

CHARACTERIZATIONS OF FRACTIONAL SOBOLEV SPACES FROM THE PERSPECTIVE OF RIEMANN-LIOUVILLE OPERATORS

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ABSTRACT. Fractional Sobolev spaces $\widehat{H}^s(\mathbb{R})$ have been playing important roles in analysis of many mathematical subjects. In this work, we re-consider fractional Sobolev spaces under the perspective of fractional operators and establish characterizations on the Fourier transform of functions of fractional Sobolev spaces in \mathbb{R} , thereby giving another equivalent definition.

1. INTRODUCTION

The space $\widehat{H}^s(\mathbb{R})$ has been exhibiting important usefulness in the study of theory of elliptic PDEs; and it is well known from the standard textbooks that fractional Sobolev spaces $\widehat{H}^s(\mathbb{R})$ can be defined in several ways, namely via Fourier transform, Gagliardo norm or interpolation spaces, and so on.

Our previous work ([1], [2]) has suggested that usual fractional Sobolev spaces have been behaving new features in analysis of fraction-order differential equations due to the simultaneous appearing of left, right and mixed Riemann-Liouville derivatives. In this work we continue to explore fractional Sobolev spaces under the perspective of fractional calculus theory; and the main result in this work is Theorem 5.1, which characterizes the Fourier transform of elements of $\widehat{H}^s(\mathbb{R})$ and thus gives another equivalent definition of $\widehat{H}^s(\mathbb{R})$. Meanwhile, it is notable that some other related work has been done (such as [3], [4], [5], [6]), which is an interesting comparison to this work. Also, during the review of this article, the results were presented in author's dissertation [7].

The material is organized as follows:

- Section 2 introduces the notations and conventions adopted throughout the material.
- Section 3 introduces the preliminary knowledge on fractional R-L operators.

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- Section 4 outlines some existing results on the characterization of $\widehat{H}^s(\mathbb{R})$, which are obtained in [1] and will provide the whole abstract setting for the work.
- Section 5 establishes the main results.

2. NOTATIONS

- All functions considered in this material are default to be real-valued unless otherwise specified.
- (f, g) and $\int_{\mathbb{R}} fg$ shall be used interchangeably. Also, we denote integration $\int_A f$ on the set A without pointing out the variable unless it is necessary to specify.
- If $f, g \in L^2(\mathbb{R})$, $f = g$ means $f = g, a.e.$, unless stated otherwise.
- $C_0^\infty(\mathbb{R})$ denotes the space of all infinitely differentiable functions with compact support in \mathbb{R} .
- \mathbb{N} represents the set of all non-negative integers.
- $\mathcal{F}(u)$ denotes the Fourier transform of u with specific expression defined in Definition 6.2, \widehat{u} denotes the Plancherel transform of u defined in Theorem 6.1, which is well known that \widehat{u} is an isometry map from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$ and coincides with $\mathcal{F}(u)$ if $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.
- u^\vee denotes the inverse of Plancherel transform, and $*$ denotes convolution.

3. PRELIMINARY

3.1. Fractional Riemann-Liouville Integrals and Properties.

Definition 3.1. Let $u : \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma > 0$. The left and right Riemann-Liouville fractional integrals of order σ are, formally respectively, defined as

$$\mathbf{D}^{-\sigma}u(x) := \frac{1}{\Gamma(\sigma)} \int_{-\infty}^x (x-s)^{\sigma-1}u(s) ds, \quad (3.1)$$

$$\mathbf{D}^{-\sigma*}u(x) := \frac{1}{\Gamma(\sigma)} \int_x^{\infty} (s-x)^{\sigma-1}u(s) ds, \quad (3.2)$$

where $\Gamma(\sigma)$ is the usual Gamma function.

Property 3.1 ([8], p. 96). Given $0 < \sigma$,

$$(\phi, \mathbf{D}^{-\sigma}\psi) = (\mathbf{D}^{-\sigma*}\phi, \psi), \quad (3.3)$$

for $\phi \in L^p(\mathbb{R})$, $\psi \in L^q(\mathbb{R})$, $p > 1, q > 1, 1/p + 1/q = 1 + \sigma$.

Property 3.2 ([8], Corollary 2.1). Let $\mu, \sigma > 0$, and $w \in C_0^\infty(\mathbb{R})$, then

$$\mathbf{D}^{-\mu}\mathbf{D}^{-\sigma}w = \mathbf{D}^{-(\mu+\sigma)}w \quad \text{and} \quad \mathbf{D}^{-\mu*}\mathbf{D}^{-\sigma*}w = \mathbf{D}^{-(\mu+\sigma)*}w. \quad (3.4)$$

