \, (t_2^\alpha - t_1^\alpha). \end{aligned}$$

As $t_2 \to t_1$, we obtain $\|K_1u(t_2) - K_1u(t_1)\|_{L^1} \to 0$. This shows that K_1u is continuous.

Now, we have to show that $(K_1u)_h \to (K_1u)$ in $L^1(J,\mathbb{R})$ for each $u \in B_r$. Consider then,

$$\begin{aligned} \|(K_1u)_h - (K_1u)\|_{L^1} &= \int_0^T |(K_1u)_h(t) - (K_1u)(t)| \, dt \\ &= \int_0^T \left| \frac{1}{h} \int_t^{t+h} (K_1u)(s) ds - (K_1u)(t) \right| dt. \end{aligned} \tag{24}$$

Since K_1u is continuous on J for all $u \in B_r$, then $\lim_{h\to 0} \frac{1}{h}\int_t^{t+h}(K_1u)(s)ds = (K_1u)(t)$. That is, the right hand side of (24) tends to zero as $h \to 0$ and $\frac{1}{h}\int_t^{t+h}(K_1u)(s)ds \to (K_1u)$ uniformly. Then, by Kolmogorov compactness criterion, $\{K_1u(t)\}$ is relatively compact. Therefore, K_1 is a compact operator.

Step 4. (K_2 is a contraction mapping)

Using (22) and (14), we have

$$\|K_2u(t) - K_2v(t)\| \leq \rho \sum_{k=1}^m |a_k| \, \|I^\alpha(A(t,u)u(t) - A(t,v)v(t))|_{t=t_k}\| + \|I^\alpha(A(.,u)u(.) - A(.,v)v(.))\|,$$

then for $u, v \in B_r$ with applying (6), we get

$$\begin{aligned} \|K_2u(t) - K_2v(t)\|_{L^1} \leq \; & \rho\gamma_1(2ar+b)\|u-v\|_{L^1} \sum_{k=1}^m |a_k| + \gamma_1(2ar+b)\|u-v\|_{L^1} \\ \leq \; & 2\gamma_1(2ar+b)\|u-v\|_{L^1}. \end{aligned} \tag{25}$$

Since $2\gamma_1(2ar+b) < 1$, K_2 is a contraction mapping. As a consequence of Kranoselskii's fixed point theorem, K has at least one fixed point. Then, the quasilinear problem (1) with condition (3) has at least one mild solution $u \in B_r \subset L^1(J,\mathbb{R})$. Therefore, the proof is completed.

Example 1 Consider the following fractional nonlocal problem

$$\begin{cases} {}^cD^{\frac{2}{5}}x(t) = 10^{-3}e^{-t}\sin(x(t))\ x(t) + \frac{1}{(e^t+9)(1+|x(t)|+|I^\alpha x(t)|)},\ t \in [0,1]; \\ \sum_{k=1}^{2} 4\ x(t_k) = 1,\ 0 < t_1 < t_2 < 1. \end{cases} \tag{26}$$

Set

$$A(t,x) = 10^{-3}e^{-t}\sin(x(t))\ I,\ (t,x) \in [0,1]\times\mathbb{R}$$

and

$$f(t,x,y) = \frac{1}{(e^t+9)(1+|x|+|y|)},\ (t,x,y) \in [0,1]\times\mathbb{R}\times\mathbb{R}.$$

Then,

$$\begin{aligned} \|A(t,x_1) - A(t,x_2)\| &= \left\|10^{-3}e^{-t}\left(\sin(x_1(t)) - \sin(x_2(t))\right)\right\| \\ &\le 10^{-3}e^{-t}\ \|x_1 - x_2\| \\ &\le 10^{-3}\ \|x_1 - x_2\|, \end{aligned}$$

and

$$\begin{aligned} |f(t,x_1,y_1) - f(t,x_2,y_2)| &= \left|\frac{1}{(e^t+9)}\left(\frac{1}{1+|x_1|+|y_1|} - \frac{1}{1+|x_2|+|y_2|}\right)\right| \\ &= \left|\frac{|x_2|-|x_1|+|y_2|-|y_1|}{(e^t+9)(1+|x_1|+|y_1|)(1+|x_2|+|y_2|)}\right| \\ &\le \frac{|x_1-x_2|+|y_1-y_2|}{(e^t+9)(1+|x_1|+|y_1|)(1+|x_2|+|y_2|)} \\ &\le \frac{1}{e^t+9}\left(|x_1-x_2|+|y_1-y_2|\right) \\ &\le \frac{1}{10}\left(|x_1-x_2|+|y_1-y_2|\right). \end{aligned}$$

