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ON GENERALIZATION OF SOME INEQUALITIES OF CHEBYSHEV'S FUNCTIONAL USING GENERALIZED KATUGAMPOLA FRACTIONAL INTEGRAL

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Abstract. In this paper, we obtain a generalization of some integral inequalities related to Chebyshev's functional by using a generalized Katugampola fractional integral. With the help of generalized Katugampola fractional integral operator, some more integral inequalities of Chebyshev's functional are derived and some special cases for the result obtained in the paper are also discussed.

1. INTRODUCTION:

The fractional calculus is generalized form of classical integrals and derivatives, which study integrals and derivatives in case of non-integer order. In recent two decades, fractional calculus theory acquires more importance due to its applications in several fields such as computer networking, biology, physics, fluid dynamics, signal processing, image processing, control theory and other fields, for some examples see ([1], [19]). One cannot deny the significance of fractional calculus, and one can observe several researchers recently have shown their keen interest in studying it deeply and subsequently, its classical concept has been extended and developed by many authors.

In classical integral and differential equations, mathematical inequalities plays very authoritative role and in the past several years, a number of useful and important mathematical inequalities invented by many authors.

In (1882), Chebyshev [3], has given the following functional

$$T(\varphi,\psi) := \frac{1}{b-a}\int_a^b \varphi(\gamma)\,\psi(\gamma)\,d\gamma - \frac{1}{b-a}\left(\int_a^b \varphi(\gamma)\,d\gamma\right)\left(\int_a^b \psi(\gamma)\,d\gamma\right). \quad (1)$$

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Its extention is obtained by Mitrinovic as (see [14])

$$\begin{aligned} T(\varphi,\psi,g,h) \quad : \quad &= \int_a^b h(\gamma)\,d\gamma \int_a^b \varphi(\gamma)\,\psi(\gamma)\,g(\gamma)\,d\gamma \\ &+ \int_a^b g(\gamma)\,d\gamma \int_a^b \varphi(\gamma)\,\psi(\gamma)\,h(\gamma)\,d\gamma \\ &- \left(\int_a^b \varphi(\gamma)\,h(\gamma)\,d\gamma\right)\left(\int_a^b \psi(\gamma)\,g(\gamma)\,d\gamma\right) \\ &- \left(\int_a^b \varphi(\gamma)\,g(\gamma)\,dx\right)\left(\int_a^b \psi(\gamma)\,h(\gamma)\,d\gamma\right), \end{aligned} \tag{2}$$

where φ and ψ are two integrable functions on $[a,b]$. Many researchers have given considerable attention to the both functionals and a number of inequalities and a number of extensions, generalizations and variants have appeared in the literature, for more details see ([9], [10], [15], [16], [17]). If φ and ψ satisfies the following condition

$$(\varphi(\tau)-\varphi(\gamma))(\psi(\tau)-\psi(\gamma)) \geq 0,$$

for any $\tau,\gamma \in [a,b]$, then φ and ψ are synchronous on $[a,b]$, moreover, $T(\varphi,\psi,g,h) \geq 0$ (see also [14]). It should be noted that the sign of this inequality is reversed if φ and ψ are synchronous on $[a,b]$, namely $(\varphi(\tau)-\varphi(\gamma))(\psi(\tau)-\psi(\gamma)) \leq 0$, for any $\tau,\gamma \in [a,b]$.

One of the most important things to note in this work is the celebrated Chebyshev functional [3]

$$\begin{aligned} T(\varphi,\psi,g) \quad : \quad &= \int_a^b g(\gamma)\,d\gamma \int_a^b \varphi(\gamma)\,\psi(\gamma)\,g(\gamma)\,d\gamma \\ &+ \int_a^b g(\gamma)\,\varphi(\gamma)\,d\gamma \int_a^b \psi(\gamma)\,g(\gamma)\,d\gamma, \end{aligned} \tag{3}$$

where φ and ψ are two differentiable functions on $[a,b]$ and p is positive and integrable functions on $[a,b]$, this functional has many applications in probability and statistics and other fields.

In [7] (see also [8]), Dragomir proved that, if φ and ψ are two differentiable functions such that $\varphi' \in L_s(a,b)$, $\psi' \in L_v(a,b)$, $r>1$, $\frac{1}{s}+\frac{1}{v}=1$, then

$$2\left|T(\varphi,\psi,g)\right| \leq \|\varphi'\|_s \|\psi'\|_v \left[\int_a^b |\tau-\gamma|\,g(\tau)\,g(\gamma)\,d\tau d\gamma\right]. \tag{4}$$

For inequality (4), Dahmani in [4] (see also [5], [6]), proved fractional version of the inequality as

$$\begin{aligned} &2\left|I^\alpha g(x)\,I^\alpha(g\varphi\psi)(x) - I^\alpha(g\varphi)(x)\,I^\alpha(g\psi)(x)\right| \\ &\leq \frac{\|\varphi'\|_s \|\psi'\|_v}{\Gamma^2(\alpha)} \int_a^b \int_a^b (x-\tau)^{\alpha-1}(x-\gamma)^{\alpha-1}|\tau-\gamma|\,g(\tau)\,g(\gamma)\,d\tau d\gamma \\ &\leq \|\varphi'\|_s \|\psi'\|_v\, x\left(I^\alpha g(x)\right)^2, \end{aligned} \tag{5}$$

for all $\alpha>0$, $x>0$.

Motivated by ([4], [5], [6]), in this paper, we establish some new fractional inequalities for Chebyshev functional involving generalized Katugampola fractional integral. Our results in this paper are organized in three sections, first and second sections are related to Chebyshev functional in case of synchronous functions and the third section is related to Chebyshev functional in case of differentiable functions whose derivatives belong to $L_p([0,\infty])$.

2. PRELIMINARIES:

Now, in this section, we give the necessary notation and basic definitions used in our subsequent discussion. For more details see ([11], [12], [18]).

Definition 2.1. *Consider the space* $X_c^p(a,b)\,(c\in\mathbb{R}, 1\le p\le\infty)$, *of those complex valued Lebesgue measurable functions* φ *on* (a,b) *for which the norm* $\|\varphi\|_{X_c^p}<\infty$, *such that*

$$\|\varphi\|_{X_c^p}=\left(\int_x^b |x^c\varphi|^p\,\frac{dx}{x}\right)^{\frac{1}{p}},\qquad (1\le p<\infty)$$

and

$$\|\varphi\|_{X_c^\infty}=\sup ess_{x\in(a,b)}\left[x^c\,|\varphi|\right].$$

In particular, when $c=1/p$, *the space* $X_c^p(a,b)$ *coincides with the space* $L^p(a,b)$.

