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ON RELATIVE DEFECTS OF SPECIAL TYPE OF DIFFERENTIAL POLYNOMIAL IN CONNECTION WITH THEIR INTEGRATED MODULI OF LOGARITHMIC DERIVATIVE

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Abstract. The paper aims at the comparison between the relative Valiron defect and relative Nevanlinna defect of special type differential polynomials from the view point of their integrated moduli of logarithmic derivative. A few examples are provided here to validate the conclusion of the results obtained.

1. Introduction, Definitions and Notations

Let f be a transcendental meromorphic function defined in the open complex plane $\mathbb{C}$. A monomial in f is an expression of the form $M[f] = (f)^{n_0} . \left(f^{(1)}\right)^{n_1} ... \left(f^{(k)}\right)^{n_k}$ where $n_0, n_1, ..., n_k$ are non negative integers. $\gamma_M = n_0 + n_1 + ... + n_k$ and $\Gamma_M =$

$n_0 + 2n_1 + ... + (k+1) n_k$ are respectively called the degree and weight of the monomial.

If $M_1[f], M_2[f], ..., M_n[f]$ denote monomials in f , then $Q[f] = a_1 M_1[f] + a_2 M_2[f] + ... + a_n M_n[f]$, where $a_i \neq 0 (i = 1, 2, ..., n)$ is called a differential polynomial generated by f of degree $\gamma_Q = Max\{\gamma_{M_j} : 1 \leq j \leq n\}$ and weight

$\Gamma_Q = Max\{\Gamma_{M_j} : 1 \leq j \leq n\}$.

Also we call numbers $\underline{\gamma_Q} = \underset{1 \leq j \leq n}{Min} \gamma_{M_j}$ and k (the order of the highest derivative of f) the lower degree and the order of $Q[f]$ respectively. If $\underline{\gamma_Q} = \gamma_Q$, $Q[f]$ is called a homogeneous differential polynomial.

For $a \in \mathbb{C} \cup \infty$, the quantity

$$\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}$$

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is called the Nevanlinna deficiency of the value a. Similarly, the Valiron deficiency $\Delta(a;f)$ of the value a is defined as

$$\Delta(a;f)=1-\liminf_{r\to\infty}\frac{N(r,a;f)}{T(r,f)}=\limsup_{r\to\infty}\frac{m(r,a;f)}{T(r,f)}.$$

Milloux [1] introduced the concept of absolute defect of a with respect to f. Later Xiong [2] extended this definition. He introduced the term

$$\delta_R^{(k)}(a;f)=1-\limsup_{r\to\infty}\frac{N(r,a;f^{(k)})}{T(r,f)},\quad \text{for } k=1,2,3,...$$

and called it the relative Nevanlinna defect of a with respect to $f^{(k)}$.

We may now recall the following definition.

If f be a meromorphic function in the complex plane, then the integrated moduli of the logarithmic derivative $I(r,f)$ is defined by

$$I(r,f)=\frac{r}{\pi}\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|d\theta,$$

for $0<r<\infty$, (cf.[3]) In this paper we call the following four terms by using the concept of $I(r,f)$

$$\delta_I(a;f)=1-\limsup_{r\to\infty}\frac{N(r,a;f)}{I(r,f)},$$

$$\Delta_I(a;f)=1-\liminf_{r\to\infty}\frac{N(r,a;f)}{I(r,f)},$$

$$\delta_I^P(a;f)=1-\limsup_{r\to\infty}\frac{N(r,a;F)}{I(r,f)}$$

and

$$\Delta_I^P(a;f)=1-\liminf_{r\to\infty}\frac{N(r,a;F)}{I(r,f)}.$$

Xiong [2] has shown various relations between the usual defects and relative defects of meromorphic functions. As a continuation of [4], [5] and [6], in the paper we consider $F=f^nQ[f]$, $Q[f]$ being a differential polynomial in f with $n=1,2,3,...$ and compare the relative Valiron defect with the relative Nevanlinna defect of F under the flavour of integrated moduli of logarithmic derivative. The term $S(r,f)$ denotes any quantity satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ through all values of r if f is of finite order and except possibly for a set of r of finite linear measure otherwise. We do not explain the standard definitions and notations of the value distribution and the Nevanlinna theory of entire and meromorphic functions as those are available in [7] and [3].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. *Let k be any positive integer and $\psi=\sum_{i=0}^{k}a_if^{(i)}$, where a_i are meromorphic functions such that $T(r,a_i)=S(r,f)$ for $i=0,1,2,...,k$. Then*

$$m\left(r,\frac{\psi}{f}\right)=S(r,f).$$

Lemma 2.2. *Let $F = f^n Q[f]$ where $Q[f]$ is a differential polynomial in f. If $n \geq 1$, then*

$$\lim_{r\to\infty} \frac{T(r,F)}{T(r,f)} = 1.$$

The proof is omitted.