Property 3.3 ([8], pp. 95, 96). Let $\mu > 0$. Given $h \in \mathbb{R}$, define the translation operator τ_h as $\tau_h u(x) = u(x-h)$. Also, given $\kappa > 0$, define the dilation operator Π_κ as $\Pi_\kappa u(x) = u(\kappa x)$. Under the assumption that $\mathbf{D}^{-\mu}u$ and $\mathbf{D}^{-\mu*}u$ are well-defined, the following is true:

$$\begin{aligned} \tau_h(\mathbf{D}^{-\mu}u) &= \mathbf{D}^{-\mu}(\tau_h u), & \tau_h(\mathbf{D}^{-\mu*}u) &= \mathbf{D}^{-\mu*}(\tau_h u), \\ \Pi_\kappa(\mathbf{D}^{-\mu}u) &= \kappa^\mu \mathbf{D}^{-\mu}(\Pi_\kappa u), & \Pi_\kappa(\mathbf{D}^{-\mu*}u) &= \kappa^\mu \mathbf{D}^{-\mu*}(\Pi_\kappa u). \end{aligned} \quad (3.5)$$

3.2. Fractional Riemann-Liouville Derivatives and Properties.

Definition 3.2. Let $u : \mathbb{R} \rightarrow \mathbb{R}$. Assume $\mu > 0$, n is the smallest integer greater than μ (i.e., $n - 1 \leq \mu < n$), and $\sigma = n - \mu$. The left and right Riemann-Liouville fractional derivatives of order μ are, formally respectively, defined as

$$\mathbf{D}^\mu u := \frac{1}{\Gamma(\sigma)} \frac{d^n}{dx^n} \int_{-\infty}^x (x-s)^{\sigma-1} u(s) ds, \quad (3.6)$$

$$\mathbf{D}^{\mu*} u := \frac{(-1)^n}{\Gamma(\sigma)} \frac{d^n}{dx^n} \int_x^{\infty} (s-x)^{\sigma-1} u(s) ds. \quad (3.7)$$

Property 3.4. Let $0 < \mu$ and $u \in C_0^\infty(\mathbb{R})$, then $\mathbf{D}^\mu u, \mathbf{D}^{\mu*} u \in L^p(\mathbb{R})$ for any $1 \leq p < \infty$.

Property 3.5 ([8], p. 137). Let $\mu > 0, u \in C_0^\infty(\mathbb{R})$, then

$$\mathcal{F}(\mathbf{D}^\mu u) = (2\pi i \xi)^\mu \mathcal{F}(u) \text{ and } \mathcal{F}(\mathbf{D}^{\mu*} u) = (-2\pi i \xi)^\mu \mathcal{F}(u), \quad \xi \neq 0, \quad (3.8)$$

where $\mathcal{F}(\cdot)$ is the Fourier Transform as defined in Definition 6.2 and the complex power functions are understood as $(\mp i \xi)^\sigma = |\xi|^\sigma e^{\mp \sigma \pi i \cdot \text{sign}(\xi)/2}$.

Property 3.6 ([1]). Consider τ_h and Π_κ defined in Property 3.3. Let $\mu > 0$, $n - 1 \leq \mu < n$, where n is a positive integer, then

$$\begin{aligned} \tau_h(\mathbf{D}^\mu u) &= \mathbf{D}^\mu(\tau_h u), & \tau_h(\mathbf{D}^{\mu*} u) &= \mathbf{D}^{\mu*}(\tau_h u), \\ \Pi_\kappa(\mathbf{D}^\mu u) &= \kappa^{-\mu} \mathbf{D}^\mu(\Pi_\kappa u), & \Pi_\kappa(\mathbf{D}^{\mu*} u) &= \kappa^{-\mu} \mathbf{D}^{\mu*}(\Pi_\kappa u). \end{aligned} \quad (3.9)$$

4. CHARACTERIZATION OF SOBOLEV SPACE $\widehat{H}^s(\mathbb{R})$

In this section, we will list the necessary concepts and results developed in [1], which characterize the classical Sobolev space $\widehat{H}^s(\mathbb{R})$ defined in 6.1. This section gives us the theoretical framework in which the main results in Section 5 will be established.

Definition 4.1 (Weak Fractional Derivatives[1]). Let $\mu > 0$, and $u, w \in L_{loc}^1(\mathbb{R})$. The function w is called weak μ -order left fractional derivative of u , written as $\mathbf{D}^\mu u = w$, provided

$$(u, \mathbf{D}^{\mu*} \psi) = (w, \psi) \quad \forall \psi \in C_0^\infty(\mathbb{R}). \quad (4.10)$$

In a similar fashion, w is called weak μ -order right fractional derivative of u , written as $\mathbf{D}^{\mu*} u = w$, provided

$$(u, \mathbf{D}^\mu \psi) = (w, \psi) \quad \forall \psi \in C_0^\infty(\mathbb{R}). \quad (4.11)$$