So, we have
$T = 1,\ \alpha = \frac{2}{5},\ m = 2,\ q = \|\omega\|_{L^1} = \frac{1}{10},\ \sum_{k=1}^{m} a_k = 8 \neq 0,\ u_0 = 1,\ a = 10^{-3},\ b = 0,\ \rho = \frac{1}{8},\ \gamma_1 = \frac{1}{\Gamma(1.4)},\ \gamma_2 = \frac{1}{\Gamma(1.8)} + \frac{1}{\Gamma(1.4)}$ and $\lambda_1 = \frac{11}{50} \in (0, \frac{1}{2})$.
The quadratic equation will be

$$0.002254\ r^2 - 0.5598\ r + 1.5504 = 0$$

which gives $r = 2.8$. Therefore, all conditions of Theorem 3 are satisfied. Then, problem (26) has a unique mild solution $x \in B_{2.8} \subset L^1([0,1],\mathbb{R})$.

4. Nonlocal quasilinear fractional implicit differential equation

In this section, we investigate the existence and uniqueness of mild integrable solutions to the nonlocal problem (2)-(3).

Let $y(t)$ be a solution of the integral equation

$$y(t) = A(t,u)\left(V_y + I^\alpha y(t)\right) + f(t,\ V_y + I^\alpha y(t),\ y(t)), \tag{27}$$

where for brevity,

$$V_y := \frac{u_0}{\sum_{k=1}^{m} a_k} - \frac{1}{\sum_{k=1}^{m} a_k}\sum_{k=1}^{m} a_k\ I^\alpha y(t)|_{t=t_k}, \tag{28}$$

Lemma 2 Let $u(t)$ be a solution of the nonlocal quasilinear problem (2)-(3). Then, $u(t)$ is a solution of the integral equation

$$u(t) = V_y + I^{\alpha} y(t). \tag{29}$$

Proof

Let $u(t)$ be a solution of the nonlocal quasilinear problem (2)-(3) and

$$^{c}D^{\alpha} u(t) = y(t). \tag{30}$$

Operating I^{α} on both sides of (30), we get

$$u(t) = u(0) + I^{\alpha} y(t). \tag{31}$$

Putting $t = t_k$ in (31) and using condition (3), we obtain

$$u_0 = \sum_{k=1}^{m} a_k \ u(t_k) = \sum_{k=1}^{m} a_k u(0) + \sum_{k=1}^{m} a_k \ I^{\alpha} y(t)|_{t=t_k},$$

then

$$u(0) = \frac{u_0}{\sum_{k=1}^{m} a_k} - \frac{1}{\sum_{k=1}^{m} a_k} \sum_{k=1}^{m} a_k \ I^{\alpha} y(t)|_{t=t_k} = V_y. \tag{32}$$

Substituting (32) into (31), we get the required.

By a mild integrable solution of the nonlocal quasilinear problem (2)-(3), we mean a function $u \in L^1(J, \mathbb{R})$ such that $\sum_{k=1}^{m} a_k \ u(t_k) = u_0$, $\sum_{k=1}^{m} a_k \neq 0$ and $u(t)$ satisfies the integral equation (29).

Consider the nonempty, convex, bounded and closed set B_σ such that

$$B_\sigma = \left\{ y \in L^1(J, \mathbb{R}) : \|y\|_{L^1} \leq \sigma, \ \sigma > 0 \right\}. \tag{33}$$

In what follows, we display some useful calculations.
For $y \in L^1(J, \mathbb{R})$,

$$\|I^{\alpha} y\|_{L^1} \leq \gamma_1 \ \|y\|_{L^1}. \tag{34}$$

Using (28) and (34), we obtain

$$\begin{aligned} \|V_y\|_{L_1} &\leq \rho \|u_0\|_{L^1} + \rho \sum_{k=1}^{m} |a_k| \ \|I^{\alpha} y(t)|_{t=t_k}\|_{L^1} \\ &\leq \rho \|u_0\|_{L^1} + \gamma_1 \|y\|_{L^1}. \end{aligned} \tag{35}$$