Lemma 2.3. [6] *Let $F = f^n Q[f]$ where $Q[f]$ is a differential polynomial in f. If $n \geq 1$ then for any α,*

$$\delta_R^F(\alpha; f) = \liminf_{r\to\infty} \frac{m(r,\alpha;F)}{T(r,f)}$$

and

$$\Delta_R^F(\alpha; f) = \limsup_{r\to\infty} \frac{m(r,\alpha;F)}{T(r,f)}.$$

Lemma 2.4. *[[3], p.41] Let f be meromorphic and non-constant in $|z| \leq R_0$. Then*

$$\lim_{r\to R_0} \frac{S(r,f)}{T(r,f)} = 0 \qquad (*)$$

with the following provisions :

(a) $(*)$ holds without restrictions if $R_0 = +\infty$ and f is of finite order in the plane.

(b) If f has infinite order in the plane, $(*)$ still holds as $r \to \infty$ outside a certain exceptional set E of finite length. Here E depends only on f.

(c) If $R_0 < +\infty$ and

$$\limsup_{r\to\infty} \frac{T(r,f)}{\log\{\frac{1}{R_0 - r}\}} = +\infty ,$$

then $(*)$ holds as $r \to R_0$ through a suitable sequence (r_n) depending only on f.

Lemma 2.5. [8] *Let f be an entire function of finite order ρ with no zeros in $\mathbb{C}$. Then*

$$\lim_{r\to\infty} \frac{I(r,f)}{T(r,f)} = \pi\rho.$$

Lemma 2.6. *Let f be a non-constant meromorphic function of finite order in $\mathbb{C}$. Then*

$$\lim_{r\to\infty} \frac{S(r,f)}{I(r,f)} = 0.$$

Proof. In view of Lemma 2.4, we get that

$$\lim_{r\to\infty} \frac{S(r,f)}{T(r,f)} = 0.$$

Now,

$$\begin{aligned} \lim_{r\to\infty} \frac{S(r,f)}{I(r,f)} &= \lim_{r\to\infty} \left\{ \frac{S(r,f)}{T(r,f)} \cdot \frac{T(r,f)}{I(r,f)} \right\} \\ &= \lim_{r\to\infty} \frac{S(r,f)}{T(r,f)} \cdot \lim_{r\to\infty} \frac{T(r,f)}{I(r,f)} = 0. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.7. *Let f be a transcendental entire function of non-zero finite order ρ having no zeros in $\mathbb{C}$. Then*

$$\delta_I(a;f) = \left(1 - \frac{1}{\pi\rho}\right) + \liminf_{r\to\infty} \frac{m(r,a;f)}{I(r,f)}$$

and

$$\Delta_I(a;f) = \left(1 - \frac{1}{\pi\rho}\right) + \limsup_{r\to\infty} \frac{m(r,a;f)}{I(r,f)}.$$

Proof. We know that

$$\begin{aligned}
\delta_I(a;f) &= 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{I(r,f)} \\
&= 1 - \limsup_{r\to\infty} \left\{ \frac{N(r,a;f)}{T(r,f)} \cdot \frac{T(r,f)}{I(r,f)} \right\} \\
&= 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} \cdot \lim_{r\to\infty} \frac{T(r,f)}{I(r,f)} \\
&= 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} \cdot \frac{1}{\pi\rho} \\
&= \frac{1}{\pi\rho} \left\{ 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} \right\} + \left(1 - \frac{1}{\pi\rho}\right) \\
&= \frac{1}{\pi\rho} \left\{ \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)} \right\} + \left(1 - \frac{1}{\pi\rho}\right) \\
&= \frac{1}{\pi\rho} \left\{ \liminf_{r\to\infty} \frac{m(r,a;f)}{I(r,f)} \cdot \frac{I(r,f)}{T(r,f)} \right\} + \left(1 - \frac{1}{\pi\rho}\right) \\
&= \frac{1}{\pi\rho} \left\{ \liminf_{r\to\infty} \frac{m(r,a;f)}{I(r,f)} \cdot \pi\rho \right\} + \left(1 - \frac{1}{\pi\rho}\right) \\
&= \left(1 - \frac{1}{\pi\rho}\right) + \liminf_{r\to\infty} \frac{m(r,a;f)}{I(r,f)}.
\end{aligned}$$

This proves the first part of the lemma.
Similarly, we can prove the second part of the lemma. □

Lemma 2.8. *Let f be an entire function of non-zero finite order ρ having no zeros in $\mathbb{C}$. Also let $F = f^n Q[f]$ where $Q[f]$ is a differential polynomial in f . Then for any α,*

$$\delta_I^F(\alpha;f) = \left(1 - \frac{1}{\pi\rho}\right) + \liminf_{r\to\infty} \frac{m(r,\alpha;F)}{I(r,f)}$$

and

$$\Delta_I^F(\alpha;f) = \left(1 - \frac{1}{\pi\rho}\right) + \limsup_{r\to\infty} \frac{m(r,\alpha;F)}{I(r,f)},$$

where $n \geq 1$.

