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INTEGRAL TRANSFORMS OF GENERALIZED M-SERIES

DHEERANDRA SHANKER SACHAN, SHAILESH JALOREE

Abstract. Integral transforms provide powerful operational methods for solving initial value problems and initial-boundary value problems for linear differential and integral equations. Keeping these features in mind, in this paper we have provided certain integral transforms of generalized M-series. The results are provided in terms of hypergeometric function and generalized Wright function.

1. Introduction

The generalized M-series [1] is defined as

$$\begin{aligned} {}_pM_q^{\alpha,\beta}(z) &= {}_pM_q^{\alpha,\beta}(a_1,\cdots,a_p;b_1,\cdots,b_q;z) \\ &= \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{\Gamma(\alpha k+\beta)} \end{aligned} \tag{1}$$

where $\alpha,\beta\in C$, $z\in C$, $Re(\alpha)>0$; $(a_i)_k$ $(i=\overline{1,p})$ and $(b_j)_k$ $(j=\overline{1,q})$ are Pochhammer symbols. The series (1) is defined when none of the parameters $(b_j)_k$ $(j=\overline{1,q})$ is a negative integer or zero; if any numerator parameter a_i is a negative integer or zero, then series terminates to a polynomial in z. The series (1) is convergent for all z if $p\le q$; it is convergent for $|z|<\delta=\alpha^{\alpha}$ if $p=q+1$ and divergent if $p>q+1$. When $p=q+1$ and $|z|=\delta$, the series is convergent on conditions depending on the parameters.The detailed account of the M-series can be found in the paper [1]

The generalized M-series is a special case of the Wright generalized hypergeometric function ${}_p\psi_q(z)$. The M- series has following relationship with various classical special functions and trigonometrical functions.
(i) When $\alpha=\beta=1$ then (1) reduces to a generalized hypergeometric function [2]

$$\begin{aligned} {}_pM_q^{1,1}(a_1,\cdots,a_p;b_1,\cdots,b_q;z) &= \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!} \\ &= {}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;z) \end{aligned} \tag{2}$$

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$$|z| < \infty \text{ if } p \leq q,\ |z| < 1 \text{ if } p = q+1$$

(ii) When $p = q = 1, a = \rho \in C, b = 1$ then (1) reduces to generalized Mittag-Leffler function introduced by Prabhakar [3]

$$_1M_1^{\alpha,\beta}(\rho;1;z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{(1)_k} \frac{z^k}{\Gamma(\alpha k+\beta)} = E_{\alpha,\beta}^{\rho}(z) \tag{3}$$

$$Re(\rho) > 0, Re(\alpha) > 0, Re(\beta) > 0$$

(iii) When there are no upper and lower parameters ($p = q = 0$) then (1) reduces to generalized Mittag-Leffler function [4]

$$_0M_0^{\alpha,\beta}(-;-;z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} = E_{\alpha,\beta}(z) \tag{4}$$

$$Re(\alpha) > 0, Re(\beta) > 0$$

(iv) When there are no upper and lower parameters ($p = q = 0$) and $\alpha = \beta = 1$ then (1) reduces to exponential series

$$_0M_0^{1,1}(-;-;z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \tag{5}$$

(v) When $p = 0, q = 1, b = \frac{3}{2}, \alpha = \beta = 1$ and z is replaced by $-\frac{z^2}{4}$ then (1) reduces to sine series

$$z\,_0M_1^{1,1}(-;\frac{3}{2};-\frac{z^2}{4}) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = sin\ z \tag{6}$$

(vi) When $p = 0, q = 1, b = \frac{1}{2}, \alpha = \beta = 1$ and z is replaced by $-\frac{z^2}{4}$ then (1) reduces to cosine series

$$_0M_1^{1,1}(-;\frac{1}{2};-\frac{z^2}{4}) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = cos\ z \tag{7}$$

(vi) When $p = 1, q = 0$ and $\alpha = \beta = 1$ then (1) reduces to binomial series

$$_1M_0^{1,1}(a;-;z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{k!} = (1-z)^{-a} \tag{8}$$

Wright generalized hypergeometric function [6, 7, 8] is defined by means of the series representation in the form

$$_p\psi_q(z) = {}_p\psi_q\left[\begin{array}{c}(a_p,\alpha_p);\\(b_q,\beta_q);\end{array}\middle|z\right] = {}_p\psi_q\left[\begin{array}{c}(a_1,\alpha_1),\cdots,(a_p,\alpha_p);\\(b_1,\beta_1),\cdots,(b_q,\beta_q);\end{array}\middle|z\right]$$
$$= \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)} \frac{z^k}{k!} \tag{9}$$

Where $z \in C,\ a_j(j = 1,2,\cdots p), b_j(j = 1,2,\cdots q) \in C,\ \alpha_j \neq 0,\ \beta_j \neq 0$ and $\alpha_j(j = 1,2,\cdots p), \beta_j(j = 1,2,\cdots q) \in R$.