Definition 4.2 ([1]). Let $s \geq 0$. Define spaces

$$\widetilde{W}_L^s(\mathbb{R}) = \{u \in L^2(\mathbb{R}), \mathbf{D}^s u \in L^2(\mathbb{R})\}, \quad (4.12)$$

$$\widetilde{W}_R^s(\mathbb{R}) = \{u \in L^2(\mathbb{R}), \mathbf{D}^{s*} u \in L^2(\mathbb{R})\}, \quad (4.13)$$

where $\mathbf{D}^s u$ and $\mathbf{D}^{s*} u$ are in the weak fractional derivative sense as defined in Definition 4.1. A semi-norm

$$|u|_L := \|\mathbf{D}^s u\|_{L^2(\mathbb{R})} \text{ for } \widetilde{W}_L^s(\mathbb{R}) \text{ and } |u|_R := \|\mathbf{D}^{s*} u\|_{L^2(\mathbb{R})} \text{ for } \widetilde{W}_R^s(\mathbb{R}), \quad (4.14)$$

is given with the corresponding norm

$$\|u\|_\star := (\|u\|_{L^2(\mathbb{R})}^2 + |u|_\star^2)^{1/2}, \quad \star = L, R. \quad (4.15)$$

Remark 4.1. Notice the special case $\widetilde{W}_L^0(\mathbb{R}) = \widetilde{W}_R^0(\mathbb{R}) = \widehat{H}^0(\mathbb{R}) = L^2(\mathbb{R})$.

Property 4.1 (Uniqueness of Weak Fractional R-L Derivatives [1]). *If $v \in L^1_{loc}(\mathbb{R})$ has a weak s -order left (or right) fractional derivative, then it is unique up to a set of zero measure.*

Now we have the following characterization of Sobolev space $\widehat{H}^s(\mathbb{R})$.

Theorem 4.1 ([1]). *Given $s \geq 0$, $\widetilde{W}_L^s(\mathbb{R})$, $\widetilde{W}_R^s(\mathbb{R})$ and $\widehat{H}^s(\mathbb{R})$ are identical spaces with equal norms and semi-norms.*

Corollary 4.1 ([1]). *$u \in \widehat{H}^s(\mathbb{R})$ if and only if there exists a sequence $\{u_n\} \subset C_0^\infty(\mathbb{R})$ such that $\{u_n\}$ and $\{\mathbf{D}^s u_n\}$ are Cauchy sequences in $L^2(\mathbb{R})$ and $\lim_{n \rightarrow \infty} u_n = u$. As a consequence, we have $\lim_{n \rightarrow \infty} \mathbf{D}^s u_n = \mathbf{D}^s u$.*

Likewise,

$u \in \widehat{H}^s(\mathbb{R})$ if and only if there exists a sequence $\{u_n\} \subset C_0^\infty(\mathbb{R})$ such that $\{u_n\}$ and $\{\mathbf{D}^{s} u_n\}$ are Cauchy sequences in $L^2(\mathbb{R})$ and $\lim_{n \rightarrow \infty} u_n = u$. As a consequence, we have $\lim_{n \rightarrow \infty} \mathbf{D}^{s*} u_n = \mathbf{D}^{s*} u$.*

5. MAIN RESULTS

Let us keep in mind that throughout the rest of paper, fractional derivatives are always understood in the weak sense defined in Section 4. The main result in this work is the following.

Theorem 5.1. *Define the sign function*

$$\chi(x) = \begin{cases} 1, & \text{if } \frac{x}{2} \in [0, \frac{1}{2}] \text{ or } [\frac{3}{2} + 2k, \frac{5}{2} + 2k], k \in \mathbb{N}, \\ -1, & \text{if } \frac{x}{2} \in (\frac{1}{2} + 2k, \frac{3}{2} + 2k), k \in \mathbb{N}. \end{cases}$$

Given $s \geq 0$, n is a positive integer. Let $f(\xi)$ be any function of the form $\prod_{i=1}^n (1 + \chi(s_i) \cdot (\pm 2\pi\xi i)^{s_i})$ and fixed, where $0 \leq s_i$ ($i = 1, \dots, n$), $\sum_{i=1}^n s_i \leq s$ and the sign \pm is arbitrary for our choice.

If we denote $\tau = s - \sum_{i=1}^n s_i$, then the following is true:

$v(x) \in \widehat{H}^s(\mathbb{R})$ if and only if there exists a function $u(x) \in \widehat{H}^\tau(\mathbb{R})$ such that $f(\xi) \cdot \widehat{v}(\xi) = \widehat{u}(\xi)$.

Remark 5.1. *As convention, the complex power functions are understood as $(\mp i\xi)^\sigma = |\xi|^\sigma e^{\mp \sigma \pi i \cdot \text{sign}(\xi)/2}$.*

Before embarking on the rigorous proof of the theorem above, let us first establish several lemmas which are of crucial importance later.