Proof. We know that

$$\begin{aligned}
\delta_I^F(\alpha; f) &= 1 - \lim_{r\to\infty}\sup \frac{N(r,a;F)}{I(r,f)} \\
&= 1 - \lim_{r\to\infty}\sup \left(\frac{N(r,a;F)}{T(r,f)} \cdot \frac{T(r,f)}{I(r,f)}\right) \\
&= 1 - \lim_{r\to\infty}\sup \frac{N(r,a;F)}{T(r,f)} \cdot \lim_{r\to\infty} \frac{T(r,f)}{I(r,f)} \\
&= 1 - \lim_{r\to\infty}\sup \frac{N(r,a;F)}{T(r,f)} \cdot \frac{1}{\pi\rho} \\
&= \left(1 - \frac{1}{\pi\rho}\right) + \frac{1}{\pi\rho}\left\{1 - \limsup_{r\to\infty} \frac{N(r,\alpha;F)}{T(r,f)}\right\} \\
&= \left(1 - \frac{1}{\pi\rho}\right) + \frac{1}{\pi\rho}\liminf_{r\to\infty} \frac{m(r,\alpha;F)}{T(r,f)} \\
&= \left(1 - \frac{1}{\pi\rho}\right) + \frac{1}{\pi\rho}\liminf_{r\to\infty} \left(\frac{m(r,\alpha;F)}{I(r,f)} \cdot \frac{I(r,f)}{T(r,f)}\right) \\
&= \left(1 - \frac{1}{\pi\rho}\right) + \frac{1}{\pi\rho} \cdot \liminf_{r\to\infty} \frac{m(r,\alpha;F)}{I(r,f)} \cdot \lim_{r\to\infty} \frac{I(r,f)}{T(r,f)} \\
&= \left(1 - \frac{1}{\pi\rho}\right) + \frac{1}{\pi\rho} \cdot \liminf_{r\to\infty} \frac{m(r,\alpha;F)}{I(r,f)} \cdot \pi\rho \\
&= \left(1 - \frac{1}{\pi\rho}\right) + \liminf_{r\to\infty} \frac{m(r,\alpha;F)}{I(r,f)}.
\end{aligned}$$

This completes the proof of the the first part of the lemma.
Similarly, we can prove the second part of the lemma. □

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let f be a transcendental entire function of non-zero finite order ρ having no zeros in $\mathbb{C}$ such that $m(r,f) = S(r,f)$. If a, b and c are any three non zero finite complex numbers then*

$$3\delta_I(a;f) + 2\delta_I(b;f) + \delta_I(c;f) + 5\Delta_I^F(\infty;f) + \frac{1}{\pi\rho} \leq 5\Delta_I(\infty;f) + 5\Delta_I^F(0;f) + 1,$$

where F is a differential polynomial in f of the form $F = f^n Q[f]$ with $n \geq 1$.

Proof. Let us consider the following identity

$$\frac{b-a}{f-a} = \left[\frac{F}{f-a}\left\{\frac{f-a}{F} - \frac{f-b}{F}\right\} - \frac{f-c}{F} \cdot \frac{F}{f} \cdot \frac{F}{f-a} \cdot \left\{\frac{f-a}{F} - \frac{f-b}{F}\right\}\right] \cdot \frac{f}{c}.$$

Since $m\left(r, \frac{1}{f-a}\right) \leq m\left(r, \frac{b-a}{f-a}\right) + O(1)$ and $m\left(r, \frac{f}{c}\right) \leq m(r, f) + O(1)$, we get from the above identity in view of Lemma 2.1 that

$$\begin{aligned} m\left(r, \frac{b-a}{f-a}\right) &\leq m\left(r, \frac{f-a}{F}\right) + m\left(r, \frac{f-b}{F}\right) + m\left(r, \frac{f-c}{F}\right) \\ &+ m\left(r, \frac{f-a}{F}\right) + m\left(r, \frac{f-b}{F}\right) + m\left(r, \frac{f}{c}\right) \\ &+ S(r, f) \end{aligned}$$

$$\begin{aligned} i.e.,\ m\left(r, \frac{1}{f-a}\right) &\leq 2m\left(r, \frac{f-a}{F}\right) + 2m\left(r, \frac{f-b}{F}\right) + m\left(r, \frac{f-c}{F}\right) \\ &+ m(r, f) + S(r, f) + O(1) \end{aligned}$$

$$\begin{aligned} i.e.,\ m\left(r, \frac{1}{f-a}\right) &\leq 2T\left(r, \frac{f-a}{F}\right) - 2N\left(r, \frac{f-a}{F}\right) + 2T\left(r, \frac{f-b}{F}\right) \\ &- 2N\left(r, \frac{f-b}{F}\right) + T\left(r, \frac{f-c}{F}\right) - N\left(r, \frac{f-c}{F}\right) \\ &+ m(r, f) + S(r, f) + O(1). \end{aligned} \tag{1}$$

Now by the relation $T\left(r, \frac{1}{f}\right) = T(r, f) + O(1)$ and in view of Lemma 2.1, it follows from (1) that