The conditions for the existence of the generalized Wright function $_p\psi_q(z)$ are given by

$$\Delta = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$$
$$\delta = \prod_{j=1}^{p} |\alpha_j|^{-\alpha_j} \prod_{j=1}^{q} |\beta_j|^{\beta_j} \tag{10}$$
$$\mu = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2}$$

If $\Delta > -1$ then the series (9) is absolutely convergent for all $z \in C$ and also if $\Delta = -1$ then the series (9) is absolutely convergent for all values of $|z| < \delta$ and $|z| = \delta$, $Re(\mu) > \frac{1}{2}$.

Various properties and applications of (1) are studied by Saxena [5], Chouhan and Saraswat [9], Gehlot [10], Singh [11] Chouhan and Khan [12] Kumar and Saxena [13] Suthar et al.[14] Khan et al.[15] etc.

For our study of integral transforms of generalized M-Series, we need the following well known facts.

(i) Beta Transform (Sneddon [16])

The Beta (Euler) transform of the function $f(z)$ is defined by

$$B\{f(z); a, b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z) dz, \; Re(a) > 0, Re(b) > 0 \tag{11}$$

(ii) Laplace Transform (Mathai et al. [18],p.48, eq. 2.11)

The Laplace transform of the function $f(z)$ is defined by

$$F(s) = L\{f(z); s\} = \int_0^\infty e^{-sz} f(z)\, dz, \; Re(s) > 0 \tag{12}$$

(iii) K-Transform (Mathai et al. [18], p. 53, eq. 2.33)

The K-transform of function $f(z)$ is defined by the following integral equation

$$R_\nu\{f(x); p\} = g(p; \nu) = \int_0^\infty (px)^{\frac{1}{2}} K_\nu(px) f(x)\, dx \tag{13}$$

where p is a complex parameter and $K_\nu(z)$ represent a modified Bessel function of third kind.

(iv) Varma Transform (Mathai et al. [18], p.55, eq. 2.38)

The Varma transform of function $f(z)$ is defined by the following integral equation

$$V(f,k,m;s)=\int_0^\infty (sx)^{m-\frac{1}{2}} exp\Big(-\frac{1}{2}sx\Big) W_{k,m}(sx)f(x)\,dx \tag{14}$$

where $W_{k,m}(z)$ represents a Whittaker function defined by (Mathai et al.[18], p.55, eq. 2.39)

(v) Hankel Transform (Mathai et al. [18], p. 56, eq. 2.43)

The Hankel transform of function $f(z)$ is defined as

$$H_\nu\{f(x);p\}=g(p;\nu)=\int_0^\infty (px)^{\frac{1}{2}} J_\nu(px)f(x)dx,\ p>0 \tag{15}$$

where $J_\nu(z)$ represents a Bessel function of first kind.

(vi) Sumudu Transform (Watugala[19] p. 38,eq. 12)

The Sumudu transform is defined as

$$G(u)=S[f(t)]=\int_0^\infty f(ut)e^{-t}dt,\ u\in(-\tau_1,\tau_2) \tag{16}$$

over the set of functions

$$A=\{f(t)\in R|\exists M,\tau_1,\tau_2>0,|f(t)|<Me^{\frac{|t|}{\tau_j}},\\ \text{if }\ t\in(-1)^j\times[0,\infty),j=1,2\}$$

(vii) Fractional Fourier Transform (Luchko et al.[20] p.460, eq.4)

The fractional Fourier transform of the order α, $0<\alpha\leq 1$ is defined as

$$\hat{u}_\alpha(\omega)=\Im_\alpha[u](\omega)=\int_R e^{i\omega^{\frac{1}{\alpha}}z}u(z)\,dz \tag{17}$$

where u is a function belonging to the Lizorkin space of functions. On setting $\alpha=1$, (17) reduces to conventional Fourier transform given by

$$\hat{u}(\omega)=\Im[u](\omega)=\int_R e^{i\omega z}u(z)\,dz \tag{18}$$

2. Integral Transforms of Generalized M-Series

In this section, several integral transforms like Laplace transform, Beta transform, Varma transform, K-transform, Hankel transform, Sumudu transform and fractional Fourier transform are discussed for the generalized M-Series under the following theorems.

