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SOME INCLUSION PROPERTIES FOR NEW SUBCLASSES OF MEROMORPHIC MULTIVALENT STRONGLY STARLIKE AND STRONGLY CONVEX FUNCTIONS ASSOCIATED WITH MITTAG-LEFFLER FUNCTION

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Abstract. The purpose of the present paper is to introduce and study some new classes of strongly starlike and strongly convex functions of order δ and type (μ_1,μ_2) by using a new operator associated with Mittag-Leffler function and study their properties.

1. Introduction

Denote by Σ_p the class of meromorphic multivalent functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{n=1-p}^{\infty} a_n z^n \ , \tag{1}$$

which are analytic in $\mathbb{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$.

A function $f(z) \in \Sigma_p$ is said to be in the class of meromorphically strongly starlike functions of order δ and type (μ_1, μ_2) $(\mathcal{S}_p^*(\mu_1, \mu_2, \delta))$ (see [11, 14]) if :

$$-\frac{\pi}{2}\mu_1 < \arg\left(\frac{zf'(z)}{f(z)} + \delta\right) < \frac{\pi}{2}\mu_2 \quad (\delta > p; 0 < \mu_1, \mu_2 \le 1) \tag{2}$$

and in the class of meromorphically strongly convex functions of order δ and type (μ_1, μ_2) $(\mathcal{C}_p(\mu_1, \mu_2, \delta))$ (see [11, 14]) if :

$$-\frac{\pi}{2}\mu_1 < \arg\left(1 + \frac{zf''(z)}{f'(z)} + \delta\right) < \frac{\pi}{2}\mu_2 \quad (\delta > p; 0 < \mu_1, \mu_2 \le 1) \tag{3}$$

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and also, in the class of meromorphically strongly $\lambda-$convex functions of order δ and type (μ_1, μ_2) $(R_p(\mu_1, \mu_2, \lambda, \delta))$ if:

$$-\frac{\pi}{2}\mu_1 < \arg\left((1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1+\frac{zf''(z)}{f'(z)}\right)+\delta\right) < \frac{\pi}{2}\mu_2$$
$$(\lambda \geq 0; \delta > p; 0 < \mu_1, \mu_2 \leq 1). \tag{4}$$

From (2) and (3), we get

$$f(z) \in \mathcal{C}_p(\mu_1, \mu_2, \delta) \Leftrightarrow -\frac{z}{p}f'(z) \in \mathcal{S}_p^*(\mu_1, \mu_2, \delta). \tag{5}$$

We note that:
(i) $R_p(\mu_1, \mu_2, 0, \delta) = \mathcal{S}_p^*(\mu_1, \mu_2, \delta)$;
(ii) $R_p(\mu_1, \mu_2, 1, \delta) = \mathcal{C}_p(\mu_1, \mu_2, \delta)$.

The Mittag-Leffler function $E_\alpha(z)$ $(z \in \mathbb{C})$ ([4, 5]) see also ([1, 2], [3], [8, 9, 10] and [12]) is defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)} z^n, \alpha \in \mathbb{C}, \Re(\alpha) > 0.$$

For $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha)$ ¿ max $\{0, \Re(k)-1\}$ and $\Re(k) > 0$, Srivastava and Tomovski [13] generalized Mittag-Leffler function and introduced the function

$$E_{\alpha,\beta}^{\gamma,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk}}{\Gamma(n\alpha+\beta)n!} z^n \tag{6}$$

and proved that it is an entire function in the complex $z-$plane, where

$$(\gamma)_\theta = \frac{\Gamma(\gamma+\theta)}{\Gamma(\gamma)} \begin{cases} 1, & \theta = 0 \\ \gamma(\gamma+1)...(\gamma+\theta-1), & \theta \neq 0 \end{cases}.$$

Mostafa and Aouf [6], used the function $E_{\alpha,\beta}^{\gamma,k}(z)$ and defined the meromorphic function

$$\begin{aligned} \mathcal{M}_{p,\alpha,\beta}^{\gamma,k}(z) &= z^{-p}\Gamma(\beta)E_{\alpha,\beta}^{\gamma,k}(z) \\ &= z^{-p} + \sum_{n=1-p}^{\infty} \frac{\Gamma(\beta)\Gamma[\gamma+(n+p)k]}{\Gamma(\gamma)\Gamma[\beta+(n+p)\alpha]\Gamma(n+p)} z^n, \\ & \quad (\Re\alpha = 0 \text{ when } \Re k = 1 \; with \; \beta \neq 0), \end{aligned} \tag{7}$$

and for $f(z) \in \Sigma_p$

$$\begin{aligned} \mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z) &= \mathcal{M}_{p,\alpha,\beta}^{\gamma,k}(z) * f(z) \\ &= z^{-p} + \sum_{n=1-p}^{\infty} \frac{\Gamma(\beta)\Gamma[\gamma+(n+p)k]}{\Gamma(\gamma)\Gamma[\beta+(n+p)\alpha]\Gamma(n+p)} a_n z^n. \end{aligned} \tag{8}$$