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) &\leq 2T\left(r, \frac{F}{f-a}\right) - 2N\left(r, \frac{f-a}{F}\right) + 2T\left(r, \frac{F}{f-b}\right) \\ &- 2N\left(r, \frac{f-b}{F}\right) + T\left(r, \frac{F}{f-c}\right) - N\left(r, \frac{f-c}{F}\right) \\ &+ m(r, f) + S(r, f) + O(1) \end{aligned}$$

$$\begin{aligned} i.e.,\ m\left(r, \frac{1}{f-a}\right) &\leq 2\left\{N\left(r, \frac{F}{f-a}\right) - N\left(r, \frac{f-a}{F}\right)\right\} \\ &+ 2\left\{N\left(r, \frac{F}{f-b}\right) - N\left(r, \frac{f-b}{F}\right)\right\} \\ &+ N\left(r, \frac{F}{f-c}\right) - N\left(r, \frac{f-c}{F}\right) \\ &+ m(r, f) + S(r, f) + O(1). \end{aligned} \tag{2}$$

In view of [[3], p. 34] , we obtain from (2) that

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) &\leq 2N(r, F) + 2N\left(r, \frac{1}{f-a}\right) - 2N(r, f) - 2N\left(r, \frac{1}{F}\right) \\ &+ 2N(r, F) + 2N\left(r, \frac{1}{f-b}\right) - 2N(r, f) - 2N\left(r, \frac{1}{F}\right) \\ &+ N(r, F) + N\left(r, \frac{1}{f-c}\right) - N(r, f) - N\left(r, \frac{1}{F}\right) \\ &+ m(r, f) + S(r, f) + O(1) \end{aligned}$$

$$\begin{aligned} i.e., \; m\left(r, \frac{1}{f-a}\right) \leq\; & 5N\left(r, F\right) - 5N\left(r, f\right) - 5N\left(r, \frac{1}{F}\right) \\ & +2N\left(r, \frac{1}{f-a}\right) + 2N\left(r, \frac{1}{f-b}\right) + N\left(r, \frac{1}{f-c}\right) \\ & +m\left(r, f\right) + S\left(r, f\right) + O\left(1\right). \end{aligned} \tag{3}$$

In view of Lemma 2.6, Lemma 2.7 and $m\left(r, f\right) = S\left(r, f\right)$, it follows from (3) that

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) \leq\; & 5N\left(r, F\right) - 5N\left(r, f\right) - 5N\left(r, \frac{1}{F}\right) \\ & +2N\left(r, \frac{1}{f-a}\right) + 2N\left(r, \frac{1}{f-b}\right) + N\left(r, \frac{1}{f-c}\right) \\ & +S\left(r, f\right) \end{aligned}$$

$$\begin{aligned} i.e., \; \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{I\left(r, f\right)} \leq\; & 5\liminf_{r\to\infty} \left\{ \frac{N\left(r, F\right)}{I\left(r, f\right)} - \frac{N\left(r, f\right)}{I\left(r, f\right)} - \frac{N\left(r, \frac{1}{F}\right)}{I\left(r, f\right)} \right\} \\ & +2\limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{I\left(r, f\right)} + 2\limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-b}\right)}{I\left(r, f\right)} \\ & +\limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-c}\right)}{I\left(r, f\right)} \end{aligned}$$

$$\begin{aligned} i.e., \; \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{I\left(r, f\right)} \leq\; & 5\left\{ \liminf_{r\to\infty} \frac{N\left(r, F\right)}{I\left(r, f\right)} - \liminf_{r\to\infty} \frac{N\left(r, f\right)}{I\left(r, f\right)} - \liminf_{r\to\infty} \frac{N\left(r, \frac{1}{F}\right)}{I\left(r, f\right)} \right\} \\ & +2\limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{I\left(r, f\right)} + 2\limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-b}\right)}{I\left(r, f\right)} \\ & +\limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-c}\right)}{I\left(r, f\right)} \end{aligned}$$

$$\begin{aligned} i.e., \; \delta_I\left(a; f\right) - \left(1 - \frac{1}{\pi\rho}\right) \leq\; & 5\left\{1 - \Delta_I^F\left(\infty; f\right)\right\} - 5\left\{1 - \Delta_I\left(\infty; f\right)\right\} \\ & -5\left\{1 - \Delta_I^F\left(0; f\right)\right\} + 2\left\{1 - \delta_I\left(a; f\right)\right\} \\ & +2\left\{1 - \delta_I\left(b; f\right)\right\} + 2\left\{1 - \delta_I\left(c; f\right)\right\} \end{aligned}$$

$$i.e., \; 3\delta_I\left(a; f\right) + 2\delta_I\left(b; f\right) + \delta_I\left(c; f\right) + 5\Delta_I^F\left(\infty; f\right) + \frac{1}{\pi\rho} \leq 5\Delta_I\left(\infty; f\right) + 5\Delta_I^F\left(0; f\right) + 1.$$

This proves the theorem. □

Remark 3.1. *The condition "a, b and c are non-zero finite complex numbers in Theorem 3.1" is essential as is evident from the following examples.*

