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FEKETE-SZEGÖ INEQUALITY FOR CERTAIN CLASSES OF CLOSE-TO-CONVEX FUNCTIONS

GAGANDEEP SINGH, GURCHARANJIT SINGH, HARJINDER SINGH

ABSTRACT. Close-to-convex functions and quasi-convex functions are of great importance in geometric function theory. In the present investigation, the authors study the subclass C_1 of close-to-convex functions and the subclasses C' and C'_1 of quasi convex functions in the open unit disc $E = \{z : |z| < 1\}$. The sharp upper bounds of the functional $|a_3 - \mu a_2^2|$, μ real, for the functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belonging to these classes are provided. This work will pave the way to investigate the upper bound of the Fekete-Szegö functional for some other subclasses of close-to-convex and quasi-convex functions.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. Let S be the class of functions of the form (1) which are analytic univalent in E.

We shall concentrate on the coefficient problem for the class S and certain of its subclasses. In 1916, Bieberbach [3] proved that $|a_2| \leq 2$ for $f(z) \in S$ as a corollary to an elementary area theorem. He conjectured that, for each function $f(z) \in S$, $|a_n| \leq n$; equality holds for the Koebe function $k(z) = z/(1-z)^2$, which maps the unit disc E onto the entire complex plane minus the slit along the negative real axis from $-\frac{1}{4}$ to $-\infty$. De Branges [5] solved the Bieberbach conjecture in 1984. The contribution of Löwner [10] in proving that $|a_3| \leq 3$ for the class S was huge.

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and a_2^2 for the class S. This thought prompted Fekete and Szegö [6] and they used Löwner's method to prove the following well-known result for the class S:

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If $f(z) \in S$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu & \text{if } \mu \leq 0, \\ 1 + 2\exp\left(\dfrac{-2\mu}{1-\mu}\right) & \text{if } 0 \leq \mu \leq 1, \\ 4\mu - 3 & \text{if } \mu \geq 1. \end{cases} \tag{2}$$

The inequality (2) plays a very important role in determining estimates of higher coefficients for some subclasses of S (see Chichra [4], Babalola [2]).

Next, we define some subclasses of S and obtain analogous of (2).

We denote by S^* the class of univalent starlike functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A$ and satisfying the condition

$$\Re\left(\frac{zg'(z)}{g(z)}\right) > 0, \ z \in E. \tag{3}$$

We denote by K the class of convex univalent functions $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in A$ which satisfy the condition

$$\Re\left(\frac{(zh'(z))'}{h'(z)}\right) > 0, \ z \in E. \tag{4}$$

A function $f(z) \in A$ is said to be close to convex if there exists a function $g(z) \in S^*$ such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \ z \in E. \tag{5}$$

The class of close to convex functions is denoted by C and was introduced by Kaplan [8], who showed that all close to convex functions are univalent.The immediate shoot of C are its following subclasses:

$$C_1 = \left\{ f(z) \in A : \Re\left(\frac{zf'(z)}{h(z)}\right) > 0, \ h(z) \in K, \ z \in E \right\}, \tag{6}$$

$$C' = \left\{ f(z) \in A : \Re\left(\frac{(zf'(z))'}{g'(z)}\right) > 0, \ g(z) \in S^*, \ z \in E \right\}, \tag{7}$$

$$C_1' = \left\{ f(z) \in A : \Re\left(\frac{(zf'(z))'}{h'(z)}\right) > 0, \ h(z) \in K, \ z \in E \right\}. \tag{8}$$

Some specific examples for the functions belonging to the classes C, C_1, C' and C_1' are

$f(z) = \frac{z}{(1-z)^2}$,

$f_1(z) = \frac{3}{16\sqrt{2}}\left[\left(1 + \frac{10\sqrt{2}}{3}z\right)^{\frac{8}{5}} - 1\right]$,

$f_2(z) = \int_0^z \frac{3\sqrt{5}}{44z}\left[\left(1 + \frac{29}{3\sqrt{5}}z\right)^{\frac{44}{29}} - 1\right] dz$

and

$f_3(z) = \int_0^z \frac{3\sqrt{3}}{28z}\left[\left(1 + \frac{19}{3\sqrt{3}}z\right)^{\frac{28}{19}} - 1\right] dz$ respectively.