Theorem 1 (Laplace Transform): If $\alpha, \beta, a, s \in C, Re(\alpha) > 0, Re(\beta) > 0, Re(s) > 0$ and $|as^{-\alpha}| < 1$ then there hold the following formula

$$\int_0^\infty z^{\beta-1} e^{-sz} {}_pM_q{}^{\alpha,\beta}(az^\alpha)\, dz = \frac{1}{s^\beta}\, {}_{p+1}F_q\left(1, a_1, ..., a_p; b_1, ..., b_q; \frac{a}{s^\alpha}\right) \tag{19}$$

Proof. In virtue of (1) and(12)

$$\int_0^\infty z^{\beta-1} e^{-sz} {}_pM_q{}^{\alpha,\beta}(az^\alpha)\, dz = \int_0^\infty z^{\beta-1} e^{-sz} \sum_{k=0}^\infty \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{a^k z^{\alpha k}}{\Gamma(\alpha k + \beta)}\, dz$$

Interchanging the order of integral and summation which is permissible under the conditions stated with the Theorem , we get

$$= \sum_{k=0}^\infty \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{a^k}{\Gamma(\alpha k + \beta)} \int_0^\infty e^{-sz} z^{\alpha k + \beta - 1}\, dz$$

substituting $sz = t$, we get

$$= \sum_{k=0}^\infty \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{a^k}{\Gamma(\alpha k + \beta)} \frac{1}{s^{\alpha k + \beta}} \int_0^\infty e^{-t} t^{\alpha k + \beta - 1}\, dt$$

on applying the definition of Gamma function, we obtain

$$= \frac{1}{s^\beta} \sum_{k=0}^\infty \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \left(\frac{a}{s^\alpha}\right)^k$$
$$= \frac{1}{s^\beta}\, {}_{p+1}F_q\left(1, a_1 ..., a_p; b_1, ..., b_q; \frac{a}{s^\alpha}\right)$$

Hence the required result.

Theorem 2 (Beta Transform): If $\alpha, \beta, \rho, \sigma, b \in C, Re(\alpha) > 0, Re(\beta) > 0, Re(\rho) > 0, Re(\sigma) > 0$ and $\nu > 0, \tau > 0$ then there hold the following formula

$$\int_0^t z^{\rho-1}(t-z)^{\sigma-1} {}_pM_q{}^{\alpha,\beta}\left(bz^\nu (t-z)^\tau\right) dz = t^{\rho+\sigma-1} \left(\frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)}\right) \times {}_{p+3}\psi_{q+2}\left[\begin{matrix} (a_1,1), \cdots, (a_p,1), (\rho,\nu), (\sigma,\tau)(1,1) \\ (b_1,1), \cdots, (b_q,1), (\beta,\alpha), (\rho+\sigma, \nu+\tau) \end{matrix} \middle| bt^{\nu+\tau}\right] \tag{20}$$

Proof. If we apply the definition (1) of the M-Series to the given integral, we have

$$\int_0^t z^{\rho-1}(t-z)^{\sigma-1}{}_pM_q{}^{\alpha,\beta}\Big(bz^{\nu}(t-z)^{\tau}\Big)\,dz$$

$$=\int_0^t z^{\rho-1}(t-z)^{\sigma-1}\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{b^kz^{\nu k}(t-z)^{\tau k}}{\Gamma(\alpha k+\beta)}\,dz$$

Interchanging the order of integral and summation which is permissible under the conditions stated with the Theorem , we get

$$=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{b^k}{\Gamma(\alpha k+\beta)}\int_0^t z^{\nu k+\rho-1}(t-z)^{\tau k+\sigma-1}\,dz$$

substituting $z=st$, we get

$$=t^{\rho+\sigma-1}\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{b^kt^{(\nu+\tau)k}}{\Gamma(\alpha k+\beta)}\int_0^1 s^{\nu k+\rho-1}(1-s)^{\tau k+\sigma-1}\,ds$$

$$=t^{\rho+\sigma-1}\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{\Gamma(\nu k+\rho)\Gamma(\tau k+\sigma)(1)_kbt^{(\nu+\tau)k}}{\Gamma(\alpha k+\beta)\Gamma((\nu+\tau)k+\rho+\sigma)k!}$$

by adjusting the terms, we obtain (20) where ${}_p\psi_q$ is Wright's generalized hypergeometric function defined by (9).