Example 1. *Let $f(z) = \exp z$ and $a = b = c = 0$. Then we see that $N(r, f) = 0$,*

$$\begin{aligned} T(r,f) &= N(r,f) + m(r,f) = \frac{1}{2\Pi}\int_0^{2\Pi} \log^+ \left|f(re^{i\theta})\right| d\theta = \frac{1}{2\Pi}\int_0^{2\Pi} \log^+ \left|e^{re^{i\theta}}\right| \\ &= \frac{1}{2\Pi}\int_0^{2\Pi} \log^+ (e^{r\cos\theta}) d\theta = \frac{1}{2\Pi}\int_{-\frac{\Pi}{2}}^{\frac{\Pi}{2}} r\cos\theta d\theta = \frac{r}{\Pi}, \end{aligned}$$

$$\begin{aligned} I(r,f) &= \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{e^{re^{i\theta}}\cdot re^{i\theta}\cdot i}{e^{re^{i\theta}}}\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi}\left|re^{i\theta}\cdot i\right| d\theta \\ &= \frac{r}{2\pi}\int_0^{2\pi}(r)\, d\theta = \frac{r^2}{2\pi}\int_0^{2\pi} d\theta = \frac{r^2}{2\pi}\cdot 2\pi = r^2 \neq 0, \end{aligned}$$

$$\rho = \limsup_{r\to\infty}\frac{\log T(r;f)}{\log r} = \limsup_{r\to\infty}\frac{\log\frac{r}{\pi}}{\log r} = 1$$

and

$$\delta_I(0;f) = \Delta_I(\infty;f) = 1.$$

Let us consider $n = 1$, $n_0 = 0$, $n_1 = ... = n_k = 0$ and $a_1 = 1$. So, $F = \exp z = f$ and

$$\Delta_I^F(\infty;f) = \Delta_I^F(0;f) = 1.$$

Thus,

$$3\delta_I(a;f) + 2\delta_I(b;f) + \delta_I(c;f) + 5\Delta_I^F(\infty;f) + \frac{1}{\pi\rho} = 3 + 2 + 1 + 5 + \frac{1}{\pi} = 11 + \frac{1}{\pi}$$

and

$$5\Delta_I(\infty;f) + 5\Delta_I^F(0;f) + 1 = 5 + 5 + 1 = 11,$$

which is contrary to Theorem 3.1.

Example 2. *Let $f(z) = \exp z$ and $a = b = c = \infty$. Then we see that $N(r, f) = 0$, $T(r, f) = \frac{r}{\Pi} \neq 0$, $I(r, f) = r^2 \neq 0$ and $\rho = 1$.*

So,

$$\delta_I(\infty;f) = \Delta_I(\infty;f) = 1.$$

Let us consider $n = 1$, $n_0 = 0$, $n_1 = ... = n_k = 0$ and $a_1 = 1$. Then we see that $F = \exp z = f$ and

$$\Delta_I^F(\infty;f) = \Delta_I^F(0;f) = 1.$$

Therefore,

$$3\delta_I(a;f) + 2\delta_I(b;f) + \delta_I(c;f) + 5\Delta_I^F(\infty;f) + \frac{1}{\pi\rho} = 3 + 2 + 1 + 5 + \frac{1}{\pi} = 11 + \frac{1}{\pi}$$

and

$$5\Delta_I(\infty;f) + 5\Delta_I^F(0;f) + 1 = 5 + 5 + 1 = 11,$$

which contradicts Theorem 3.1.

Remark 3.2. *The condition "f has no zeros in $\mathbb{C}$ in Theorem 3.1" is essential as is evident from the following example.*

Example 3. *Let $f = \exp(3z)$. Then we see that $N(r,f) = 0$,*

$$\begin{aligned} I(r,f) &= \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|d\theta = \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{e^{3re^{i\theta}}\cdot 3re^{i\theta}\cdot i}{e^{3re^{i\theta}}}\right|d\theta = \frac{3r}{2\pi}\int_0^{2\pi}\left|re^{i\theta}\cdot i\right|d\theta \\ &= \frac{3r}{2\pi}\int_0^{2\pi}(r)\,d\theta = \frac{3r^2}{2\pi}\int_0^{2\pi}d\theta = \frac{3r^2}{2\pi}\cdot 2\pi = 3r^2 \neq 0 \end{aligned}$$

and

$$\rho = \limsup_{r\to\infty}\frac{\log^{[2]} M(r;f)}{\log r} = \limsup_{r\to\infty}\frac{\log^{[2]} e^{3r}}{\log r} = 1.$$

.Now

$$\delta_I(a;f) = \delta_I(b;f) = \delta_I(c;f) = \Delta_I(\infty;f) = 1.$$

Let us consider $n = 1$, $n_0 = 0$, $n_1 = ... = n_k = 0$ and $a_1 = 1$. Then we get that $F = \exp(3z) = f$ and

$$\Delta_I^F(\infty;f) = \Delta_I^F(0;f) = 1.$$

Hence,

$$3\delta_I(a;f) + 2\delta_I(b;f) + \delta_I(c;f) + 5\Delta_I^F(\infty;f) + \frac{1}{\pi\rho} = 3+2+1+5+\frac{1}{\pi} = 11 + \frac{1}{\pi}$$

and

$$5\Delta_I(\infty;f) + 5\Delta_I^F(0;f) + 1 = 5+5+1 = 11.$$

So, we arrive at a contradiction.