Definition 2.2. *The left and right-sided fractional integrals of a function* φ *where* $\varphi\in X_c^p(a,b)$, $\alpha>0$ *and* $\beta,\rho,\eta,k\in\mathbb{R}$, *are defined respectively by*

$$^{\rho}\mathcal{I}^{\alpha,\beta}_{a+;\eta,k}\varphi(x)=\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_a^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\varphi(\tau)\,d\tau,\qquad 0\le a<x<b\le\infty \quad (6)$$

and

$$^{\rho}\mathcal{I}^{\alpha,\beta}_{b-;\eta,k}\varphi(x)=\frac{\rho^{1-\beta}x^{\rho\eta}}{\Gamma(\alpha)}\int_x^b\frac{\tau^{k+\rho-1}}{(\tau^\rho-x^\rho)^{1-\alpha}}\varphi(\tau)\,d\tau,\qquad 0\le a<x<b\le\infty, \quad (7)$$

if the integral exist.

In this paper we use the left-sided fractional integrals (6) to present and discuss our new results, also we consider $a=0$ in (6), to obtain

$$^{\rho}\mathcal{I}^{\alpha,\beta}_{\eta,k}\varphi(x)=\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\varphi(\tau)\,d\tau.$$

Now, we define the following function as in [18]: let $x>0$, $\alpha>0$, $\rho,k,\beta,\eta\in\mathbb{R}$, then

$$\Lambda^{\rho,\beta}_{x,k}(\alpha,\eta)=\frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha+1)}\rho^{-\beta}x^{k+\rho(\eta+\alpha)}.$$

Note that, the Definition (2.2) is more generalized and can be reduce to six cases by change its parameters with convenient choice as follows:

- Liouville fractional integral, for $\eta=0$, $a=0$, $k=0$ and taking the limit $\rho\to 1$, [[13], p. 79].

- Weyl fractional integral, for $\eta=0$, $a=-\infty$, $k=0$ and taking the limit $\rho\to 1$, [[2], p. 50].

-Riemann-Liouville fractional integral, for $\eta=0$, $k=0$ and taking the limit $\rho\to 1$,[[13], p. 69].

-Katugampola fractional integral, for $\beta=\alpha$, $k=0$, $\eta=0$, [11].

-Erdélyi-Kober fractional integral, for $\beta=0$, $k=-\rho(\alpha+\eta)$, [[13], p.105].

-Hadamard fractional integral, for $\beta = \alpha$, $k = 0$, $\eta = 0^+$ and taking the limit $\rho \to 1$, [[13], p. 110].

3. Generalized fractional inequality for Chebyshev's functional:

In this section, we establish inequalities for Chebyshev functional [3], which deals with same parameters

Theorem 3.1. Let φ and ψ be two integrable and synchronous functions on $[0, \infty)$. Then for all $x > 0$, $\alpha > 0$, $\rho > 0$, $k, \beta, \eta \in \mathbb{R}$ we have:

$$ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(\varphi\psi\right)(x) \geq \frac{1}{\Lambda_{x,k}^{\rho,\beta}\left(\alpha,\eta\right)}\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\varphi\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\psi\left(x\right). \tag{8} $$

Proof. For a synchronous functions φ, ψ on $[0, \infty)$, we have for all $\tau \geq 0$, $\gamma \geq 0$:

$$ \left(\varphi\left(\tau\right) - \varphi\left(\gamma\right)\right)\left(\psi\left(\tau\right) - \psi\left(\gamma\right)\right) \geq 0. $$

Therefore

$$ \varphi\left(\tau\right)\psi\left(\tau\right) + \varphi\left(\gamma\right)\psi\left(\gamma\right) \geq \varphi\left(\tau\right)\psi\left(\gamma\right) + \varphi\left(\gamma\right)\psi\left(\tau\right). \tag{9} $$

Multiplying both sides of (9) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}$, where $\tau \in (0, x)$ and integrating the resulting inequality over $(0, x)$, with respect to the variable τ, we obtain

$$ \begin{aligned} &\frac{\rho^{1-\beta}x^k}{\Gamma\left(\alpha\right)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{\left(x^\rho - \tau^\rho\right)^{1-\alpha}}\varphi\left(\tau\right)\psi\left(\tau\right)d\tau \\ &+\varphi\left(\gamma\right)\psi\left(\gamma\right)\frac{\rho^{1-\beta}x^k}{\Gamma\left(\alpha\right)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{\left(x^\rho - \tau^\rho\right)^{1-\alpha}}d\tau \\ &\geq \psi\left(\gamma\right)\frac{\rho^{1-\beta}x^k}{\Gamma\left(\alpha\right)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{\left(x^\rho - \tau^\rho\right)^{1-\alpha}}\varphi\left(\tau\right)d\tau \\ &+\varphi\left(\gamma\right)\frac{\rho^{1-\beta}x^k}{\Gamma\left(\alpha\right)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{\left(x^\rho - \tau^\rho\right)^{1-\alpha}}\psi\left(\tau\right)d\tau. \end{aligned} \tag{10} $$

So we have

$$ \begin{aligned} &{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(\varphi\psi\right)(x) + \Lambda_{x,k}^{\rho,\beta}\left(\alpha,\eta\right)\varphi\left(\gamma\right)\psi\left(\gamma\right) \\ &\geq \psi\left(\gamma\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\varphi\left(x\right) + \varphi\left(\gamma\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\psi\left(x\right). \end{aligned} \tag{11} $$

Now multiplying both sides of (11) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\alpha}}$, where $\gamma \in (0, x)$, over $(0, x)$, then integrating the resulting inequality over $(0, x)$, with respect to the variable γ, we get:

$$ \begin{aligned} &{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(\varphi\psi\right)(x)\frac{\rho^{1-\beta}x^k}{\Gamma\left(\alpha\right)}\int_0^x \frac{\gamma^{\rho(\eta+1)-1}}{\left(x^\rho - \gamma^\rho\right)^{1-\alpha}}d\gamma \\ &+\Lambda_{x,k}^{\rho,\beta}\left(\alpha,\eta\right)\frac{\rho^{1-\beta}x^k}{\Gamma\left(\alpha\right)}\int_0^x \frac{\gamma^{\rho(\eta+1)-1}}{\left(x^\rho - \gamma^\rho\right)^{1-\alpha}}\varphi\left(\gamma\right)\psi\left(\gamma\right)d\gamma \\ &\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\varphi\left(x\right)\frac{\rho^{1-\beta}x^k}{\Gamma\left(\alpha\right)}\int_0^x \frac{\gamma^{\rho(\eta+1)-1}}{\left(x^\rho - \gamma^\rho\right)^{1-\alpha}}\psi\left(\gamma\right)d\gamma \\ &+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\psi\left(x\right)\frac{\rho^{1-\beta}x^k}{\Gamma\left(\alpha\right)}\int_0^x \frac{\gamma^{\rho(\eta+1)-1}}{\left(x^\rho - \gamma^\rho\right)^{1-\alpha}}\varphi\left(\gamma\right)d\gamma. \end{aligned} $$