Theorem 3 (Varma Transform): If $\alpha,\beta,\rho,\nu,\lambda,a,b\in C,\sigma>0$ and $Re(\alpha)>0,Re(\beta)>0,Re(\rho)>0,Re(\nu)>0,Re(\lambda)>0,Re(a)>0$ then there hold the following formula

$$\int_0^{\infty} z^{\rho-1}exp\Big(-\frac{1}{2}az\Big)W_{\lambda,\nu}(az){}_pM_q{}^{\alpha,\beta}(bz^{\sigma})\,dz=a^{-\rho}\left(\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}\right)$$
$$\times\,{}_{p+3}\psi_{q+2}\left[\begin{array}{c}(a_1,1),\cdots,(a_p,1),\Big(\rho+\nu+\frac{1}{2},\sigma\Big),\Big(\rho-\nu+\frac{1}{2},\sigma\Big)(1,1)\\(b_1,1),\cdots,(b_q,1),(\beta,\alpha),(1-\lambda+\rho,\sigma)\end{array}\Bigg|\frac{b}{a^{\sigma}}\right]\tag{21}$$

Proof. If we apply the definition (1) of the M-Series to the given integral, we have

$$\int_0^{\infty} z^{\rho-1}exp\Big(-\frac{1}{2}az\Big)W_{\lambda,\nu}(az){}_pM_q{}^{\alpha,\beta}(bz^{\sigma})\,dz$$

$$=\int_0^{\infty} z^{\rho-1}exp\Big(-\frac{1}{2}az\Big)W_{\lambda,\nu}(az)\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{b^kz^{\sigma k}}{\Gamma(\alpha k+\beta)}\,dz$$

Interchanging the order of integral and summation which is permissible under the conditions stated with the Theorem , we get

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{b^k}{\Gamma(\alpha k+\beta)} \int_0^{\infty} z^{\sigma k+\rho-1} exp\Big(-\frac{1}{2}az\Big) W_{\lambda,\nu}(az)\, dz$$

Now using the integral of Mathai and Saxena ([18], p.56, eq. 2.41)

$$\int_0^{\infty} z^{\rho-1} exp\Big(-\frac{1}{2}az\Big) W_{k,\nu}(az)\, dz = a^{-\rho} \frac{\Gamma(\rho+\nu+\frac{1}{2})\Gamma(\rho-\nu+\frac{1}{2})}{\Gamma(1-k+\rho)}$$

in the above equation, we obtain

$$= a^{-\rho} \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{\Gamma(\sigma k+\rho+\nu+\frac{1}{2})\Gamma(\sigma k+\rho-\nu+\frac{1}{2})(1)_k}{\Gamma(\alpha k+\beta)\Gamma(1-\lambda+\rho+\sigma k)k!} \Big(\frac{b}{a^{\sigma}}\Big)^k$$

by adjusting the terms, we obtain (21).

Theorem 4 (K-Transform): If $\alpha, \beta, \rho, \nu, a, b \in C, \sigma > 0$ and $Re(\alpha) > 0, Re(\beta) > 0, Re(\rho) > 0, Re(\nu) > 0$ then there hold the following formula

$$\int_0^{\infty} z^{\rho-1} K_{\nu}(az) {}_pM_q^{\alpha,\beta}(bz^{\sigma})\, dz = \frac{1}{4}\Big(\frac{2}{a}\Big)^{\rho} \left(\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}\right) \times {}_{p+2}\psi_{q+1}\left[\begin{matrix} (a_1,1),\cdots,(a_p,1),\Big(\frac{\rho\pm\nu}{2},\frac{\sigma}{2}\Big),(1,1) \\ (b_1,1),\cdots,(b_q,1),(\beta,\alpha)\end{matrix} \Bigg| b\Big(\frac{2}{a}\Big)^{\sigma}\right] \tag{22}$$