From (8) it is easy to have

$$\mathcal{H}_{p,0,\beta}^{1,1} f(z) = f(z),$$
$$\mathcal{H}_{p,0,\beta}^{2,1} f(z) = (p+1)\, f(z) + zf'(z), \tag{9}$$
$$kz(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z))' = \gamma \mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z) - (\gamma+pk)\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z) \tag{10}$$

and

$$\alpha z\left(\mathcal{H}_{p,\alpha,\beta+1}^{\gamma,k} f(z)\right)' = \beta\; \mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z) - (p\alpha+\beta)\; \mathcal{H}_{p,\alpha,\beta+1}^{\gamma,k} f(z), \alpha \neq 0. \tag{11}$$

Using the operator $\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)$ defined by (8), we introduce and study the properties of some new classes of meromorphicly analytic functions, defined as follows:

$$\mathcal{S}_p^{\gamma,k}(\alpha,\beta,\mu_1,\mu_2,\delta) = \left\{ \begin{array}{c} f\in\Sigma_p:\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)\in\mathcal{S}_p^*(\mu_1,\mu_2,\delta), \\ \frac{z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)}\neq\delta \end{array} \right\}, \tag{12}$$

$$\mathcal{N}_p^{\gamma,k}(\alpha,\beta,\mu_1,\mu_2,\delta) = \left\{ \begin{array}{c} f\in\Sigma_p:\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)\in\mathcal{C}_p(\mu_1,\mu_2,\delta), \\ 1+\frac{z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)\right)''}{\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}\neq\delta \end{array} \right\}, \tag{13}$$

$$\mathcal{V}_p^{\gamma,k}(\alpha,\beta,\lambda,\mu_1,\mu_2,\delta) = \left\{ \begin{array}{c} f\in\Sigma_p:\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}(z)\in R_p(\mu_1,\mu_2,\lambda,\delta) \\ \left((1-\lambda)\frac{z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)}+\lambda\left(1+\frac{z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)\right)''}{\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}\right)\right) \\ \neq\delta \end{array} \right\}. \tag{14}$$

We note that:
(i) $\mathcal{V}_p^{\gamma,k}(\alpha,\beta,0,\mu_1,\mu_2,\delta)=\mathcal{S}_p^{\gamma,k}(\alpha,\beta,\mu_1,\mu_2,\delta)$;
(ii) $\mathcal{V}_p^{\gamma,k}(\alpha,\beta,1,\mu_1,\mu_2,\delta)=\mathcal{N}_p^{\gamma,k}(\alpha,\beta,\mu_1,\mu_2,\delta)$.
From (12) and (13), we get

$$f(z)\in\mathcal{N}_p^{\gamma,k}(\alpha,\beta,\mu_1,\mu_2,\delta)\Leftrightarrow-\frac{z}{p}f'(z)\in\mathcal{S}_p^{\gamma,k}(\alpha,\beta,\mu_1,\mu_2,\delta). \tag{15}$$

2. **Preliminary results**

The following lemmas will be required in our investigation.
Lemma 1[7] . Let the function $q(z)$ given by

$$q(z)=1+\sum_{k=1}^{\infty}c_kz^k, \tag{16}$$

be analytic in $\mathbb{U}$ with $q(z)\neq0\ \ (z\in\mathbb{U})$.
If there exists a points $z_{1,},z_{2,}\in\mathbb{U}$ such that

$$\begin{aligned} -\frac{\pi}{2}\mu_1 &= \arg q(z_1)<\arg q(z)<\arg q(z_2)=\frac{\pi}{2}\mu_2 \\ & (\mu_1,\mu_2>0,|z|<|z_1|=|z_2|), \end{aligned} \tag{17}$$

then we have

$$\frac{z_1q'(z_1)}{q(z_1)}=-i\frac{\mu_1+\mu_2}{2}l, \tag{18}$$

and

$$\frac{z_2q'(z_2)}{q(z_2)}=i\frac{\mu_1+\mu_2}{2}l, \tag{19}$$

where

$$l\geq\frac{1-|a|}{1+|a|}$$

and

$$a=i\tan\left\{\frac{\pi}{4}\left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)\right\}.$$

3. Main Inclusion Relationships

Unless otherwise mentioned, we assume throughout this paper that $\delta > p, 0 < \mu_1, \mu_2 \leq 1, \lambda \geq 0, \alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha)$ ¿ max $\{0, \Re(k) - 1\}$ and $\Re(k) > 0$ and $z \in \mathbb{U}$.