So we have

$$\begin{aligned} &{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(\varphi\psi\right)(x)^{\rho}\,\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(1\right)+\Lambda_{x,k}^{\rho,\beta}\left(\alpha,\eta\right)^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(\varphi\psi\right)(x) \\ &\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\varphi\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\psi\left(x\right)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\psi\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\varphi\left(x\right). \end{aligned}$$

Hence

$${}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(\varphi\psi\right)(x)\geq\frac{1}{\Lambda_{x,k}^{\rho,\beta}\left(\alpha,\eta\right)}\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\varphi\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\psi\left(x\right).$$

This proves the result. □

Our next theorem on Chebyshev functional deals with different fractional parameters:

Theorem 3.2. Let φ and ψ be two integrable and synchronous functions on $[0,\infty)$ .Then for all $x>0,\ \alpha>0,\ \delta>0,\ \rho>0,\ k,\beta,\lambda,\eta\in\mathbb{R}$, we have:

$$\begin{aligned} &\Lambda_{x,k}^{\rho,\lambda}\left(\delta,\eta\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(\varphi\psi\right)(x)+\Lambda_{x,k}^{\rho,\beta}\left(\alpha,\eta\right)^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(\varphi\psi\right)(x) \\ &\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\varphi\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\psi\left(x\right)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\psi\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\varphi\left(x\right). \end{aligned}$$

Proof. Since φ and ψ are synchronous functions on $[0,\infty)$, by similar arguments as in the proof of Theorem (3.1) we can write

$$\begin{aligned} &{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(\varphi\psi\right)(x)+\Lambda_{x,k}^{\rho,\beta}\left(\alpha,\eta\right)\varphi\left(\gamma\right)\psi\left(\gamma\right) \\ &\geq\psi\left(\gamma\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\varphi\left(x\right)+\varphi\left(\gamma\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\psi\left(x\right). \end{aligned} \tag{12}$$

Now, multiplying both sides of (12) by $\frac{\rho^{1-\lambda}x^{k}}{\Gamma(\delta)}\frac{\gamma^{\rho(\eta+1)-1}}{(x^{\rho}-\gamma^{\rho})^{1-\delta}}$, where $\gamma\in(0,x)$, then integrating the resulting inequality over $(0,x)$, with respect to the variable γ, we obtain

$$\begin{aligned} &{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(\varphi\psi\right)(x)\frac{\rho^{1-\lambda}x^{k}}{\Gamma\left(\delta\right)}\int_{0}^{x}\frac{\gamma^{\rho(\eta+1)-1}}{\left(x^{\rho}-\gamma^{\rho}\right)^{1-\delta}}d\gamma \\ &+\Lambda_{x,k}^{\rho,\beta}\left(\alpha,\eta\right)\frac{\rho^{1-\lambda}x^{k}}{\Gamma\left(\delta\right)}\int_{0}^{x}\frac{\gamma^{\rho(\eta+1)-1}}{\left(x^{\rho}-\gamma^{\rho}\right)^{1-\delta}}\varphi\left(\gamma\right)\psi\left(\gamma\right)d\gamma \\ &\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\varphi\left(x\right)\frac{\rho^{1-\lambda}x^{k}}{\Gamma\left(\delta\right)}\int_{0}^{x}\frac{\gamma^{\rho(\eta+1)-1}}{\left(x^{\rho}-\gamma^{\rho}\right)^{1-\delta}}\psi\left(\gamma\right)d\gamma \\ &+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\psi\left(x\right)\frac{\rho^{1-\lambda}x^{k}}{\Gamma\left(\delta\right)}\int_{0}^{x}\frac{\gamma^{\rho(\eta+1)-1}}{\left(x^{\rho}-\gamma^{\rho}\right)^{1-\delta}}\varphi\left(\gamma\right)d\gamma. \end{aligned} \tag{13}$$

So we have

$$\begin{aligned} &\Lambda_{x,k}^{\rho,\lambda}\left(\delta,\eta\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(\varphi\psi\right)(x)+\Lambda_{x,k}^{\rho,\beta}\left(\alpha,\eta\right)^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(\varphi\psi\right)(x) \\ &\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\varphi\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\psi\left(x\right)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\psi\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\varphi\left(x\right). \end{aligned}$$

This proves the result. □

4. Generalized fractional inequality for extended Chebyshev's functional:

In this section, we consider the extended Chebyshev functional in case of synchronous functions (2). To prove our theorem in this section, we need the following lemma:

Lemma 4.1. Let φ and ψ be two integrable and synchronous functions on $[0,\infty)$. Suppose $s,v:[0,\infty)\to[0,\infty)$. Then for all $x>0$, $\alpha>0$, $\rho>0$, $k,\beta,\eta\in\mathbb{R}$ we have:

$$\begin{aligned}&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\varphi\psi\right)(x)^{\rho}\,\mathcal{I}_{\eta,k}^{\alpha,\beta}v\left(x\right)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}s\left(x\right)^{\rho}\,\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(v\varphi\psi\right)(x)\\&\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\varphi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(v\psi\right)(x)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\psi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(v\varphi\right)(x).\end{aligned}$$

Proof. For the synchronous functions φ,ψ on $[0,\infty)$, we have for all $\tau\geq0$, $\gamma\geq0$:

$$\left(\varphi(\tau)-\varphi(\gamma)\right)\left(\psi(\tau)-\psi(\gamma)\right)\geq0.$$

Therefore

$$\varphi(\tau)\psi(\tau)+\varphi(\gamma)\psi(\gamma)\geq\varphi(\tau)\psi(\gamma)+\varphi(\gamma)\psi(\tau).\tag{14}$$

Now, multiplying both sides of (14) by $\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}s(\tau)}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}$, where $\tau\in(0,x)$, then integrating the resulting inequality over $(0,x)$, with respect to the variable τ, we get

$$\begin{aligned}&\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}s(\tau)\varphi(\tau)\psi(\tau)\,d\tau\\&+\varphi(\gamma)\psi(\gamma)\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}s(\tau)\,d\tau\\&\geq\psi(\gamma)\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}s(\tau)\varphi(\tau)\,d\tau\\&+\varphi(\gamma)\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}s(\tau)\psi(\tau)\,d\tau.\end{aligned}$$

So we have

$$\begin{aligned}&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\varphi\psi\right)(x)+\varphi(\gamma)\psi(\gamma)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}s(x)\\&\geq\psi(\gamma)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\varphi\right)(x)+\varphi(\gamma)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\psi\right)(x).\end{aligned}\tag{15}$$