Proof. If we apply the definition (1) of the M-Series to the given integral, we have

$$\int_0^{\infty} z^{\rho-1} K_{\nu}(az) {}_pM_q^{\alpha,\beta}(bz^{\sigma})\, dz = \int_0^{\infty} z^{\rho-1} K_{\nu}(az) \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{b^k z^{\sigma k}}{\Gamma(\alpha k+\beta)}\, dz$$

Interchanging the order of integral and summation which is permissible under the conditions stated with the Theorem , we get

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{b^k}{\Gamma(\alpha k+\beta)} \int_0^{\infty} z^{\sigma k+\rho-1} K_{\nu}(az)\, dz$$

using the integral of Mathai and Saxena ([18],p. 54, eq. 2.37)

$$\int_0^{\infty} z^{\rho-1} K_{\nu}(az)\, dz = 2^{\rho-2} a^{-\rho} \Gamma\Big(\frac{\rho\pm\nu}{2}\Big)$$

in the above equation, we obtain

$$= \frac{1}{4}\Big(\frac{2}{a}\Big)^{\rho} \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k \Gamma(\frac{\sigma k}{2} + (\frac{\rho\pm\nu}{2}))(1)_k}{(b_1)_k \cdots (b_q)_k \Gamma(\alpha k+\beta)\, k!} \Big\{ b\Big(\frac{2}{a}\Big)^{\sigma}\Big\}^k$$

by adjusting the terms, we obtain (22).

Theorem 5 (Hankel Transform): If $\alpha, \beta, \rho, \nu, a, b \in C, \sigma > 0$ and $Re(\alpha) > 0, Re(\beta) > 0, Re(\rho) > 0$ then there hold the following formula

$$\int_0^\infty z^{\rho-1} J_\nu(az)_p M_q{}^{\alpha,\beta}(bz^\sigma)\,dz = \frac{1}{2}\left(\frac{2}{a}\right)^\rho \left(\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}\right) \times {}_{p+2}\psi_{q+2}\left[\begin{array}{c} (a_1,1),\cdots,(a_p,1),\left(\frac{\rho+\nu}{2},\frac{\sigma}{2}\right),(1,1) \\ (b_1,1),\cdots,(b_q,1),(\beta,\alpha),\left(\frac{2+\nu-\rho}{2},-\frac{\sigma}{2}\right) \end{array}\Bigg| b\left(\frac{2}{a}\right)^\sigma\right] \tag{23}$$

Proof. If we apply the definition (1) of the M-Series to the given integral, we have

$$\int_0^\infty z^{\rho-1} J_\nu(az)_p M_q{}^{\alpha,\beta}(bz^\sigma)\,dz = \int_0^\infty z^{\rho-1} J_\nu(az) \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{b^k z^{\sigma k}}{\Gamma(\alpha k+\beta)}\,dz$$

Interchanging the order of integral and summation which is permissible under the conditions stated with the Theorem , we get

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{b^k}{\Gamma(\alpha k+\beta)} \int_0^\infty z^{\sigma k+\rho-1} J_\nu(az)\,dz$$

using the integral of Mathai and Saxena ([18], p. 57, eq. 2.46)

$$\int_0^\infty z^{\rho-1} J_\nu(az)\,dz = 2^{\rho-1} a^{-\rho} \frac{\Gamma(\frac{\rho+\nu}{2})}{\Gamma(1+\frac{\nu-\rho}{2})}$$

in the above equation, we obtain

$$= \frac{1}{2}\left(\frac{2}{a}\right)^\rho \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k \Gamma(\frac{\sigma k}{2} + (\frac{\rho+\nu}{2}))(1)_k}{(b_1)_k \cdots (b_q)_k \Gamma(\alpha k+\beta)\Gamma(\frac{2+\nu-\rho}{2} - \frac{\sigma k}{2})k!} \left\{b\left(\frac{2}{a}\right)^\sigma\right\}^k$$

by adjusting the terms, we obtain (23).