Abdel Gawad and Thomas [1] investigated the class C_1 and also obtained (2) for $-\infty < \mu \leq 1$ (although this result seems to be doubtful).

Let U be the class of analytic bounded functions of the form

$$w(z) = \sum_{n=1}^{\infty} d_n z^n, \; z \in E, \tag{9}$$

and satisfying the conditions $w(0) = 0, |w(z)| < 1$. It is known (see [11]) that

$$|d_1| \leq 1, \; |d_2| \leq 1 - |d_1|^2. \tag{10}$$

We shall apply the subordination principle due to Rogosinski [12], which states that if $f(z) \prec F(z)$, then $f(z) = F(w(z))$, $w(z) \in U$ (where $\prec$ stands for subordination).

Hummel [7] proved a conjecture of V. Singh that $|c_3 - c_2^2| \leq \frac{1}{3}$ for the class K. Keogh and Merkes [9] obtained the estimates (2) for the classes S^*, K and C. Estimates (2) for the classes C_1, $C^{'}$ and $C_1^{'}$ have been waiting to be determined for the last 60 years.

Lemma 1 Let $g(z) \in S^*$. Then

$$|b_3 - \frac{3\mu}{4} b_2^2| \leq \begin{cases} 3(1-\mu) & \text{if } \mu \leq \frac{2}{3}, \\ 1 & \text{if } \frac{2}{3} \leq \mu \leq \frac{4}{3}, \\ 3(\mu - 1) & \text{if } \mu \geq \frac{4}{3}. \end{cases}$$

This lemma is a direct consequence of the result of Keogh and Merkes [9] which states that for $g(z) \in S^*$,

$$|b_3 - \mu b_2^2| \leq \begin{cases} 3 - 4\mu & \text{if } \mu \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq \mu \leq 1, \\ 4\mu - 3 & \text{if } \mu \geq 1. \end{cases}$$

Lemma 2 Let $h(z) \in K$. Then

$$|c_3 - \frac{3\mu}{4} c_2^2| \leq \begin{cases} 1 - \frac{3}{4}\mu & \text{if } \mu \leq \frac{8}{9}, \\ \frac{1}{3} & \text{if } \frac{8}{9} \leq \mu \leq \frac{16}{9}, \\ \frac{3}{4}\mu - 1 & \text{if } \mu \geq \frac{16}{9}. \end{cases}$$

This lemma is a direct consequence of a result of Keogh and Merkes [9], which states that for $h(z) \in K$,

$$|c_3 - \mu c_2^2| \leq \begin{cases} 1 - \mu & \text{if } \mu \leq \frac{2}{3}, \\ \frac{1}{3} & \text{if } \frac{2}{3} \leq \mu \leq \frac{4}{3}, \\ \mu - 1 & \text{if } \mu \geq \frac{4}{3}. \end{cases}$$

Unless mentioned otherwise, throughout the paper we assume the following notations:
$w(z) \in U, \; z \in E$.
For $0 < c < 1$, we write $w(z) = z(\frac{c+z}{1+cz})$ so that $\frac{1+w(z)}{1-w(z)} = 1 + 2cz + 2z^2 + ..., \, z \in E$.

2. Main Results

Theorem 1 Let $f(z) \in C'$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{19}{9} - \frac{9\mu}{4} & \text{if } \mu \leq \frac{16}{27}, \\ \frac{64}{81\mu} - \frac{5}{9} & \text{if } \frac{16}{27} \leq \mu \leq \frac{2}{3}, \\ \frac{5}{9} + \frac{(8-9\mu)^2}{81\mu} & \text{if } \frac{2}{3} \leq \mu \leq \frac{8}{9}, \\ \frac{5}{9} + \frac{(9\mu-8)^2}{16-9\mu} & \text{if } \frac{8}{9} \leq \mu \leq \frac{32}{27}, \\ \frac{5\mu}{4} - \frac{7}{9} & \text{if } \frac{32}{27} \leq \mu \leq \frac{4}{3}, \\ \frac{9\mu}{4} - \frac{19}{9} & \text{if } \mu \geq \frac{4}{3}. \end{cases} \tag{11}$$