Now, multiplying both sides of (15) by $\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\frac{\gamma^{\rho(\eta+1)-1}v(\gamma)}{(x^{\rho}-\gamma^{\rho})^{1-\alpha}}$, where $\gamma\in(0,x)$ and integrating the resulting inequality over $(0,x)$, with respect to the variable γ, we obtain

$$\begin{aligned}&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\varphi\psi\right)(x)\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\gamma^{\rho(\eta+1)-1}}{(x^{\rho}-\gamma^{\rho})^{1-\alpha}}v(\gamma)\,d\gamma\\&+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}s(x)\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\gamma^{\rho(\eta+1)-1}}{(x^{\rho}-\gamma^{\rho})^{1-\alpha}}v(\gamma)\varphi(\gamma)\psi(\gamma)\,d\gamma\\&\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\varphi\right)(x)\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\gamma^{\rho(\eta+1)-1}}{(x^{\rho}-\gamma^{\rho})^{1-\alpha}}v(\gamma)\psi(\gamma)\,d\gamma\\&+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\psi\right)(x)\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\gamma^{\rho(\eta+1)-1}}{(x^{\rho}-\gamma^{\rho})^{1-\alpha}}v(\gamma)\varphi(\gamma)\,d\gamma.\end{aligned}$$

Therefore

$$\begin{aligned}&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\varphi\psi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}v(x)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}s(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(v\varphi\psi\right)(x)\\&\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\varphi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(v\psi\right)(x)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\psi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(v\varphi\right)(x).\end{aligned}$$

Hence the proof. □

Theorem 4.1. Let φ and ψ be two integrable and synchronous functions on $[0,\infty)$ and suppose $f,g,h:[0,\infty)\to[0,\infty)$. Then for all $x>0$, $\alpha>0$, $\rho>0$, $k,\beta,\eta\in\mathbb{R}$ we have:

$$\begin{aligned}
&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)\,[{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(f\varphi\psi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g(x)+2\,{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(g\varphi\psi)(x)\\
&+{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(f\varphi\psi)(x)]+2\,{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi\psi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g(x)\\
&\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)\left[{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(f\varphi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(g\psi)(x)+{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(f\psi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(g\varphi)(x)\right]\\
&\ +{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f(x)\left[{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(g\psi)(x)+{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\psi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(g\varphi)(x)\right]\\
&\ +{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g(x)\left[{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(f\psi)(x)+{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\psi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(f\varphi)(x)\right].
\end{aligned}\tag{16}$$

Proof. In lemma (4.1), putting $s=f$, $v=g$ and multiplying both sides of the resulting inequality by ${}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)$, we get

$$\begin{aligned}
&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(f\varphi\psi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g(x)+{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(g\varphi\psi)(x)\\
&\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(f\varphi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(g\psi)(x)\\
&\ +{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(f\psi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(g\varphi)(x).
\end{aligned}\tag{17}$$

Now, putting $s=h$, $v=g$ in lemma (4.1) and multiplying both sides of the resulting inequality by ${}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f(x)$, we obtain

$$\begin{aligned}
&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi\psi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g(x)+{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(g\varphi\psi)(x)\\
&\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(g\psi)(x)\\
&\ +{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\psi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(g\varphi)(x).
\end{aligned}\tag{18}$$

Now, putting $s=h$, $v=f$ in lemma (4.1) and multiplying both sides of the resulting inequality by ${}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g(x)$, we get

$$\begin{aligned}
&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g(x)\,{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi\psi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f(x)+{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(f\varphi\psi)(x)\\
&\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(f\psi)(x)\\
&\ +{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\psi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(f\varphi)(x).
\end{aligned}\tag{19}$$

By adding the inequalities (17), (18), (19) we get the inequality (16). □

Now, we give the lemma required for proving our next theorem for different parameter.

Lemma 4.2. Let φ and ψ be two integrable and synchronous functions on $[0,\infty)$. Suppose that $s,v:[0,\infty)\to[0,\infty)$. Then for all $x>0$, $\alpha>0$, $\delta>0$, $\rho>0$, $k,\beta,\lambda,\eta\in\mathbb{R}$, we have:

$$\begin{aligned}
&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(s\varphi\psi)(x)\,{}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}v(x)+{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}s(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}(v\varphi\psi)(x)\\
&\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(s\varphi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}(v\psi)(x)+{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}(s\psi)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}(v\varphi)(x).
\end{aligned}$$

Proof. Since φ and ψ are synchronous functions on $[0,\infty)$, by similar arguments as in the proof of lemma (4.1), we can write

$$\begin{aligned}&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\varphi\psi\right)(x)+\varphi\left(\gamma\right)\psi\left(\gamma\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}s\left(x\right)\\&\geq\psi\left(\gamma\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\varphi\right)(x)+\varphi\left(\gamma\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\psi\right)(x).\end{aligned}\tag{20}$$

Multiplying both sides of (20) by $\frac{\rho^{1-\lambda}x^{k}}{\Gamma(\delta)}\frac{\gamma^{\rho(\eta+1)-1}v(\gamma)}{(x^{\rho}-\gamma^{\rho})^{1-\delta}}$, where $\gamma\in(0,x)$ and integrating resulting inequality over $(0,x)$, with respect to the variable γ, we obtain

$$\begin{aligned}&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\varphi\psi\right)(x)\frac{\rho^{1-\lambda}x^{k}}{\Gamma\left(\delta\right)}\int_{0}^{x}\frac{\gamma^{\rho(\eta+1)-1}}{\left(x^{\rho}-\gamma^{\rho}\right)^{1-\delta}}v\left(\gamma\right)d\gamma\\&+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}s\left(x\right)\frac{\rho^{1-\lambda}x^{k}}{\Gamma\left(\delta\right)}\int_{0}^{x}\frac{\gamma^{\rho(\eta+1)-1}}{\left(x^{\rho}-\gamma^{\rho}\right)^{1-\delta}}v\left(\gamma\right)\varphi\left(\gamma\right)\psi\left(\gamma\right)d\gamma\\&\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\varphi\right)(x)\frac{\rho^{1-\lambda}x^{k}}{\Gamma\left(\delta\right)}\int_{0}^{x}\frac{\gamma^{\rho(\eta+1)-1}}{\left(x^{\rho}-\gamma^{\rho}\right)^{1-\delta}}v\left(\gamma\right)\psi\left(\gamma\right)d\gamma\\&+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\psi\right)(x)\frac{\rho^{1-\lambda}x^{k}}{\Gamma\left(\delta\right)}\int_{0}^{x}\frac{\gamma^{\rho(\eta+1)-1}}{\left(x^{\rho}-\gamma^{\rho}\right)^{1-\delta}}v\left(\gamma\right)\varphi\left(\gamma\right)d\gamma\ .\end{aligned}$$

Therefore

$$\begin{aligned}&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\varphi\psi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}v\left(x\right)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}s\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(v\varphi\psi\right)(x)\\&\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\varphi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(v\psi\right)(x)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(s\psi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(v\varphi\right)(x)\ .\end{aligned}$$