Theorem 6 (Sumudu Transform): If $\alpha,\ \beta \in C, Re(\alpha) > 0,\ Re(\beta) > 0$ and $|u^\alpha| < 1$ then there hold the following formula

$$\int_0^\infty (uz)^{\beta-1} e^{-z}{}_p M_q{}^{\alpha,\beta}(uz)^\alpha\,dz = u^{\beta-1}\ {}_{p+1}F_q(1, a_1, ..., a_p; b_1, ..., b_q; u^\alpha) \tag{24}$$

Proof. In virtue of (1) and (16)

$$\int_0^\infty (uz)^{\beta-1} e^{-z}{}_p M_q{}^{\alpha,\beta}(uz)^\alpha\,dz = \int_0^\infty (uz)^{\beta-1} e^{-z} \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{u^{\alpha k} z^{\alpha k}}{\Gamma(\alpha k+\beta)}\,dz$$

Interchanging the order of integral and summation which is permissible under the conditions stated with the Theorem , we get

$$= u^{\beta-1}\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{u^{\alpha k}}{\Gamma(\alpha k+\beta)}\int_0^{\infty} e^{-z}z^{\alpha k+\beta-1}\,dz$$

on applying the definition of Gamma function, we obtain

$$= u^{\beta-1}\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}u^{\alpha k}$$

$$= u^{\beta-1}{}_{p+1}F_q\Big(1, a_1, ..., a_p; b_1, ..., b_q; u^{\alpha})$$

Hence the required result.

Theorem 7 (Fractional Fourier Transform): If σ, $\beta \in C, Re(\sigma) > 0$, $Re(\beta) > 0, z < 0$ and $\left|\left(\frac{-a}{i\omega^{\frac{1}{\alpha}}}\right)^{\sigma}\right| < 1$ then there hold the following formula

$$\int_R z^{\beta-1}e^{i\omega^{\frac{1}{\alpha}}z}{}_pM_q{}^{\sigma,\beta}(az)^{\sigma}\,dz = \frac{(-1)^{\beta-1}}{(i\omega^{\frac{1}{\alpha}})^{\beta}}\,{}_{p+1}F_q\left(1, a_1, ..., a_p; b_1, ..., b_q; \left(\frac{-a}{i\omega^{\frac{1}{\alpha}}}\right)^{\sigma}\right) \tag{25}$$

Proof. If we apply the definition (1) of the M-Series to the given integral, we have

$$\int_R z^{\beta-1}e^{i\omega^{\frac{1}{\alpha}}z}{}_pM_q{}^{\sigma,\beta}(az)^{\sigma}\,dz = \int_R z^{\beta-1}e^{i\omega^{\frac{1}{\alpha}}z}\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{a^{\sigma k}z^{\sigma k}}{\Gamma(\sigma k+\beta)}\,dz$$

Interchanging the order of integral and summation which is permissible under the conditions stated with the Theorem , we get

$$= \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{a^{\sigma k}}{\Gamma(\sigma k+\beta)}\int_{-\infty}^{0} e^{i\omega^{\frac{1}{\alpha}}z}z^{\sigma k+\beta-1}\,dz$$

changing the variable $i\omega^{\frac{1}{\alpha}}z = -t$, we get

$$= \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{a^{\sigma k}}{\Gamma(\sigma k+\beta)}\frac{(-1)^{\sigma k+\beta-1}}{(i\omega^{\frac{1}{\alpha}})^{\sigma k+\beta}}\int_0^{\infty} e^{-t}t^{\sigma k+\beta-1}\,dt$$

on applying the definition of Gamma function, we obtain

$$= \frac{(-1)^{\beta-1}}{(i\omega^{\frac{1}{\alpha}})^{\beta}}\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{(-1)^{\sigma k}a^{\sigma k}}{(i\omega^{\frac{1}{\alpha}})^{\sigma k}}$$

$$= \frac{(-1)^{\beta-1}}{(i\omega^{\frac{1}{\alpha}})^{\beta}}\,{}_{p+1}F_q\left(1, a_1, ..., a_p; b_1, ..., b_q; \left(\frac{-a}{i\omega^{\frac{1}{\alpha}}}\right)^{\sigma}\right)$$

Hence the required result.

3. Conclusion

In this study, we have developed various integral transforms of the generalized M-series.The results obtained in this article are general in nature and are likely to find useful in applied problems of science, engineering and technology.

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D.S. Sachan

St. Mary's PG College, Vidisha, (M.P.), INDIA, 464001.

E-mail address: sachan.dheerandra17@gmail.com

S. Jaloree

Samrat Ashok Technological Institute,Vidisha,(M.P.) INDIA, 464001.

E-mail address: shailesh_jaloree@rediffmail.com