Lemma 5.1. *Fix $s \geq 0$. The following sets are dense in $L^2(\mathbb{R})$ respectively:*

- $M_1 = \{w : w = \psi + \mathbf{D}^s \psi, \forall \psi \in C_0^\infty(\mathbb{R})\}$,
- $M_2 = \{w : w = \psi - \mathbf{D}^s \psi, \forall \psi \in C_0^\infty(\mathbb{R})\}$,
- $M_3 = \{w : w = \psi + \mathbf{D}^{s*} \psi, \forall \psi \in C_0^\infty(\mathbb{R})\}$,
- $M_4 = \{w : w = \psi - \mathbf{D}^{s*} \psi, \forall \psi \in C_0^\infty(\mathbb{R})\}$.

Proof. 1. The proof is provided for M_1 only and the results for the other sets follow analogously without essential differences.

2. To use Theorem 6.3, first notice that $L^2(\mathbb{R})$ is a Hilbert space and $M_1 \subset L^2(\mathbb{R})$ by Property 3.4. Furthermore, it is effortless to verify M_1 is a subspace of $L^2(\mathbb{R})$. Thus all the hypotheses of Theorem 6.3 are met.

3. Let us assume that $g \in L^2(\mathbb{R})$ and $(g, w) = 0$ for any $w \in M_1$. The proof is done if this implies $g = 0$ a.e..

4. Pick a non-zero function $\psi \in C_0^\infty(\mathbb{R})$, then by Plancherel Theorem we know $\widehat{\psi}(\xi)$ is a non-zero function, namely, $\widehat{\psi}(\xi) \neq 0$. On account of continuity of $\widehat{\psi}(\xi)$, there exists a non-empty open interval $(a, b) \subset \mathbb{R}$ such that $\widehat{\psi}(\xi) \neq 0$ on (a, b) .

5. Let $v(x) = \psi(\epsilon x)$, where ϵ is any fixed positive number. It's clear that $v(x) \in C_0^\infty(\mathbb{R})$ as well. Computing the Fourier transform of $w(x) = v(x) + \mathbf{D}^s v$ by Property 3.5 gives

$$\widehat{w}(\xi) = (1 + (2\pi i\xi)^s)\widehat{v}(\xi) = (1 + (2\pi i\xi)^s) \cdot \frac{1}{\epsilon} \widehat{\psi}\left(\frac{\xi}{\epsilon}\right). \quad (5.16)$$

Notice $|1 + (2\pi i\xi)^s| \neq 0$ a.e. and $\widehat{\psi}\left(\frac{\xi}{\epsilon}\right) \neq 0$ on $(a\epsilon, b\epsilon)$. It follows that $\widehat{w}(\xi) \neq 0$ a.e. on $(a\epsilon, b\epsilon)$.

6. Set a new function $G(-y) := \int_{\mathbb{R}} g(x)w(x-y) dx = \int_{\mathbb{R}} g(x) \tau_y w(x) dx$. Using Property 3.6 gives $\tau_y w(x) \in M_1$ and therefore $G(-y) = 0$ for any $y \in \mathbb{R}$ by our assumption in Step 3. And thus by Plancherel Theorem we know $\widehat{G}(\xi) = 0$ a.e. as well.

7. Note that $G(y) = g(-x) * w(x)$ and $g, w \in L^2(\mathbb{R})$. Using convolution theorem ([9], Theorem 1.2, p.12) gives $\widehat{G} = \widehat{g(-x)} \cdot \widehat{w(x)} = 0$, which implies $\widehat{g(-x)}(\xi) = 0$ a.e. on $(\epsilon a, \epsilon b)$ by recalling $\widehat{w}(\xi) \neq 0$ a.e. on $(a\epsilon, b\epsilon)$.

8. Because of the arbitrariness of ϵ , we deduce $\widehat{g(-x)}(\xi) = 0$ a.e. in \mathbb{R} and therefore $g(x) = 0$ a.e. in \mathbb{R} by another use of inverse Plancherel Theorem. This completes the whole proof. \square

Lemma 5.2. *Given $0 \leq s, \psi \in C_0^\infty(\mathbb{R})$, then*

$$\|\psi \pm \mathbf{D}^s \psi\|_{L^2(\mathbb{R})}^2 = \|\psi\|_{L^2(\mathbb{R})}^2 \pm 2 \cos\left(\frac{s}{2}\pi\right) \|\mathbf{D}^{s/2} \psi\|_{L^2(\mathbb{R})}^2 + \|\mathbf{D}^s \psi\|_{L^2(\mathbb{R})}^2, \quad (5.17)$$

$$\|\psi \pm \mathbf{D}^{s*} \psi\|_{L^2(\mathbb{R})}^2 = \|\psi\|_{L^2(\mathbb{R})}^2 \pm 2 \cos\left(\frac{s}{2}\pi\right) \|\mathbf{D}^{s/2} \psi\|_{L^2(\mathbb{R})}^2 + \|\mathbf{D}^{s*} \psi\|_{L^2(\mathbb{R})}^2. \quad (5.18)$$

Proof. 1. The proof is established for one of the four identities above only, namely $\|\psi + \mathbf{D}^s \psi\|_{L^2(\mathbb{R})}^2$, and the others can be shown analogously by repeating the same procedure (even though involve left derivative and right derivative in different cases).