Theorem 3.2. *Let f be an entire function of non-zero finite order ρ with no zeros in $\mathbb{C}$ such that $m(r,f) = S(r,f)$. Also let $F = f^n Q[f]$ is a differential polynomial in f with $n \geq 1$. If a, b, c and d are any four distinct complex numbers then*

$$\delta_I(d;f) + \delta_I^F(b;f) + \delta_I^F(c;f) + \frac{1}{\pi\rho} \leq 3.$$

Proof. Let us consider the following identity

$$\frac{1}{f-d} = \left[\frac{1}{a}\left\{\frac{F}{f-a} - \frac{F-a}{f^n}\cdot\frac{f^n}{f-a}\right\}\cdot\left\{\frac{F}{f-d}\cdot\frac{1}{F}\right\}\right]\cdot(f-a).$$

Since $m(r,f-a) \leq m(r,f) + O(1)$, we get from the above identity in view of Lemma 2.1 that

$$m\left(r,\frac{1}{f-d}\right) \leq m\left(r,\frac{1}{F}\right) + m(r,f) + S(r,f) + O(1)$$

$$\text{i.e., } m\left(r,\frac{1}{f-d}\right) \leq T\left(r,\frac{1}{F}\right) - N\left(r,\frac{1}{F}\right) + m(r,f) + S(r,f) + O(1). \quad (4)$$

Now by the relation $T(r,F) = T\left(r,\frac{1}{F}\right) + O(1)$, we obtain from (4) that

$$m\left(r,\frac{1}{f-d}\right) \leq T(r,F) - N\left(r,\frac{1}{F}\right) + m(r,f) + S(r,f) + O(1). \quad (5)$$

Now by Nevanlinna's second fundamental theorem, it follows from (5) that

$$\begin{aligned} m\left(r,\frac{1}{f-d}\right) &\leq \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-b}\right) + \overline{N}\left(r,\frac{1}{F-c}\right) - N\left(r,\frac{1}{F}\right) \\ &\quad + m(r,f) + S(r,f) + O(1). \end{aligned} \quad (6)$$

In view of Lemma 2.2, Lemma 2.6, Lemma 2.7, $\overline{N}\left(r,\frac{1}{F}\right)-N\left(r,\frac{1}{F}\right)\leq 0$ and $m\left(r,f\right)=S\left(r,f\right)$, we get from (6) that

$$m\left(r,\frac{1}{f-d}\right)\leq\overline{N}\left(r,\frac{1}{F-b}\right)+\overline{N}\left(r,\frac{1}{F-c}\right)+S\left(r,f\right)$$

$$i.e.,\ m\left(r,\frac{1}{f-d}\right)\leq N\left(r,\frac{1}{F-b}\right)+N\left(r,\frac{1}{F-c}\right)+S\left(r,f\right)$$

$$i.e.,\ m\left(r,\frac{1}{f-d}\right)\leq T\left(r,\frac{1}{F-b}\right)-m\left(r,\frac{1}{F-b}\right)+T\left(r,\frac{1}{F-c}\right)-m\left(r,\frac{1}{F-c}\right)+S\left(r,f\right)$$

$$i.e.,\ m\left(r,\frac{1}{f-d}\right)\leq 2T\left(r,F\right)-m\left(r,\frac{1}{F-b}\right)-m\left(r,\frac{1}{F-c}\right)+S\left(r,f\right)$$

$$i.e.,\ \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-d}\right)}{I\left(r,f\right)}\leq 2\liminf_{r\to\infty}\frac{T\left(r,F\right)}{I\left(r,f\right)}-\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{F-b}\right)}{I\left(r,f\right)}-\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{F-c}\right)}{I\left(r,f\right)}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-d}\right)}{I\left(r,f\right)}\leq 2\liminf_{r\to\infty}\left(\frac{T\left(r,F\right)}{T\left(r,f\right)}\cdot\frac{T\left(r,f\right)}{I\left(r,f\right)}\right)-\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{F-b}\right)}{I\left(r,f\right)}-\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{F-c}\right)}{I\left(r,f\right)}$$

$$i.e.,\ \delta_I\left(d;f\right)-\left(1-\frac{1}{\pi\rho}\right)\leq\frac{2}{\pi\rho}-\delta_I^F\left(b;f\right)+\left(1-\frac{1}{\pi\rho}\right)-\delta_I^F\left(c;f\right)+\left(1-\frac{1}{\pi\rho}\right)$$

$$i.e.,\ \delta_I\left(d;f\right)+\delta_I^F\left(b;f\right)+\delta_I^F\left(c;f\right)+\frac{1}{\pi\rho}\leq 3.$$

This proves the theorem. □

Remark 3.3. *The condition "b, c and d are non-zero finite complex numbers in Theorem 3.2" is essential as we see in the following examples.*