These results are sharp.
Proof. By definition of C',

$$\frac{(zf'(z))'}{g'(z)} = \frac{1+w(z)}{1-w(z)},$$

which on expansion yields

$$1 + 4a_2 z + 9a_3 z^2 + \cdots = (1 + 2b_2 z + 3b_3 z^2 + \cdots)(1 + 2d_1 z + 2(d_2 + d_1^2)z^2 + \cdots).$$

Identifying terms in the above expansion,

$$a_2 = \frac{1}{2}(b_2 + d_1), \tag{12}$$

$$a_3 = \frac{b_3}{3} + \frac{4}{9}b_2 d_1 + \frac{2}{9}(d_2 + d_1^2). \tag{13}$$

From (12) and (13)and using (10),it is easily established that

$$|a_3 - \mu a_2^2| \leq \frac{1}{3}\left|b_3 - \frac{3}{4}\mu b_2^2\right| + \frac{1}{18}|8-9\mu||b_2||d_1| + \frac{1}{36}(8(1-|d_1|^2) + |8-9\mu||d_1|^2). \tag{14}$$

$$|a_3 - \mu a_2^2| \leq \frac{2}{9} + \frac{1}{3}\left|b_3 - \frac{3}{4}\mu b_2^2\right| + \frac{1}{18}|8-9\mu|xy + \frac{1}{36}(|8-9\mu| - 8)x^2, \tag{15}$$

where $x = |d_1| \leq 1$ and $y = |b_2| \leq 2$.

Case I. Suppose that $\mu \leq \frac{2}{3}$. By Lemma 1, (15) can be written as

$$|a_3 - \mu a_2^2| \leq \frac{2}{9} + (1-\mu) + \frac{1}{9}(8-9\mu)x - \frac{\mu}{4}x^2 = H_0(x), \text{ say,}$$

and

$$H_0^{'}(x) = \frac{1}{9}(8-9\mu) - \frac{\mu}{2}x, \quad H_0^{''}(x) = -\frac{\mu}{2}.$$

Subcase I(i). For $\mu \leq 0$, since $x \geq 0$ we have $H_0'(x) > 0$.
$H_0(x)$ is an increasing function in $[0,1]$ and $\max H_0(x) = H_0(1) = \frac{19}{9} - \frac{9\mu}{4}$.

Subcase I(ii). Suppose $0 < \mu \leq \frac{2}{3}$. $H_0^{'}(x) = 0$ when $x = \frac{2(8-9\mu)}{9\mu} = x_0$.
$x_0 > 1$ if and only if $\mu < \frac{16}{27}$ and we have $max H_0(x) = H_0(1) = \frac{19}{9} - \frac{9\mu}{4}$.
Combining the above two subcases, we obtain first result of (11).

Subcase I(iii). For $\frac{16}{27} \leq \mu \leq \frac{2}{3}(x_0 < 1)$,since $H_0^{''}(x) < 0$, therefore we have $\max H_0(x) = H_0(x_0) = \frac{64}{81\mu} - \frac{5}{9}$.

Case II. Suppose that $\frac{2}{3} \leq \mu \leq \frac{8}{3}$, then by Lemma 1,(15) takes the form

$$|a_3 - \mu a_2^2| \leq \frac{2}{9} + \frac{1}{3} + \frac{1}{9}|8 - 9\mu|x - \frac{\mu}{4}x^2.$$

Subcase II(i). $\frac{2}{3} < \mu < \frac{8}{9}$.
Under the above condition, from (15), we get

$$|a_3 - \mu a_2^2| \leq \frac{2}{9} + \frac{1}{3} + \frac{1}{9}(8 - 9\mu)x - \frac{\mu}{4}x^2 = H_1(x), \text{say}.$$

$$H_1^{'}(x) = \frac{1}{9}(8 - 9\mu) - \frac{\mu}{2}x, H_1^{''}(x) = -\frac{\mu}{2} < 0$$

$H_1^{'}(x) = 0$ implies that $x = \frac{2(8-9\mu)}{9\mu} = x_1$ and $\max H_1(x) = H_1(x_1) = \frac{5}{9} + \frac{(8-9\mu)^2}{81\mu}$.