2. Since $\|\psi + \mathbf{D}^s \psi\|_{L^2(\mathbb{R})}^2 = \|\psi\|_{L^2(\mathbb{R})}^2 + \|\mathbf{D}^s \psi\|_{L^2(\mathbb{R})}^2 + 2(\psi, \mathbf{D}^s \psi)$, we only need to show $(\psi, \mathbf{D}^s \psi) = \cos\left(\frac{s}{2}\pi\right) \|\mathbf{D}^{s/2} \psi\|_{L^2(\mathbb{R})}^2$.

3. Note that if we could show $(\psi, \mathbf{D}^s \psi) = (\mathbf{D}^{s/2*} \psi, \mathbf{D}^{s/2} \psi)$, then $(\psi, \mathbf{D}^s \psi) = \cos\left(\frac{s}{2}\pi\right) \|\mathbf{D}^{s/2} \psi\|_{L^2(\mathbb{R})}^2$ follows immediately from the application of the second identity of Theorem 4.1 in [1] and so the proof is done.

4. Now we intend to show $(\psi, \mathbf{D}^s \psi) = (\mathbf{D}^{s/2*} \psi, \mathbf{D}^{s/2} \psi)$. If $s = 0$, it is trivially true, and if s is an integer, it can be directly verified by using integration by parts. Otherwise, we can always write s as $s = n - \delta$ with $0 < \delta < 1$, where n is a positive integer. Notice $\psi \in C_0^\infty(\mathbb{R})$, if n is even, using the definition of R-L derivative and Lemma 2.2 ([9], p.73) gives

$$(\psi, \mathbf{D}^s \psi) = \left(\psi, \frac{d^{(n/2)}}{dx^{(n/2)}} \mathbf{D}^{n/2-\delta} \psi\right) = \left(\psi, \frac{d^{(n/2)}}{dx^{(n/2)}} \mathbf{D}^{-\delta} \psi^{(n/2)}\right). \quad (5.19)$$

Using integration by parts and Property 3.2, the last term becomes

$$\left(\psi, \frac{d^{(n/2)}}{dx^{(n/2)}} \mathbf{D}^{-\delta} \psi^{(n/2)}\right) = (\mathbf{D}^{n/2*} \psi, \mathbf{D}^{-\delta/2} \mathbf{D}^{-\delta/2} \psi^{(n/2)}).$$

To simplify the right-hand side, notice the fact that $\mathbf{D}^{-\delta/2} \psi^{(n/2)} = \mathbf{D}^{n/2-\delta/2} \psi$ by applying Lemma 2.2 ([9], p.73), and this gives $\mathbf{D}^{-\delta/2} \psi^{(n/2)} \in L^p(\mathbb{R})$ for $p \geq 1$ by Property 3.4. Thus

$$(\mathbf{D}^{n/2*} \psi, \mathbf{D}^{-\delta/2} \mathbf{D}^{-\delta/2} \psi^{(n/2)}) = (\mathbf{D}^{s/2*} \psi, \mathbf{D}^{s/2} \psi)$$

follows immediately from Property 3.1 and another use of Lemma 2.2 ([9], p.73). Therefore $(\psi, \mathbf{D}^s \psi) = (\mathbf{D}^{s/2*} \psi, \mathbf{D}^{s/2} \psi)$.

5. If n is odd, using definition of R-L derivative and Lemma 2.2 ([9], p.73), it is not difficult to verify

$$(\psi, \mathbf{D}^s \psi) = \left(\psi, \frac{d^{((n+1)/2)}}{dx^{((n+1)/2)}} \mathbf{D}^{-(\delta+1)} \psi^{((n+1)/2)}\right).$$

Now $n+1$ is even. Repeating above procedure starting from equation (5.19) gives us the desired result, which is omitted here. Thus we are done. \square

Lemma 5.3. (a). *Given $0 \leq s$, there exists a one to one and onto map $T : v \mapsto u$ from $\widehat{H}^s(\mathbb{R})$ to $L^2(\mathbb{R})$ such that*

$$v + \chi(s) \cdot \mathbf{D}^s v = u. \quad (5.20)$$

Analogously, there exists a one to one and onto map $T^* : v \mapsto u$ from $\widehat{H}^s(\mathbb{R})$ to $L^2(\mathbb{R})$ such that

$$v + \chi(s) \cdot \mathbf{D}^{s*} v = u. \quad (5.21)$$

(b). *Furthermore, given $t > 0$, for each case in (a), $v \in \widehat{H}^{s+t}(\mathbb{R})$ if and only if $u \in \widehat{H}^t(\mathbb{R})$.*

Proof. 1. For part (a), the proof is shown only for (5.20) since (5.21) can be shown in a similar manner. Also, without loss of generality, we assume $\chi(s) = 1$, namely we suppose that $\frac{s}{2} \in [0, \frac{1}{2}]$ or $[\frac{3}{2} + 2k, \frac{5}{2} + 2k]$, $k \in \mathbb{N}$ (recall the definition in Theorem 5.1). For $\chi(s) = -1$, the proof can be performed in the same way.