Example 4. *Let $f=\exp\left(2z\right)$ and $b=c=d=0$. Then $N(r,f)=0$,*

$$\begin{aligned}I\left(r,f\right) &= \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right|d\theta=\frac{r}{2\pi}\int_0^{2\pi}\left|\frac{e^{2re^{i\theta}}\cdot 2re^{i\theta}\cdot i}{e^{2re^{i\theta}}}\right|d\theta=\frac{r}{\pi}\int_0^{2\pi}\left|re^{i\theta}\cdot i\right|d\theta\\ &= \frac{r}{\pi}\int_0^{2\pi}\left(r\right)d\theta=\frac{r^2}{\pi}\int_0^{2\pi}d\theta=\frac{r^2}{\pi}\cdot 2\pi=2r^2\neq 0,\end{aligned}$$

$$\rho=\limsup_{r\to\infty}\frac{\log^{[2]}M\left(r;f\right)}{\log r}=\limsup_{r\to\infty}\frac{\log^{[2]}e^{2r}}{\log r}=1$$

.and

$$\delta_I\left(d;f\right)=\delta_I\left(0;f\right)=1.$$

Let us consider $n=1$, $n_0=0$, $n_1=...=n_k=0$ and $a_1=1$. Then we get that $F=\exp\left(2z\right)=f$ and

$$\delta_I^F\left(b;f\right)=\delta_I^F\left(c;f\right)=\delta_I^F\left(0;f\right)=1.$$

Hence,

$$\delta_I\left(d;f\right)+\delta_I^F\left(b;f\right)+\delta_I^F\left(c;f\right)+\frac{1}{\pi\rho}=1+1+1+\frac{1}{\pi}=3+\frac{1}{\pi}>3,$$

which is contrary to Theorem 3.2.

Example 5. *Let* $f = \exp(2z)$ *and* $b = c = d = \infty$. *Then* $N(r, f) = 0$, $I(r, f) = 2r^2 \neq 0$, $\rho = 1$ *and*

$$\delta_I(d; f) = \delta_I(\infty; f) = 1.$$

Let us consider $n = 1$, $n_0 = 0$, $n_1 = \ldots = n_k = 0$ *and* $a_1 = 1$. *Then we get that* $F = \exp(2z) = f$ *and*

$$\delta_I^F(b; f) = \delta_I^F(c; f) = \delta_I^F(\infty; f) = 1.$$

Hence,

$$\delta_I(d; f) + \delta_I^F(b; f) + \delta_I^F(c; f) + \frac{1}{\pi\rho} = 1 + 1 + 1 + \frac{1}{\pi} = 3 + \frac{1}{\pi} > 3,$$

which contradicts Theorem 3.2.

Remark 3.4. *The condition that "f has no zeros in* $\mathbb{C}$ *in Theorem 3.2" is essential and is evident from the following example.*

Example 6. *Let* $f = \exp(3z)$. *Then we see that* $N(r, f) = 0$, $I(r, f) = 3r^2 \neq 0$, $\rho = 1$ *and*

$$\delta_I(d; f) = 1.$$

Let us consider $n = 1$, $n_0 = 0$, $n_1 = \ldots = n_k = 0$ *and* $a_1 = 1$. *Then we see that* $F = \exp(3z) = f$ *and*

$$\delta_I^F(b; f) = \delta_I^F(c; f) = 1.$$

Hence,

$$\delta_I(d; f) + \delta_I^F(b; f) + \delta_I^F(c; f) + \frac{1}{\pi\rho} = 1 + 1 + 1 + \frac{1}{\pi} = 3 + \frac{1}{\pi} > 3.$$

So, we arrive at a contradiction.

Theorem 3.3. *Let* f *be a transcendental entire function of finite order* ρ *having no zeros in* $\mathbb{C}$ *such that* $m(r, f) = S(r, f)$. *If* a & c *are any two distinct complex numbers and let* $F = f^n Q[f]$ *is a polynomial in* f *with* $n \geq 1$ *then*

$$\delta_I(0; f) + \delta_I(c; f) + \Delta_I^F(\infty; f) \leq \Delta_I(\infty; f) + 2\Delta_I^F(0; f).$$

Proof. Let us consider the following identity

$$\frac{c}{f} = \left[\left\{1 - \frac{f-c}{F} \cdot \frac{F}{f}\right\} \cdot \left\{\frac{F}{f-a} \cdot \frac{1}{F}\right\}\right] \cdot (f - a).$$

Since $m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{c}{f}\right) + O(1)$ and $m(r, f - a) \leq m(r, f) + O(1)$, we get from the above identity in view of Lemma 2.1 that

$$m\left(r, \frac{c}{f}\right) \leq m\left(r, \frac{f-c}{F}\right) + m\left(r, \frac{1}{F}\right) + m(r, f) + S(r, f) + O(1)$$

$$\begin{aligned} i.e.,\ m\left(r, \frac{1}{f}\right) \leq\ & T\left(r, \frac{f-c}{F}\right) - N\left(r, \frac{f-c}{F}\right) + T\left(r, \frac{1}{F}\right) - N\left(r, \frac{1}{F}\right) \\ & + m(r, f) + S(r, f) + O(1). \end{aligned} \tag{7}$$