Subcase II(ii). For $\frac{8}{9} \leq \mu \leq \frac{32}{27}$, by Lemma 1, (15) reduces to

$$|a_3 - \mu a_2^2| \leq \frac{5}{9} + (9\mu - 8)x + \frac{(16 - 9\mu)}{36}x^2 = H_2(x), \text{say}.$$

$$H_2^{'}(x) = (9\mu - 8) - \frac{1}{18}(9\mu - 16)x, H_2^{''}(x) < 0.$$

$H_2^{'}(x)$ vanishes when $x = \frac{2(9\mu-8)}{(16-9\mu)} = x_2 < 1$ and
$\max H_2(x) = H_2(x_2) = \frac{5}{9} + \frac{(8-9\mu)^2}{(16-9\mu)}$.

Subcase II(iii). $\frac{32}{27} \leq \mu \leq \frac{4}{3}$. (15) can be expressed as

$$|a_3 - \mu a_2^2| \leq \frac{5}{9} + \frac{1}{9}(9\mu - 8)x - \frac{(16 - 9\mu)}{36}x^2 = H_3(x), \text{say}.$$

$$H_3^{'}(x) = \frac{1}{9}(9\mu - 8) - \frac{1}{18}(16 - 9\mu)x.$$

$H_3^{'}(x) = 0$ yields $x = \frac{2(9\mu-8)}{(16-9\mu)} = x_3 \geq 1$ and
$\max H_3(x) = H_3(1) = \frac{5\mu}{4} - \frac{7}{9}$.

Case III. $\mu \geq \frac{4}{3}$. By Lemma 1, (15) can be put in the form

$$|a_3 - \mu a_2^2| \leq \frac{2}{9} + (\mu - 1) + \frac{1}{9}(9\mu - 8)x - \frac{(16 - 9\mu)}{36}x^2 = H_4(x), \text{say}.$$

$$H_4^{'}(x) = \frac{1}{9}(9\mu - 8) - \frac{1}{18}(16 - 9\mu)x$$

which vanishes at $x = \frac{2(9\mu-8)}{(16-9\mu)} = x_4 \geq 1$ and therefore $\max H_4(x) = H_4(1) = \frac{9\mu}{4} - \frac{19}{9}$.

The first and second inequalities of (11) coincide at $\mu = \frac{16}{27}$ and each is equal to $\frac{7}{9}$.
The second and third inequalities of (11) coincide at $\mu = \frac{2}{3}$ and each is equal to $\frac{17}{27}$.
The third and fourth inequalities of (11) coincide at $\mu = \frac{8}{9}$ and each is equal to $\frac{5}{9}$.
The fourth and fifth inequalities of (11) coincide at $\mu = \frac{32}{27}$ and each is equal to $\frac{19}{27}$.
The fifth and last inequalities of (11) coincide at $\mu = \frac{4}{3}$ and each is equal to $\frac{8}{9}$.

Results of (11)are sharp for the functions defined by their respective derivatives in order as follows:
$f_1'(z) = \frac{1}{z}\left[\left(\int_0^z \frac{(1+t)^2}{(1-t)^4}dt\right)\right]$.

$f_2'(z) = \frac{1}{z}\left[\left(\int_0^z \frac{(1+t)(1+2ct+2t^2+\ldots)}{(1-t)^3}dt\right)\right]$ where $c = \frac{2(8-9\mu)}{9\mu}$.

$f_3'(z) = \frac{1}{z}\left[\left(\int_0^z \frac{(1+t)(1+2dt+2t^2+\ldots)}{(1-t)^3}dt\right)\right]$ where $d = \frac{2(8-9\mu)}{9\mu}$.

$f_4'(z) = \frac{1}{z}\left[\left(\int_0^z \frac{(1+t)(1+2et+2t^2+\ldots)}{(1-t)^3}dt\right)\right]$ where $e = \frac{2(9\mu-8)}{(16-9\mu)}$.

$f_5'(z) = \frac{1}{z}\left[\left(\int_0^z [(1+\frac{29}{3\sqrt{5}}t)^{\frac{15}{29}}dt]\right)\right]$ where $|t| < \frac{3\sqrt{5}}{29}$.