From (29),

$$\begin{aligned} \|u\|_{L_1} &\leq \|V_y\|_{L_1} + \|I^{\alpha} y\|_{L_1} \\ &\leq \rho \|u_0\|_{L^1} + 2\gamma_1 \|y\|_{L^1}. \end{aligned} \tag{36}$$

Applying (H_1) with (36), we get

$$\begin{aligned} \|A(.,u)\|_{L_1} &\leq \|A(.,u) - A(.,0)\|_{L_1} + \|A(.,0)\|_{L_1} \\ &\leq a\|u\|_{L_1} + b \\ &\leq a \left(\rho \|u_0\|_{L^1} + 2\gamma_1 \|y\|_{L^1} \right) + b. \end{aligned} \tag{37}$$

Moreover, as in (6), we get

$$\begin{aligned}\|A(.,u)u - A(.,v)v\|_{L^1} &\le (a(\|u\|_{L^1} + \|v\|_{L^1}) + b)\,\|u-v\|_{L^1}\\ &\le [2a(\rho\|u_0\|_{L^1} + 2\gamma_1\|y\|_{L^1}) + b]\ \|u-v\|_{L^1}. \end{aligned} \tag{38}$$

Applying (H_3), (34) and (35), we obtain

$$\begin{aligned}\|f\,(.,\ V_y + I^\alpha y(.),\ y(.))\,\|_{L_1} &\le q\,[\|u\|_{L^1} + \|y\|_{L^1}] + \|\omega\|_{L_1}\\ &\le q(\rho\|u_0\|_{L^1} + (2\gamma_1+1)\|y\|_{L^1}) + \|\omega\|_{L_1}.\end{aligned} \tag{39}$$

Moreover,

$$\begin{aligned}&\|f\,(.,\ V_{y_2} + I^\alpha y_2(.),\ y_2(.)) - f\,(.,\ V_{y_1} + I^\alpha y_1(.),\ y_1(.))\,\|_{L_1}\\ &\le \rho\ q\sum_{k=1}^{m}|a_k|\ \|I^\alpha\,(y_2(t)-y_1(t))\,|_{t=t_k}\|_{L^1} + q\ \|I^\alpha\,(y_2-y_1)\,\|_{L^1} + q\ \|y_2-y_1\|_{L^1}\\ &\le\ q(2\gamma_1+1)\ \|y_2-y_1\|_{L_1}.\end{aligned} \tag{40}$$

The following theorem shows the uniqueness result.

Theorem 5 Let the assumptions (H_1)-(H_3) be satisfied. The nonlocal quasilinear problem (2)-(3) has a unique mild solution $u \in L^1(J,\mathbb{R})$ if for $\lambda_2 := \gamma_1(2a\rho\|u_0\|_{L^1} + b + q) + q \in (0, \frac{1}{2})$, we have

$$a\gamma_1^2[\rho\|u_0\|_{L^1}(a\rho\|u_0\|_{L^1}+b+q)+\|\omega\|_{L^1}] \le \frac{1}{16}(1-\lambda_2)^2 \text{ and } \sigma < \frac{1-[(b+q)\,\gamma_1+\lambda_2]}{4a\gamma_1^2}$$

where $\sigma > 0$ is the solution of the quadratic equation

$$4a\gamma_1^2\sigma^2+\{[2\gamma_1(2a\rho\|u_0\|_{L^1}+b+q)+q]-1\}\sigma+\rho\|u_0\|_{L^1}(a\rho\|u_0\|_{L^1}+b+q)+\|\omega\|_{L^1} = 0. \tag{41}$$

Proof

Suppose that the operator $G : L^1(J,\mathbb{R}) \to L^1(J,\mathbb{R})$ is defined by

$$Gy(t) = A(t,u)\,(V_y + I^\alpha y(t)) + f\,(t,\ V_y + I^\alpha y(t),\ y(t))\,. \tag{42}$$

We divide the proof into two steps.

Step 1. $(GB_\sigma \subset B_\sigma)$

Using (42) for $y \in L^1(J,\mathbb{R})$, we have

$$\|Gy(t)\|\ \le\ \|A(.,u)\|\,(\ \|V_y\| + \|I^\alpha y(t)\|) + \|f(t,\ V_y + I^\alpha y(t),\ y(t))\|.$$

Then, for $y \in B_\sigma$, by using (34),(35),(37) and (39), it is easy to see that $\|Gy\|_{L^1} \le \sigma$ where $\sigma > 0$ is the solution of the quadratic equation (41). Thus, G maps B_σ into itself.