Hence the proof. □

Theorem 4.2. Let φ and ψ be two integrable and synchronous functions on $[0,\infty)$ and suppose $f,g,h:[0,\infty)\to[0,\infty)$. Then for all $x>0$, $\alpha>0$, $\delta>0$, $\rho>0$, $k,\beta,\lambda,\eta\in\mathbb{R}$, we have:

$$\begin{aligned}&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h\left(x\right)\left[\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(f\varphi\psi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}g\left(x\right)+2\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(g\varphi\psi\right)(x)\right.\\&\left.+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(f\varphi\psi\right)(x)\right]\\&+\left[\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}g\left(x\right)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}f\left(x\right)\right]\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(h\varphi\psi\right)(x)\\&\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h\left(x\right)\left[\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(f\varphi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(g\psi\right)(x)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(f\psi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(g\varphi\right)(x)\right]\\&+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f\left(x\right)\left[\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(h\varphi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(g\psi\right)(x)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(h\psi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(g\varphi\right)(x)\right]\\&+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g\left(x\right)\left[\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(h\varphi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(f\psi\right)(x)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(h\psi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(f\varphi\right)(x)\right].\end{aligned}\tag{21}$$

Proof. In lemma (4.2), putting $s=f$, $v=g$, we can write

$$\begin{aligned}&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(f\varphi\psi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}g\left(x\right)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(g\varphi\psi\right)(x)\\&\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(f\varphi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(g\psi\right)(x)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(f\psi\right)(x)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(g\varphi\right)(x)\ .\end{aligned}\tag{22}$$

Multiplying both sides of (22) by ${}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h\left(x\right),$ we get

$$\begin{aligned}&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(f\varphi\psi\right)\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}g\left(x\right)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(g\varphi\psi\right)\left(x\right)\\ &\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(f\varphi\right)\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(g\psi\right)\left(x\right)\\ &+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(f\psi\right)\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(g\varphi\right)\left(x\right).\end{aligned}\tag{23}$$

Now, putting $s=h$, $v=g$ in lemma (4.2) and multiplying both sides of the resulting inequality by ${}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f\left(x\right),$ we obtain

$$\begin{aligned}&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(h\varphi\psi\right)\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}g\left(x\right)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(g\varphi\psi\right)\left(x\right)\\ &\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(h\varphi\right)\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(g\psi\right)\left(x\right)\\ &+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}f\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(h\psi\right)\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(g\varphi\right)\left(x\right).\end{aligned}\tag{24}$$

Now, putting $s=h$, $v=f$ in lemma (4.2) and multiplying both sides of the resulting inequality by ${}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g\left(x\right),$ we get

$$\begin{aligned}&{}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(h\varphi\psi\right)\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}f\left(x\right)+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(f\varphi\psi\right)\left(x\right)\\ &\geq\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(h\varphi\right)\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(f\psi\right)\left(x\right)\\ &+\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}g\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(h\psi\right)\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\delta,\lambda}\left(f\varphi\right)\left(x\right).\end{aligned}\tag{25}$$

By adding the inequalities (23), (24), (25) we get the required inequality (21). □

5. Generalized fractional inequality for Chebyshev functional for differentiable functions:

Our results in present section are for Chebyshev functional in case of differentiable functions whose derivatives belong to $L_p\left([0,\infty]\right)$ [3].

First, we give the following lemma

Lemma 5.1. Let h be a positive function on $[0,\infty)$, and let φ,ψ be two differentiable functions on $[0,\infty)$. Then for all $x>0$, $\alpha>0$, $\rho>0$, $k,\beta,\eta\in\mathbb{R}$ we have:

$$\begin{aligned}&\frac{\rho^{2(1-\beta)}x^{2k}}{\Gamma^{2}\left(\alpha\right)}\int_{0}^{x}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{\left(x^{\rho}-\tau^{\rho}\right)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{\left(x^{\rho}-\gamma^{\rho}\right)^{1-\alpha}}h\left(\tau\right)h\left(\gamma\right)H\left(\tau,\gamma\right)d\tau d\gamma\\ &=\left[\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(h\varphi\psi\right)\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}h\left(x\right)-\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(h\psi\right)\left(x\right)\ {}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}\left(h\varphi\right)\left(x\right)\right].\end{aligned}\tag{26}$$

Proof. Define

$$H\left(\tau,\gamma\right)=\left(\varphi\left(\tau\right)-\varphi\left(\gamma\right)\right)\left(\psi\left(\tau\right)-\psi\left(\gamma\right)\right);\ \tau,\gamma\in\left(0,x\right),\ x>0.\tag{27}$$

Multiplying both sides of (27) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}h(\tau)}{(x^\rho-\tau^\rho)^{1-\alpha}}$, where $\tau\in(0,x)$, we get

$$\begin{aligned}\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}h(\tau)H(\tau,\gamma) &= \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}h(\tau)\varphi(\tau)\psi(\tau)\\ &-\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}h(\tau)\varphi(\tau)\psi(\gamma)\\ &-\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}h(\tau)\varphi(\gamma)\psi(\tau)\\ &+\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}h(\tau)\varphi(\gamma)\psi(\gamma).\end{aligned} \tag{28}$$

Now, integrating 28 over $(0,x)$, with respect to the variable τ, we obtain

$$\begin{aligned}&\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}h(\tau)H(\tau,\gamma)\,d\tau\\ &= {}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi\psi)(x)-\psi(\gamma)\ {}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi)(x)\\ &\quad-\varphi(\gamma)\ {}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\psi)(x)+\varphi(\gamma)\psi(\gamma)\ {}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x).\end{aligned} \tag{29}$$

Now, multiplying both sides of (29) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\gamma^{\rho(\eta+1)-1}h(\gamma)}{(x^\rho-\gamma^\rho)^{1-\alpha}}$, where $\gamma\in(0,x)$ and integrating the resulting identity with respect to γ from 0 to x, we get

$$\begin{aligned}&\frac{\rho^{2(1-\beta)}x^{2k}}{\Gamma^2(\alpha)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\alpha}}h(\tau)h(\gamma)H(\tau,\gamma)\,d\tau d\gamma\\ &=2\left[{}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi\psi)(x)\ {}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)-{}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\psi)(x)\ {}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi)(x)\right].\end{aligned}$$