$f_6'(z) = f_1'(z)$.
Proof of the theorem is complete.

Theorem 2 Let $f(z) \in C_1'$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1-\mu & \text{if } \mu \leq \dfrac{4}{9}, \\ \dfrac{16}{81\mu} + \dfrac{1}{9} & \text{if } \dfrac{4}{9} \leq \mu \leq \dfrac{8}{9}, \\ \dfrac{1}{3} + \dfrac{(9\mu-8)^2}{36(16-9\mu)} & \text{if } \dfrac{8}{9} \leq \mu \leq \dfrac{4}{3}, \\ \dfrac{3\mu}{4} - \dfrac{5}{9} & \text{if } \dfrac{4}{3} \leq \mu \leq \dfrac{16}{9}, \\ \mu - 1 & \text{if } \mu \geq \dfrac{16}{9}. \end{cases}$$

The results are sharp.

Proof. Proceeding as in Theorem 1, we have

$$|a_3 - \mu a_2^2| \leq \frac{2}{9} + \frac{1}{3}|c_3 - \frac{3}{4}\mu c_2^2| + \frac{1}{18}|8-9\mu||c_2||d_1| + \frac{1}{36}(|8-9\mu|-8)|d_1|^2. \quad (16)$$

Case I. Suppose that $\mu \leq \frac{8}{9}$. By Lemma 2, and putting $x = |d_1| \leq 1$ and $y = |c_2| \leq 1$,(16) reduces to

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{2}{9} + \frac{1}{3}\left(1 - \frac{3\mu}{4}\right) + \frac{1}{18}(8-9\mu)xy - \frac{\mu}{4}x^2 \\ &= \left(\frac{5}{9} - \frac{\mu}{4}\right) + \frac{1}{18}(8-9\mu)x - \frac{\mu}{4}x^2 = H_6(x), \text{ say.} \end{aligned}$$

Then

$$H_6'(x) = \frac{8-9\mu}{18} - \frac{\mu}{2}x, \quad H_6''(x) = -\frac{\mu}{2}.$$

When $H_6'(x) = 0$, we have $8 - 9\mu = 9\mu x = 9\mu x_6$, say.

Subcase I(i). For $\mu \leq 0$, since $x \geq 0$ we have $H_6'(x) \geq 0$. Suppose $\mu > 0$. Since $x \leq 1$, $H_6'(x) \geq 4/9 - \mu > 0$ if and only if $\mu < 4/9$. Then for $\mu < 4/9$, we have $H_6(x) \leq H_6(1) = 1 - \mu$.

Subcase I(ii). Suppose that $\frac{4}{9} \leq \mu \leq \frac{8}{9}$. Then $\max H_6(x) = H_6(x_6) = 16/81\mu + 1/9$.

Case II. Suppose that $\frac{8}{9} \leq \mu \leq \frac{16}{9}$. By Lemma 2 and (16),

$$|a_3 - \mu a_2^2| \leq \frac{1}{3} + \frac{1}{18}(9\mu - 8)x - \frac{1}{36}(16 - 9\mu)x^2 = H_7(x), \text{ say.}$$

Then $H_7'(x) = 0$ when $x = (9\mu - 8)/(16 - 9\mu) = x_7$, say, and $H_7''(x) = -(16 - 9\mu)/18 < 0$. Since $x_7 \leq 1$, this is relevant only for $\mu \leq \frac{4}{3}$.

Subcase II(i). Suppose that $\frac{8}{9} \leq \mu \leq \frac{4}{3}$. Then

$$\max H_7(x) = H_7(x_7) = \frac{1}{3} + \frac{(9\mu - 8)^2}{36(16 - 9\mu)}.$$

Subcase II(ii). If $\frac{4}{3} \leq \mu \leq \frac{16}{9}$, then $H_7'(x) \geq 0$, so $H_7(x)$ is a monotonically increasing function of x and $\max H_7(x) = H_7(1) = 3\mu/4 - 5/9$.