Step 2. (G is a contraction mapping)

Using (42) for all $y, z \in L^1(J,\mathbb{R})$, we have

$$\begin{aligned}&\|Gy(t) - Gz(t)\|\\ &\quad\le\ \|A(.,u)\|\left(\rho\sum_{k=1}^{m}|a_k|\ \|I^\alpha(y(t)-z(t))|_{t=t_k}\| + \|I^\alpha(y(t)-z(t))\|\right)\\ &\quad\quad + \|f\,(t,\ V_y + I^\alpha y(t),\ y(t)) - f\,(t,\ V_z + I^\alpha z(t),\ z(t))\,\|.\end{aligned}$$

For $y, z \in L^1(J, B_\sigma)$ with using (34), (37) and (40), one can get

$$\|Gy - Gz\|_{L^1} \le \{2\gamma_1\,[a(\rho\|u_0\|_{L^1} + 2\sigma\gamma_1) + b + q] + q\}\ \ \|y-z\|_{L^1}.$$

Since $2\gamma_1\left[a(\rho\|u_0\|_{L^1}+2\sigma\gamma_1)+b+q\right]+q<1$, G is a contraction mapping and it has a unique fixed point $u\in B_\sigma\subset L^1(J,\mathbb{R})$ which is, by Lemma 2, the unique mild solution of the nonlocal quasilinear problem (2)-(3).

What follows deals with the existence result.

Theorem 6 Let the assumptions (H_1)-(H_3) be satisfied. Then, the nonlocal quasilinear problem (2)-(3) has at least one mild solution $u\in L^1(J,\mathbb{R})$ if for $\lambda_2\in(0,\frac{1}{2})$, we have

$$a\gamma_1^2[\rho\|u_0\|_{L^1}(a\rho\|u_0\|_{L^1}+b+q)+\|\omega\|_{L^1}]\leq\frac{1}{16}(1-\lambda_2)^2$$

and

$$\sigma<\frac{1-\left[2\gamma_1\left(a\rho\|u_0\|_{L^1}+b\right)\right]}{4a\gamma_1^2}$$

where $\sigma>0$ is the solution of the quadratic equation (41).

Proof

Suppose that the operator G is defined such that $Gy(t)=G_1(t)+G_2(t)$ where

$$G_1y(t)=f(t,\ V_y+I^\alpha y(t),\ y(t)),\tag{43}$$

and

$$G_2y(t)=A(t,u)\left(V_y+I^\alpha y(t).\right)\tag{44}$$

The proof will be given in four steps.

Step 1. ($G_1y+G_2z\in B_\sigma$ whenever $y,z\in B_\sigma$)

From (43) and (44),

$$\|G_1y(t)+G_2z(t)\|\ \leq\ \|f(t,\ V_y+I^\alpha y(t),\ y(t))\|+\|A(.,u)\|\left(\ \|V_z\|+\|I^\alpha z(t)\|\right).$$

For $y,z\in B_\sigma$, by using (34), (35), (37) and (39), we get that $\|G_1y+G_2z\|_{L^1}\leq\sigma$ where where $\sigma>0$ is the solution of the quadratic equation (41). Thus, $G_1y+G_2z\in B_\sigma$ whenever $y,z\in B_\sigma$.

Step 2. (G_1 is continuous)

Assumption (H_2) implies that G_1 is continuous.

Step 3. (G_1 is compact)

Clearly that G_1B_r is bounded in $L^1(J,\mathbb{R})$ which is the first condition of Kolmogorov compactness criterion. Let $y\in B_\sigma$. Using (43) with applying Theorem 1, we have

$$\begin{aligned}
&\|(G_1y)_h-(G_1y)\|_{L^1}\\
&=\int_0^T|(G_1y)_h(t)-G_1y(t)|dt\\
&=\int_0^T|\frac{1}{h}\int_t^{t+h}(G_1y)_h(s)ds-G_1y(t)|dt\\
&\leq\int_0^T\left\{\frac{1}{h}\int_t^{t+h}|(G_1y)_h(s)-G_1y(t)|ds\right\}dt\\
&\leq\int_0^T\left(\frac{1}{h}\int_t^{t+h}|f(s,\ V_y+I^\alpha y(t)|_{t=s},\ y(s))-f(t,\ V_y+I^\alpha y(t),\ y(t))|\ ds\right)dt.
\end{aligned}$$