Which is (26) . □

Theorem 5.1. Let h be a positive function on $[0,\infty)$, and let φ,ψ be two differentiable functions on $[0,\infty)$. Suppose that $\varphi'\in L_s([0,\infty))$, $\psi'\in L_v([0,\infty))$, $s>1$, $\frac{1}{s}+\frac{1}{v}=1$. Then for all $x>0$, $\alpha>0$, $\rho>0$, $k,\beta,\eta\in\mathbb{R}$ we have:

$$\begin{aligned}&2\left|{}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi\psi)(x)\ {}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)-{}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\psi)(x)\ {}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi)(x)\right|\\ &\le\frac{\|\varphi'\|_s\|\psi'\|_v\rho^{2(1-\beta)}x^{2k}}{\Gamma^2(\alpha)}\int_0^x\int_0^x\left[\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}h(\tau)\right.\\ &\quad\left.\times\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\alpha}}h(\gamma)|\tau-\gamma|\right]d\tau d\gamma\le\|\varphi'\|_s\|\psi'\|_v\,x\left({}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)\right)^2.\end{aligned} \tag{30}$$

Proof. In lemma (5.1), from the identity (27), we can write

$$H(\tau,\gamma)=\int_\tau^\gamma\int_\tau^\gamma\varphi'(t)\psi'(r)\,dtdr.$$

By applying Holder inequality for double integral, we obtain

$$\begin{aligned} |H(\tau,\gamma)| &\leq \left|\int_{\tau}^{\gamma}\int_{\tau}^{\gamma}|\varphi'(t)|^{s}\,dtdr\right|^{\frac{1}{s}}\left|\int_{\tau}^{\gamma}\int_{\tau}^{\gamma}|\psi'(r)|^{v}\,dtdr\right|^{\frac{1}{v}} \\ &\leq \left(|\tau-\gamma|^{\frac{1}{s}}\left|\int_{\tau}^{\gamma}|\varphi'(t)|^{s}\,dt\right|^{\frac{1}{s}}\right)\left(|\tau-\gamma|^{\frac{1}{v}}\left|\int_{\tau}^{\gamma}|\psi'(r)|^{v}\,dr\right|^{\frac{1}{v}}\right) \\ &\leq |\tau-\gamma|\left|\int_{\tau}^{\gamma}|\varphi'(t)|^{s}\,dt\right|^{\frac{1}{s}}\left|\int_{\tau}^{\gamma}|\psi'(r)|^{v}\,dr\right|^{\frac{1}{v}}. \end{aligned} \tag{31}$$

Using inequality (31) in left-hand side of lemma (5.1), we can write

$$\begin{aligned} &\frac{\rho^{2(1-\beta)}x^{2k}}{\Gamma^{2}(\alpha)}\int_{0}^{x}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^{\rho}-\gamma^{\rho})^{1-\alpha}}h(\tau)\,h(\gamma)\,|H(\tau,\gamma)|\,d\tau d\gamma \\ &\leq \frac{\rho^{2(1-\beta)}x^{2k}}{\Gamma^{2}(\alpha)}\int_{0}^{x}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^{\rho}-\gamma^{\rho})^{1-\alpha}}h(\tau)\,h(\gamma)\,|\tau-\gamma| \\ &\quad\times\left|\int_{\tau}^{\gamma}|\varphi'(t)|^{s}\,dt\right|^{\frac{1}{s}}\left|\int_{\tau}^{\gamma}|\psi'(r)|^{v}\,dr\right|^{\frac{1}{v}}d\tau d\gamma. \end{aligned} \tag{32}$$

Again, applying Holder inequality to the right-hand side of inequality (32), we obtain

$$\begin{aligned} &\frac{\rho^{2(1-\beta)}x^{2k}}{\Gamma^{2}(\alpha)}\int_{0}^{x}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^{\rho}-\gamma^{\rho})^{1-\alpha}}h(\tau)\,h(\gamma)\,|H(\tau,\gamma)|\,d\tau d\gamma \\ &\leq \left[\frac{\rho^{2(1-\beta)}x^{2k}}{\Gamma^{2}(\alpha)}\int_{0}^{x}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^{\rho}-\gamma^{\rho})^{1-\alpha}}\right. \\ &\qquad\left.\times h(\tau)\,h(\gamma)\,|\tau-\gamma|\left|\int_{\tau}^{\gamma}|\varphi'(t)|^{s}\,dt\right|d\tau d\gamma\right]^{\frac{1}{s}} \\ &\times\left[\frac{\rho^{2(1-\beta)}x^{2k}}{\Gamma^{2}(\alpha)}\int_{0}^{x}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^{\rho}-\gamma^{\rho})^{1-\alpha}}\right. \\ &\qquad\left.\times h(\tau)\,h(\gamma)\,|\tau-\gamma|\left|\int_{\tau}^{\gamma}|\psi'(r)|^{v}\,dr\right|d\tau d\gamma\right]^{\frac{1}{v}}. \end{aligned} \tag{33}$$

Since

$$\left|\int_{\tau}^{\gamma}|\varphi'(t)|^{s}\,dt\right| \leq \|\varphi'\|_{s}^{s},$$

$$\left|\int_{\tau}^{\gamma}|\psi'(r)|^{v}\,dr\right| \leq \|\psi'\|_{v}^{v}. \tag{34}$$

Then, from (33), we have

$$\begin{aligned}&\frac{\rho^{2(1-\beta)}x^{2k}}{\Gamma^2(\alpha)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\alpha}}h(\tau)h(\gamma)\left|H(\tau,\gamma)\right|d\tau d\gamma\\&\leq\left[\frac{\|\varphi'\|_s^s\rho^{2(1-\beta)}x^{2k}}{\Gamma^2(\alpha)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\alpha}}h(\tau)h(\gamma)\left|\tau-\gamma\right|d\tau d\gamma\right]^{\frac{1}{s}}\\&\times\left[\frac{\|\psi'\|_v^v\rho^{2(1-\beta)}x^{2k}}{\Gamma^2(\alpha)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\alpha}}h(\tau)h(\gamma)\left|\tau-\gamma\right|d\tau d\gamma\right]^{\frac{1}{v}}.\end{aligned}$$

So we have

$$\begin{aligned}&\frac{\rho^{2(1-\beta)}x^{2k}}{\Gamma^2(\alpha)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\alpha}}h(\tau)h(\gamma)\left|H(\tau,\gamma)\right|d\tau d\gamma\\&\leq\frac{\|\varphi'\|_s\|\psi'\|_v\rho^{2(1-\beta)}x^{2k}}{\Gamma^2(\alpha)}\\&\times\left[\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\alpha}}h(\tau)h(\gamma)\left|\tau-\gamma\right|d\tau d\gamma\right]^{\frac{1}{s}}\\&\times\left[\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\alpha}}h(\tau)h(\gamma)\left|\tau-\gamma\right|d\tau d\gamma\right]^{\frac{1}{v}}.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{\rho^{2(1-\beta)}x^{2k}}{\Gamma^2(\alpha)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\alpha}}h(\tau)h(\gamma)\left|H(\tau,\gamma)\right|d\tau d\gamma\\&\leq\frac{\|\varphi'\|_s\|\psi'\|_v\rho^{2(1-\beta)}x^{2k}}{\Gamma^2(\alpha)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\alpha}}h(\tau)h(\gamma)\left|\tau-\gamma\right|d\tau d\gamma.\end{aligned}\tag{35}$$