Now by Nevanlinna's first fundamental theorem and in view of Lemma 2.1, it follows from (7) that

$$\begin{aligned} m\left(r,\frac{1}{f}\right) &\leq T\left(r,\frac{F}{f-c}\right)-N\left(r,\frac{f-c}{F}\right)+T\left(r,F\right)-N\left(r,\frac{1}{F}\right) \\ &\quad +m\left(r,f\right)+S\left(r,f\right)+O\left(1\right) \end{aligned}$$

$$\begin{aligned} i.e.,\ m\left(r,\frac{1}{f}\right) &\leq N\left(r,\frac{F}{f-c}\right)-N\left(r,\frac{f-c}{F}\right)-N\left(r,\frac{1}{F}\right)+T\left(r,F\right) \\ &\quad +m\left(r,f\right)+S\left(r,f\right)+O\left(1\right). \end{aligned} \tag{8}$$

In view of Lemma 2.2, Lemma 2.6, Lemma 2.7 and [[3], p. 34] , we obtain from (8) that

$$\begin{aligned} m\left(r,\frac{1}{f}\right) &\leq N\left(r,F\right)+N\left(r,\frac{1}{f-c}\right)-N\left(r,f-c\right)-N\left(r,\frac{1}{F}\right)-N\left(r,\frac{1}{F}\right)+T\left(r,F\right) \\ &\quad +m\left(r,f\right)+S\left(r,f\right)+O\left(1\right) \end{aligned}$$

$$\begin{aligned} i.e., \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f}\right)}{I\left(r,f\right)} &\leq \liminf_{r\to\infty}\frac{N\left(r,F\right)}{I\left(r,f\right)}-\liminf_{r\to\infty}\frac{N\left(r,f\right)}{I\left(r,f\right)}-2\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{F}\right)}{I\left(r,f\right)} \\ &\quad +\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-c}\right)}{I\left(r,f\right)}+\limsup_{r\to\infty}\frac{T\left(r,F\right)}{I\left(r,f\right)} \end{aligned}$$

$$\begin{aligned} i.e., \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f}\right)}{I\left(r,f\right)} &\leq \liminf_{r\to\infty}\frac{N\left(r,F\right)}{I\left(r,f\right)}-\liminf_{r\to\infty}\frac{N\left(r,f\right)}{I\left(r,f\right)}-2\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{F}\right)}{I\left(r,f\right)} \\ &\quad +\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-c}\right)}{I\left(r,f\right)}+\limsup_{r\to\infty}\left(\frac{T\left(r,F\right)}{T\left(r,f\right)}\cdot\frac{T\left(r,f\right)}{I\left(r,f\right)}\right) \end{aligned}$$

$$i.e.,\ \delta_I\left(0;f\right)-\left(1-\frac{1}{\pi\rho}\right)\leq\left\{1-\Delta_I^F\left(\infty;f\right)\right\}-\left\{1-\Delta_I\left(\infty;f\right)\right\}-2\left\{1-\Delta_I^F\left(0;f\right)\right\}+\left\{1-\delta_I\left(c;f\right)\right\}+\frac{1}{\pi\rho}$$

$$i.e.,\ \delta_I\left(0;f\right)+\delta_I\left(c;f\right)+\Delta_I^F\left(\infty;f\right)\leq\Delta_I\left(\infty;f\right)+2\Delta_I^F\left(0;f\right).$$

Thus the theorem is established. □

Remark 3.5. *The sign* $\leq$ *in Theorem* 3.3 *can not be replaced by the sign* $<$ *as was seen in the following example.*

Example 7. *Let* $f\left(z\right)=\exp z$ *and* $n=1$. *Then* $N(r,f)=0$, $T(r,f)=\frac{r}{\Pi}\neq 0$, $I\left(r,f\right)=r^2\neq 0$ *and* $\rho=1$.

So,

$$\delta_I\left(0;f\right)=\delta_I\left(c;f\right)=\Delta_I\left(\infty;f\right)=1.$$

Let us consider $n=1$, $n_0=0$, $n_1=...=n_k=0$ *and* $a_1=1$. *Then we see that* $F=\exp z=f$ *and*

$$\Delta_I^F\left(\infty;f\right)=\Delta_I^F\left(0;f\right)=1.$$

Therefore,

$$\delta_I\left(0;f\right)+\delta_I\left(c;f\right)+\Delta_I^F\left(\infty;f\right)=1+1+1=3$$

and

$$\Delta_I\left(\infty;f\right)+2\Delta_I^F\left(0;f\right)=1+2=3.$$

Hence,

$$\delta_I(0;f)+\delta_I(c;f)+\Delta_I^F(\infty;f)=3=\Delta_I(\infty;f)+2\Delta_I^F(0;f).$$

4. Future Prospect

In the line of the works as carried out in the paper one may think of finding out relative deficiencies of higher index in case of meromorphic functions with respect to another one on the basis of sharing of values of them and this treatment can be done under the flavour of bicomplex analysis. As a consequence, the derivation of relevant results is still open to the future workers of this branch.

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