Since $y\in B_\sigma\subset L^1(J,\mathbb{R})$ and assumption (H_3) holds which implies $f\in L^1(J,\mathbb{R})$, the right hand side of the above inequality tends to zero as h tends to zero. Thus, $(G_1y)_h\to(G_1y)$ uniformly as $h\to 0$. Then, by Kolmogorov compactness criterion, the class of $\{G_1y(t)\}$ is relatively compact and therefore G_1 is a compact operator.

Step 4. (G_2 is a contraction mapping)
Let $y, z \in B_\sigma$. Using (44), we have

$$\|G_2 y(t) - G_2 z(t)\| \leq \|A(.,u)\| \left(\rho \sum_{k=1}^{m} |a_k| \; \|I^\alpha (y(t) - z(t))|_{t=t_k}\| + \|I^\alpha (y(t) - z(t))\| \right)$$

then

$$\|G_2 y - G_2 z\|_{L^1} \;\leq\; 2\gamma_1 [a(\rho \|u_0\|_{L^1} + 2\sigma\gamma_1) + b] \; \|y - z\|_{L^1}.$$

Since $2\gamma_1 [a(\rho \|u_0\|_{L^1} + 2\sigma\gamma_1) + b] < 1$, G_1 is a contraction mapping.

As a consequence of Kranoselskii's fixed point theorem, G has at least one fixed point in B_σ. Thus, the nonlocal quasilinear problem (2)-(3) has at least one solution in B_σ. Therefore, the proof is completed.

Example 2 Consider the following fractional nonlocal problem

$$\begin{cases} {}^c D^{\frac{1}{4}} x(t) = \frac{2^{-3} e^{-t}}{(99+e^t)(1+|x(t)|)} \; x(t) + \frac{1}{(49+e^t)(1+|x(t)|+|{}^c D^\alpha x(t)|)}, \; t \in [0,1]; \\ \sum_{k=1}^{2} 10^{-2} x(t_k) = 1, \; 0 < t_1 < t_2 < 1. \end{cases} \tag{45}$$

Set

$$A(t,x) = \frac{2^{-3} e^{-t}}{(99 + e^t)(1 + |x|)} I, \; (t,x) \in [0,1] \times \mathbb{R}$$

and

$$f(t,x,y) = \frac{1}{(49 + e^t)(1 + x + y)}, \; (t,x,y) \in [0,1] \times \mathbb{R} \times \mathbb{R}.$$

Then,

$$\begin{aligned} \|A(t,x_1) - A(t,x_2)\| &= \left\| \frac{2^{-3} e^{-t}}{99 + e^t} \left(\frac{1}{1 + |x_1|} - \frac{1}{1 + |x_2|} \right) \right\| \\ &\leq \frac{2^{-3} e^{-t}}{99 + e^t} \; \||x_2| - |x_1|\| \\ &\leq \frac{2^{-3} e^{-t}}{99 + e^t} \; \|x_1 - x_2\| \\ &\leq \frac{1}{800} \; \|x_1 - x_2\|, \end{aligned}$$

and similar to Example 1, we obtain

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq \frac{1}{50} \left(|x_1 - x_2| + |y_1 - y_2| \right).$$

So, we have
$T = 1, \; \alpha = \frac{1}{4}, \; m = 2, \; q = \|\omega\|_{L^1} = \sum_{k=1}^{m} a_k = \frac{1}{50}, \; u_0 = 1, \; a = \frac{1}{800}, \; b = 0, \; \rho = 50, \; \gamma_1 = \frac{1}{\Gamma(\frac{5}{4})}, \; \gamma_2 = \frac{1}{\Gamma(\frac{5}{4})} + \frac{1}{\Gamma(\frac{3}{2})}$ and $\lambda_2 = \frac{9}{50} \in (0, \frac{1}{2})$. The quadratic equation will be

$$0.00608 \; \sigma^2 - 0.66013 \; \sigma + 4.145 = 0.$$

Therefore, all conditions of Theorem 5 are satisfied. Then, problem (45) has a unique mild solution $x \in B_{6.7} \subset L^1([0,1], \mathbb{R})$.

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