Case III. Suppose that $\mu \geq \frac{16}{9}$. By Lemma 2, from (16),

$$|a_3 - \mu a_2^2| \leq \frac{2}{9} + \frac{1}{3}\left(\frac{3\mu}{4} - 1\right) + \frac{1}{18}(9\mu - 8)x + \frac{1}{36}(9\mu - 16)x^2 = H_8(x), \text{ say.}$$

We have $H_8'(x) > 0$ and $\max H_8(x) = H_8(1) = \mu - 1$.

This completes the proof.

Extremal function $f_1(z)$ for the first and the last results is defined by $f_1'(z) = \frac{1}{z}\left[\left(\int_0^z \frac{(1+t)}{(1-t)^2}dt\right)\right]$.

Extremal function $f_2(z)$ for the second bound is defined by $f_2'(z) = \frac{1}{z}\left[\left(\int_0^z \frac{(1+2ct+2t^2+\ldots)}{(1-t)^2}dt\right)\right]$, where $c = \frac{(8-9\mu)}{9\mu}$.

Extremal function $f_3(z)$ for the third bound is defined by $f_3'(z) = \frac{1}{z}\left[\left(\int_0^z \frac{(1+2ct+2t^2+\ldots)}{(1-t^2)^2}dt\right)\right]$, where $c = \frac{(9\mu-8)}{16-9\mu}$.

Extremal function $f_4(z)$ for the fourth bound is defined by $f_4'(z) = \frac{1}{z}\left[\left(\int_0^z (1 + \frac{19t}{3\sqrt{3}})^{\frac{9}{19}}dt\right)\right]$, $|t| \leq \frac{3\sqrt{3}}{19}$.

Proceeding as in Theorem 2 and using elementary calculus, we can easily prove the following theorem.

Theorem 3 Let $f(z) \in C_1$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{5}{3} - \frac{9\mu}{4} & \text{if } \mu \leq \frac{2}{9}, \\ \frac{2}{3} + \frac{1}{9\mu} & \text{if } \frac{2}{9} \leq \mu \leq \frac{2}{3}, \\ 1 - \frac{\mu}{4} + \frac{(3\mu-2)^2}{12(4-3\mu)} & \text{if } \frac{2}{3} \leq \mu \leq \frac{8}{9}, \\ \frac{7}{9} + \frac{(3\mu-2)^2}{12(4-3\mu)} & \text{if } \frac{8}{9} \leq \mu \leq \frac{10}{9}, \\ \frac{7}{9} + 2(\mu - 1) & \text{if } \frac{10}{9} \leq \mu \leq \frac{16}{9}, \\ \frac{9\mu}{4} - \frac{5}{3} & \text{if } \mu \geq \frac{16}{9}. \end{cases}$$

The results are sharp. Extremal function $f_1(z)$ for the first and the last results is defined by $f_1(z) = \left[\left(\int_0^z \frac{(1+t)}{(1-t)^2} dt\right)\right]$.
Extremal function $f_2(z)$ for the second bound is defined by $f_2(z) = \left[\left(\int_0^z \frac{(1+2ct+2t^2+...)}{(1-t)} dt\right)\right]$, where $c = \frac{(2-3\mu)}{3\mu}$.
Extremal function $f_3(z)$ for the third and fourth bound is defined by $f_3(z) = \left[\left(\int_0^z \frac{(1+2ct+2t^2+...)}{(1-t)} dt\right)\right]$, where $c = \frac{(3\mu-2)}{2(4-3\mu)}$.
Extremal function $f_4(z)$ for the fifth bound is defined by $f_4(z) = \left[\left(\int_0^z (1 + \frac{10\sqrt{2}}{3} t)^{\frac{3}{5}} dt\right)\right]$, where $|t| \leq \frac{3}{10\sqrt{2}}$.

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Gagandeep Singh
Department of Mathematics, Khalsa College, Amritsar, Punjab, India
E-mail address: kamboj.gagandeep@yahoo.in

Gurcharanjit Singh
Department of Mathematics, G.N.D.U. College, Chungh(Tarn Taran), Punjab, India
E-mail address: dhillongs82@yahoo.com

Harjinder Singh
Department of Mathematics, Govt. College, Mohali, Punjab, India
E-mail address: harjindpreet@gmail.com