Using lemma (5.1) and the inequality (35), with the properties of the modulus, we get

$$\begin{aligned}&2\left|{}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi\psi)(x)\ {}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)-{}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\psi)(x)\ {}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi)(x)\right|\\&\leq\frac{\|\varphi'\|_s\|\psi'\|_v\rho^{2(1-\beta)}x^{2k}}{\Gamma^2(\alpha)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\alpha}}h(\tau)h(\gamma)\left|\tau-\gamma\right|d\tau d\gamma.\end{aligned}\tag{36}$$

Which proves frist part of (30). To prove the second inequality of (30), we have

$$0\leq\tau\leq x,0\leq\gamma\leq x.$$

Then

$$0\leq|\tau-\gamma|\leq x.$$

Hence

$$\frac{\rho^{2(1-\beta)}x^{2k}}{\Gamma^2(\alpha)}\int_0^x\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\alpha}}h(\tau)\,h(\gamma)\,|H(\tau,\gamma)|\,d\tau d\gamma$$
$$\le \frac{\|\varphi'\|_s\,\|\psi'\|_v\,\rho^{2(1-\beta)}x^{1+2k}}{\Gamma^2(\alpha)}\int_0^x\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\alpha}}h(\tau)\,h(\gamma)\,d\tau d\gamma.$$
$$= \|\varphi'\|_s\,\|\psi'\|_v\,x\left({}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)\right)^2. \tag{37}$$

Which proves second inequality of (30). Hence, theorem (5.1) is proved. □

Now, we give the following theorem with different parameters

Theorem 5.2. Let h be a positive function on $[0,\infty)$ and let φ,ψ be two differentiable functions on $[0,\infty)$. Suppose that $\varphi'\in L_s([0,\infty))$, $\psi'\in L_v([0,\infty))$, $s>1$, $\frac{1}{s}+\frac{1}{v}=1$. Then for all $x>0$, $\alpha>0$, $\delta>0$, $\rho>0$, $k,\beta,\lambda,\eta\in\mathbb{R}$, we have:

$$\Big|{}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi\psi)(x)\;{}^\rho\mathcal{I}_{\eta,k}^{\delta,\lambda}h(x) - {}^\rho\mathcal{I}_{\eta,k}^{\delta,\lambda}(h\psi)(x)\;{}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi)(x)$$
$$- {}^\rho\mathcal{I}_{\eta,k}^{\delta,\lambda}(h\varphi)(x)\;{}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\psi)(x) + {}^\rho\mathcal{I}_{\eta,k}^{\delta,\lambda}(h\varphi\psi)(x)\;{}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)\Big|$$
$$\le \frac{\|\varphi'\|_s\,\|\psi'\|_v\,\rho^{2-(\beta-\lambda)}x^{2k}}{\Gamma(\alpha)\Gamma(\delta)}\int_0^x\int_0^x\Bigg[\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}h(\tau)$$
$$\times\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\delta}}h(\gamma)\,|\tau-\gamma|\Bigg]d\tau d\gamma \le \|\varphi'\|_s\,\|\psi'\|_v\,x\left({}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)\;{}^\rho\mathcal{I}_{\eta,k}^{\delta,\lambda}h(x)\right). \tag{38}$$

Proof. In lemma (5.1), multiplying both sides of (29) by $\frac{\rho^{1-\lambda}x^k}{\Gamma(\delta)}\frac{\gamma^{\rho(\eta+1)-1}h(\gamma)}{(x^\rho-\gamma^\rho)^{1-\delta}}$, where $\gamma\in(0,x)$ and integrating the resulting identity with respect to γ from 0 to x, we can write

$$\frac{\rho^{2-(\beta-\lambda)}x^{2k}}{\Gamma(\alpha)\Gamma(\delta)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\delta}}h(\gamma)\,h(\tau)\,H(\tau,\gamma)\,d\tau$$
$$= {}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi\psi)(x)\;{}^\rho\mathcal{I}_{\eta,k}^{\delta,\lambda}h(x) - {}^\rho\mathcal{I}_{\eta,k}^{\delta,\lambda}(h\psi)(x)\;{}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi)(x)$$
$$- {}^\rho\mathcal{I}_{\eta,k}^{\delta,\lambda}(h\varphi)(x)\;{}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\psi)(x) + {}^\rho\mathcal{I}_{\eta,k}^{\delta,\lambda}(h\varphi\psi)(x)\;{}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x). \tag{39}$$

Using inequality (31) in left-hand side of (39), we can write

$$\frac{\rho^{2-(\beta-\lambda)}x^{2k}}{\Gamma(\alpha)\Gamma(\delta)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\delta}}h(\gamma)\,h(\tau)\,|H(\tau,\gamma)|\,d\tau$$
$$\le\frac{\rho^{2-(\beta-\lambda)}x^{2k}}{\Gamma(\alpha)\Gamma(\delta)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\delta}}h(\gamma)\,h(\tau)$$
$$\times|\tau-\gamma|\left|\int_\tau^\gamma|\varphi'(t)|^s\,dt\right|^{\frac{1}{s}}\left|\int_\tau^\gamma|\psi'(r)|^v\,dr\right|^{\frac{1}{v}}d\tau d\gamma. \tag{40}$$

By applying Holder inequality for double integral to above inequality, we obtain

$$\begin{aligned}&\frac{\rho^{2-(\beta-\lambda)}x^{2k}}{\Gamma(\alpha)\Gamma(\delta)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\delta}}h(\tau)h(\gamma)\left|H(\tau,\gamma)\right|d\tau d\gamma\\&\le\left[\frac{\rho^{2-(\beta-\lambda)}x^{2k}}{\Gamma(\alpha)\Gamma(\delta)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\delta}}\right.\\&\qquad\left.\times h(\tau)h(\gamma)\left|\tau-\gamma\right|\left|\int_\tau^\gamma\left|\varphi'(t)\right|^s dt\right|d\tau d\gamma\right]^{\frac{1}{s}}\\&\times\left[\frac{\rho^{2-(\beta-\lambda)}x^{2k}}{\Gamma(\alpha)\Gamma(\delta)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\delta}}\right.\\&\qquad\left.\times h(\tau)h(\gamma)\left|\tau-\gamma\right|\left|\int_\tau^\gamma\left|\psi'(r)\right|^v dr\right|d\tau d\gamma\right]^{\frac{1}{v}}.\end{aligned}\tag{41}$$