2. First we show that T is a map and is one to one. Observe that $\mathbf{D}^s v \in L^2(\mathbb{R})$ if $v \in \widehat{H}^s(\mathbb{R})$ by the definition of weak fractional derivative, therefore $v + \mathbf{D}^s v \in L^2(\mathbb{R})$, which means $T : v \mapsto u$ is indeed a map by the uniqueness of weak fractional derivative 4.1. To see it is one to one, assume there exists another \tilde{v} such that $\tilde{v} + \mathbf{D}^s \tilde{v} = v + \mathbf{D}^s v$, then $(\tilde{v} - v) + \mathbf{D}^s(\tilde{v} - v) = 0$. By using a standard norm estimate argument and application of Lemma 5.2, it is easy to obtain $\|\tilde{v} - v\|_{L^2(\mathbb{R})}^2 = 0$, which implies $\tilde{v} = v$. Thus $T : v \mapsto u$ is a one to one map.

3. Second we show the map is onto. Fix a function $u(x) \in L^2(\mathbb{R})$. Invoking Lemma 5.1, we know that there exists a Cauchy sequence $\{v_n + \mathbf{D}^s v_n\}$ converging to u in $L^2(\mathbb{R})$, where $\{v_n\} \subset C_0^\infty(\mathbb{R})$. This implies that

$$\|(v_m - v_n) + (\mathbf{D}^s v_m - \mathbf{D}^s v_n)\|_{L^2(\mathbb{R})}^2 \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

Employing Lemma 5.2, we equivalently obtain

$$\begin{aligned} & \|v_m - v_n\|_{L^2(\mathbb{R})}^2 + 2 \cos\left(\frac{s}{2}\pi\right) \|\mathbf{D}^{s/2} v_m - \mathbf{D}^{s/2} v_n\|_{L^2(\mathbb{R})}^2 + \|\mathbf{D}^s v_m - \mathbf{D}^s v_n\|_{L^2(\mathbb{R})}^2 \\ & \rightarrow 0, \text{ as } m, n \rightarrow \infty. \end{aligned} \quad (5.22)$$

Now notice that $2\cos(\frac{s}{2}\pi) \geq 0$ since we assume $\chi(s) = 1$. Therefore, $\{v_n\}$ and $\{\mathbf{D}^s v_n\}$ are Cauchy sequences respectively. If we denote $v = \lim_{n \rightarrow \infty} v_n$, this concludes that $v + \chi(s) \cdot \mathbf{D}^s v = u$ and $v \in \widehat{H}^s(\mathbb{R})$ by Corollary 4.1. Thus it is onto and we complete the proof of part (a).

4. We remain to show part (b). Without loss of generality, we only consider the case (5.20).

For forward direction, first suppose $v \in \widehat{H}^{s+t}(\mathbb{R})$, we need to show that the right-hand side of equation (5.20) belongs to $\widehat{H}^t(\mathbb{R})$, namely $u(x) \in \widehat{H}^t(\mathbb{R})$. Obviously $v \in \widehat{H}^t(\mathbb{R})$ by the embedding $\widehat{H}^{s+t}(\mathbb{R}) \subset \widehat{H}^t(\mathbb{R})$. Now we claim $\mathbf{D}^s v \in \widehat{H}^t(\mathbb{R})$. To see this, using Corollary 4.1 to the fact that $v \in \widehat{H}^{s+t}(\mathbb{R})$, there exist Cauchy sequences $\{v_n\} \subset C_0^\infty(\mathbb{R})$ and $\{\mathbf{D}^{s+t} v_n\}$ in $L^2(\mathbb{R})$ such that $v_n \rightarrow v, \mathbf{D}^{s+t} v_n \rightarrow \mathbf{D}^{s+t} v$ as $n \rightarrow \infty$. On the other hand, $v \in \widehat{H}^t(\mathbb{R})$ further implies $\mathbf{D}^s v_n \rightarrow \mathbf{D}^s v$. Thus for any $\psi \in C_0^\infty(\mathbb{R})$, we have

$$(\mathbf{D}^s v, \mathbf{D}^{t*} \psi) = \lim_{n \rightarrow \infty} (\mathbf{D}^s v_n, \mathbf{D}^{t*} \psi) = \lim_{n \rightarrow \infty} (\mathbf{D}^{s+t} v_n, \psi) = (\mathbf{D}^{s+t} v, \psi).$$

By the definition of $\widetilde{W}_L^t(\mathbb{R})$ and Theorem 4.1, this concludes $\mathbf{D}^s v \in \widehat{H}^t(\mathbb{R})$ and thus $u(x) = v + \chi(s) \cdot \mathbf{D}^s v \in \widehat{H}^t(\mathbb{R})$.

5. The last step is to show the backward direction, namely, if $u(x) \in \widehat{H}^t(\mathbb{R})$ in equation (5.20), then $v \in \widehat{H}^{s+t}(\mathbb{R})$. Let's see that it is always possible to rewrite $t = t_1 + \dots + t_n$ such that $0 < t_i \leq s$ ($i = 1, \dots, n$), where n is an integer that depends on t . We next inductively prove that v belongs to $\widehat{H}^{s+t_1}(\mathbb{R}), \dots, \widehat{H}^{s+t_1+\dots+t_n}(\mathbb{R}) = \widehat{H}^{s+t}(\mathbb{R})$.

First, from equation (5.20), we see $\mathbf{D}^s v \in \widehat{H}^{t_1}(\mathbb{R})$ since $v \in \widehat{H}^s(\mathbb{R}) \subset \widehat{H}^{t_1}$ and $u \in \widehat{H}^t(\mathbb{R}) \subset \widehat{H}^{t_1}$. According to Theorem 4.1, $\mathbf{D}^s v \in \widetilde{W}_L^{t_1}(\mathbb{R})$, which by definition means that there exists a function $Q(x) \in L^2(\mathbb{R})$ such that

$$(\mathbf{D}^s v, \mathbf{D}^{t_1*} \psi) = (Q, \psi), \quad \forall \psi \in C_0^\infty(\mathbb{R}). \quad (5.23)$$

Again, by invoking Corollary 4.1 to the fact that $v \in \widehat{H}^s(\mathbb{R})$, there exist Cauchy sequences $\{\tilde{v}_n\} \subset C_0^\infty(\mathbb{R})$ and $\{\mathbf{D}^s \tilde{v}_n\}$ in $L^2(\mathbb{R})$ such that $\tilde{v}_n \rightarrow v, \mathbf{D}^s \tilde{v}_n \rightarrow \mathbf{D}^s v$ as $n \rightarrow \infty$. Then left-hand side of equation (5.23) becomes

$$\begin{aligned} (\mathbf{D}^s v, \mathbf{D}^{t_1*} \psi) &= \lim_{n \rightarrow \infty} (\mathbf{D}^s \tilde{v}_n, \mathbf{D}^{t_1*} \psi) \\ &= \lim_{n \rightarrow \infty} (\tilde{v}_n, \mathbf{D}^{(s+t_1)*} \psi) \\ &= (v, \mathbf{D}^{(s+t_1)*} \psi). \end{aligned} \quad (5.24)$$

This concludes that $v \in \widehat{H}^{s+t_1}(\mathbb{R})$ by another utilization of definition of $\widetilde{W}_L^{s+t_1}(\mathbb{R})$ and Theorem 4.1. By repeating this procedure for considering t_2, \dots, t_n , we can increase the regularity of v gradually, namely, we can show that $v \in \widehat{H}^{s+t_1+t_2}(\mathbb{R}), \dots, v \in \widehat{H}^{s+t_1+\dots+t_n}(\mathbb{R}) = \widehat{H}^{s+t}(\mathbb{R})$, as desired. Thus the whole proof is completed. \square

Lemma 5.4. (a). Given $0 \leq s$, there exists a one to one and onto map $T : v \mapsto u$ from $\widehat{H}^s(\mathbb{R})$ to $L^2(\mathbb{R})$ such that

$$(1 + \chi(s) \cdot (2\pi\xi i)^s) \cdot \widehat{v}(\xi) = \widehat{u}(\xi). \quad (5.25)$$

Analogously, there exists a one to one and onto map $T^* : v \mapsto u$ from $\widehat{H}^s(\mathbb{R})$ to $L^2(\mathbb{R})$ such that

$$(1 + \chi(s) \cdot (-2\pi\xi i)^s) \cdot \widehat{v}(\xi) = \widehat{u}(\xi). \quad (5.26)$$

(b). Furthermore, given $t > 0$, for each case in (a), $v \in \widehat{H}^{s+t}(\mathbb{R})$ if and only if $u \in \widehat{H}^t(\mathbb{R})$.

Proof. 1. This Lemma is an direct application of Lemma 5.3 by taking the Plancherel transform at both sides of equation (5.20) and equation (5.21) respectively. The justification for

$$\begin{aligned}\widehat{D^s v}(\xi) &= (2\pi\xi i)^s \cdot \widehat{v}(\xi) \\ \widehat{D^{s*} v}(\xi) &= (-2\pi\xi i)^s \cdot \widehat{v}(\xi)\end{aligned}\tag{5.27}$$

is seen in the proof of Theorem 3.3 ([1]) and will not be repeated here. \square

Now we are in the position to prove Theorem 5.1.