Using (34) and (41), we can write

$$\begin{aligned}&\frac{\rho^{2-(\beta-\lambda)}x^{2k}}{\Gamma(\alpha)\Gamma(\delta)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\delta}}h(\tau)h(\gamma)\left|H(\tau,\gamma)\right|d\tau d\gamma\\&\le\frac{\|\varphi'\|_s\|\psi'\|_v\rho^{2-(\beta-\lambda)}x^{2k}}{\Gamma(\alpha)\Gamma(\delta)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\delta}}h(\tau)h(\gamma)\left|\tau-\gamma\right|d\tau d\gamma.\end{aligned}\tag{42}$$

Using (39) and (42), with the properties of the modulus, we get

$$\begin{aligned}&\Big|{}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi\psi)(x)\ {}^\rho\mathcal{I}_{\eta,k}^{\delta,\lambda}h(x)-{}^\rho\mathcal{I}_{\eta,k}^{\delta,\lambda}(h\psi)(x)\ {}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\varphi)(x)\\&-{}^\rho\mathcal{I}_{\eta,k}^{\delta,\lambda}(h\varphi)(x)\ {}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}(h\psi)(x)+{}^\rho\mathcal{I}_{\eta,k}^{\delta,\lambda}(h\varphi\psi)(x)\ {}^\rho\mathcal{I}_{\eta,k}^{\alpha,\beta}h(x)\Big|\\&\le\frac{\|\varphi'\|_s\|\psi'\|_v\rho^{2-(\beta-\lambda)}x^{2k}}{\Gamma(\alpha)\Gamma(\delta)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\gamma^{\rho(\eta+1)-1}}{(x^\rho-\gamma^\rho)^{1-\delta}}h(\tau)h(\gamma)\left|\tau-\gamma\right|d\tau d\gamma.\end{aligned}$$

Which proves first inequality of (15). Second inequality can be proved similarly. □

Remark 5.1. If $k=0$, $\eta=0$ and taking $\rho\to1$ in the theorems (5.1) and (5.2), we get

$$\begin{aligned}&2\left|\mathcal{I}^\alpha h(x)\mathcal{I}^\alpha(h\varphi\psi)(x)-\mathcal{I}^\alpha(h\varphi)(x)\mathcal{I}^\alpha(h\psi)(x)\right|\\&\le\frac{\|\varphi'\|_s\|\psi'\|_v}{\Gamma^2(\alpha)}\int_a^b\int_a^b(x-\tau)^{\alpha-1}(x-\gamma)^{\alpha-1}\left|\tau-\gamma\right|h(\tau)h(\gamma)d\tau d\gamma\\&\le\|\varphi'\|_s\|\psi'\|_v\,x\left(\mathcal{I}^\alpha h(x)\right)^2\end{aligned}$$

and

$$\begin{aligned}&\Big|\mathcal{I}^\alpha(h\varphi\psi)(x)\mathcal{I}^\delta h(x)-\mathcal{I}^\delta(h\psi)(x)\mathcal{I}^\alpha(h\varphi)(x)\\&-\mathcal{I}^\delta(h\varphi)(x)\mathcal{I}^\alpha(h\psi)(x)+\mathcal{I}^\delta(h\varphi\psi)(x)\mathcal{I}^\alpha h(x)\Big|\\&\le\frac{\|\varphi'\|_s\|\psi'\|_v}{\Gamma(\alpha)\Gamma(\delta)}\int_0^x\int_0^x(x-\tau)^{\alpha-1}(x-\gamma)^{\delta-1}h(\tau)h(\gamma)\left|\tau-\gamma\right|d\tau d\gamma\\&\le\|\varphi'\|_s\|\psi'\|_v\,x\left(\mathcal{I}^\alpha h(x)\mathcal{I}^\delta h(x)\right),\end{aligned}$$

respectively, as in (see [4]). Similarly we can get the remaining five cases of generalized fractional integral mentioned at the preliminaries.

References

[1] D. Baleanu and O. G. Mustafa, On the global existence of solutions to a class of fractional differential equations, Comput. Math. Appl. Vol. 59 , no. 5, 1835–1841, 2010.

[2] R.F. Camargo, E.C. Oliveira, Fractional Calculus (in Portuguese), Editora Livraria da Física, São Paulo 2015.

[3] P.L. Chebyshev, Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites, Proc. Math. Soc. Charkov, Vol. 2, 93–98, 1882.

[4] Z. Dahmani, O. Mechouar, S. Brahami, certain inequalities related to the Chebyshev's functional involving a Riemann-liouville operator, Bull. Math. Anal. Appl. Vol. 4, 1821–1291, 2011.

[5] Z. Dahmani, New inequalities in fractional integrals, IJNS, Vol. 9, 493–497, 2010.

[6] Z. Dahmani, On some new fractional inequalities, Inequal, Pure and Appl. Math., Vol. 10, 1821–1291, 2009.

[7] S.S. Dragomir, Some integral inequalities of Gruss type, Indian J. Pure Appl. Math. Vol. 31, 397–415, 2000.

[8] S.S. Dragomir, A generalization of Gruss's inequality in inner product spaces and applications, J. Math. Annal. Appl. Vol. 237, 74–82, 1999.

[9] I. Gavrea, On Chebyshev type inequalities involving functions whose derivatives belog to L_p spaces via isotanic functional. J. Inequal. Pure and Appl. Math. Vol. 7, 121–128, (2006).

[10] R. Gorenflo, F. Mainardi, Fractional calculus integral and differential equations of fractional order, Springer Verlag, 1997.

[11] U.N. Katugampola, A new approach to generalized fractional derivatives, Bull. Math. Anal. Appl. Vol. 6, 1–15, 2014.

[12] U.N. Katugampola, New fractional integral unifying six existing fractional integrals, pp. 6. 2016, arXiv:1612.08596 (eprint).

[13] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of the Fractional Differential Equations, Vol 204, 2006.

[14] D. S. Mitrinovic, Analytic inequalities, Springer Verlag, Berlin, 1970.

[15] S.M. Malamud, Some complements to the Jenson and Chebyshev inequalities and a problem of W. Walter, Proc. Amer. Math. Soc. Vol. 129, 2671–2678, 2001.

[16] S. Marinkovic, P. Rajkovic, M. Stankovic, The inequalities for some types qintegrals, Comput. Math. Appl. Vol. 56, 2490–2498, 2008.

[17] B.G. Pachpatte, A note on Chebyshev-Grüss type inequalities for differential functions, Tamsui Oxf. J. Manag. Sci. Vol. 22, 29–36, 2006.

[18] J. Sousa, D. S. Oliveira, E. Capelas de Oliveira , Grüss-Type Inequalities by Means of Generalized Fractional Integrals, Bull Braz Math Soc. Vol. 2, 1–19, 2019.

[19] H. G. Sun, Y. Zhang, W. Chen, and D. M. Reeves, Use of a variable-index fractional derivative model to capture transient dispersion in heterogeneous media, J. Contaminant Hydrology Vol. 157, 47–58, 2014.

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