Proof. 1. We intend to construct a one to one and onto map $v \mapsto u$ from $\widehat{H}^s(\mathbb{R})$ to $\widehat{H}^\tau(\mathbb{R})$ such that

$$f(\xi) \cdot \widehat{v}(\xi) = \widehat{u}(\xi).$$

Then the proof is done.

2. Utilizing both part (a) and (b) of Lemma 5.4, we know there exists a one to one and onto map $T_1 : v \mapsto u_1$ from $\widehat{H}^s(\mathbb{R})$ to $\widehat{H}^{s-s_1}(\mathbb{R})$ such that

$$(1 + \chi(s) \cdot (2\pi\xi i)^{s_1}) \cdot \widehat{v}(\xi) = \widehat{u}_1(\xi).\tag{5.28}$$

By a second application of Lemma 5.4 for u_1 , we know there exists a one to one and onto map $T_2 : u_1 \mapsto u_2$ from $\widehat{H}^{s-s_1}(\mathbb{R})$ to $\widehat{H}^{s-s_1-s_2}(\mathbb{R})$ such that

$$(1 + \chi(s) \cdot (2\pi\xi i)^{s_2}) \cdot \widehat{u}_1(\xi) = \widehat{u}_2(\xi).\tag{5.29}$$

By repeating the same procedure, we obtain one to one and onto maps T_2, \dots, T_n , where $T_n : u_{n-1} \mapsto u_n$ from $\widehat{H}^{s-s_1-\dots-s_{n-1}}(\mathbb{R})$ to $\widehat{H}^{s-s_1-\dots-s_n}(\mathbb{R})$ such that

$$(1 + \chi(s) \cdot (2\pi\xi i)^{s_n}) \cdot \widehat{u}_{n-1}(\xi) = \widehat{u}_n(\xi).\tag{5.30}$$

Recall $\tau = s - \sum_{i=1}^n s_i$, therefore, $T = T_1 \circ T_2 \cdots \circ T_n : v \mapsto u_n$ is a one to one and onto map $v \mapsto u_n$ from $\widehat{H}^s(\mathbb{R})$ to $\widehat{H}^\tau(\mathbb{R})$, satisfying $f(\xi) \cdot \widehat{v}(\xi) = \widehat{u}_n(\xi)$. This completes Step 1 above by regarding u_n as u , and thus completes the whole proof for Theorem 5.1. \square

6. APPENDIX

Definition 6.1 (Sobolev Spaces Via Fourier Transform). *Let $\mu \geq 0$. Define*

$$\widehat{H}^\mu(\mathbb{R}) = \left\{ w \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1 + |2\pi\xi|^{2\mu}) |\widehat{w}(\xi)|^2 d\xi < \infty \right\},\tag{6.31}$$

where \widehat{w} is Plancherel transform defined in Theorem 6.1. The space is endowed with semi-norm

$$|u|_{\widehat{H}^\mu(\mathbb{R})} := \|(2\pi\xi|^\mu \widehat{u})\|_{L^2(\mathbb{R})},\tag{6.32}$$

and norm

$$\|u\|_{\widehat{H}^\mu(\mathbb{R})} := \left(\|u\|_{L^2(\mathbb{R})}^2 + |u|_{\widehat{H}^\mu(\mathbb{R})}^2 \right)^{1/2}.\tag{6.33}$$

And it is well-known that $\widehat{H}^\mu(\mathbb{R})$ is a Hilbert space.

Definition 6.2 (Fourier Transform). *Given a function $f(x) \in L^1(\mathbb{R}) : \mathbb{R} \rightarrow \mathbb{R}$, the Fourier Transform of f is defined as*

$$\mathcal{F}(f)(\xi) := \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x) \, dx \quad \forall \xi \in \mathbb{R}.$$

Theorem 6.1 (Plancherel Theorem ([10] p. 187)). *One can associate to each $f \in L^2(\mathbb{R})$ a function $\widehat{f} \in L^2(\mathbb{R})$ so that the following properties hold:*

- *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then \widehat{f} is the defined Fourier transform of f in Definition 6.2.*
- *For every $f \in L^2(\mathbb{R})$, $\|f\|_2 = \|\widehat{f}\|_2$.*
- *The mapping $f \rightarrow \widehat{f}$ is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.*

Theorem 6.2. ([11], p. 189) *Assume $u, v \in L^2(\mathbb{R}^n)$. Then*

- $\int_{\mathbb{R}^n} u \bar{v} = \int_{\mathbb{R}^n} \widehat{u} \overline{\widehat{v}}.$
- $u = (\widehat{\widehat{u}})^{\vee}.$

Theorem 6.3 ([12], Theorem 4.3-2, p. 191). *Let $(X, (\cdot, \cdot))$ be a Hilbert space and let Y be a subspace of X , then $\bar{Y} = X$ if and only if the following is true: If the element $x \in X$ satisfies $(x, y) = 0$ for all $y \in Y$, then it has to be zero, namely $x = 0$.*

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