Theorem 1. The following inclusion relation holds:

$$\mathcal{S}_p^{\gamma+1,k}(\alpha, \beta, \mu_1, \mu_2, \delta) \subset \mathcal{S}_p^{\gamma,k}(\alpha, \beta, \mu_1, \mu_2, \delta). \tag{20}$$

Proof. Let $f(z) \in \mathcal{S}_p^{\gamma+1,k}(\alpha, \beta, \mu_1, \mu_2, \delta)$, and

$$\frac{z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f\left(z\right)\right)^{'}}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f\left(z\right)} = -\left[\left(p-\delta\right) q(z) + \delta\right], \tag{21}$$

where $q(z)$ is analytic in $\mathbb{U}$ with $q(0) = 1$ and $q(z) \neq 0$. From (10) and (21), we get

$$\frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f\left(z\right)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f\left(z\right)} = \frac{\left(\delta-p\right) q(z) - \delta + \frac{\gamma+pk}{k}}{\left(\frac{\gamma}{k}\right)}. \tag{22}$$

Now, differentiating (22), we obtain

$$\begin{aligned} \frac{z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)\right)^{'}}{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)} + \delta &= \frac{z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)\right)^{'}}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)} + \frac{(\delta-p)zq^{'}(z)}{(\delta-p)q(z)-\delta+\frac{\gamma+pk}{k}} + \delta \\ &= \left(\delta-p\right) q(z) + \frac{(\delta-p)zq^{'}(z)}{(\delta-p)q(z)-\delta+\frac{\gamma+pk}{k}}. \end{aligned} \tag{23}$$

Suppose that there exists two point $z_1, z_2 \in \mathbb{U}$ such that

$$-\frac{\pi}{2}\mu_1 = \arg q(z_1) < \arg q(z) < \arg q(z_2) = \frac{\pi}{2}\mu_2 \quad \left(|z| < |z_1| = |z_2|\right).$$

Then, applying Lemma 1, we can write that

$$\frac{z_1 q^{'}(z_1)}{q(z_1)} = -\frac{il\left(\mu_1+\mu_2\right)\left(1+t_1^2\right)}{4t_1} \tag{24}$$

and

$$\frac{z_2 q^{'}(z_2)}{q(z_2)} = \frac{il\left(\mu_1+\mu_2\right)\left(1+t_2^2\right)}{4t_2}, \tag{25}$$

where

$$q(z_1) = \left(-it_1\right)^{\frac{(\mu_1+\mu_2)}{2}} \exp\left\{i\frac{\pi}{4}\left(\mu_2-\mu_1\right)\right\} \quad t_1 > 0, \tag{26}$$

$$q(z_2) = \left(it_2\right)^{\frac{(\mu_1+\mu_2)}{2}} \exp\left\{i\frac{\pi}{4}\left(\mu_2-\mu_1\right)\right\} \quad t_2 > 0 \tag{27}$$

and

$$l \geq \frac{1-|a|}{1+|a|}.$$

Replacing z by z_2 in (23) and using (25) and (27), we get

$$\begin{aligned}
\frac{z_2\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z_2)\right)'}{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z_2)}+\delta &= (\delta-p)\, q(z_2)\left[1+\frac{\frac{z_2 q'(z_2)}{q(z_2)}}{(\delta-p)q(z_2)-\delta+\frac{\gamma+pk}{k}}\right] \\
&= (\delta-p)\, t_2^{\frac{(\mu_1+\mu_2)}{2}} \exp\left(i\frac{\pi}{2}\mu_2\right) \\
&\quad \times\left\{1+\frac{il(\mu_1+\mu_2)\left(1+t_2^2\right)}{4t_2\left[(\delta-p)t_2^{\frac{(\mu_1+\mu_2)}{2}}\exp\left(i\frac{\pi}{2}\mu_2\right)+\left(\frac{\gamma+pk}{k}-\delta\right)\right]}\right\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\arg\left\{\frac{z_2\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z_2)\right)'}{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z_2)}+\delta\right\} &= \frac{\pi}{2}\mu_2+\arg\left\{1+\frac{il(\mu_1+\mu_2)\left(t_2^{-1}+t_2\right)}{4\left[(\delta-p)t_2^{\frac{(\mu_1+\mu_2)}{2}}\exp\left(i\frac{\pi}{2}\mu_2\right)+\left(\frac{\gamma+pk}{k}-\delta\right)\right]}\right\} \\
&= \frac{\pi}{2}\mu_2+\tan^{-1}\left\{\frac{\epsilon_1(\mu_1,\mu_2,t_2)}{\epsilon_2(\mu_1,\mu_2,t_2)}\right\}\geq\frac{\pi}{2}\mu_2, \qquad (28)
\end{aligned}$$

where

$$\begin{aligned}
\epsilon_1(\mu_1,\mu_2,t_2) &= l(\mu_1+\mu_2)\left(t_2^{-1}+t_2\right)(\delta-p)\, t_2^{\frac{(\mu_1+\mu_2)}{2}}\cos\left(\frac{\pi}{2}\mu_2\right) \\
&\quad +l\left(\frac{\gamma+pk}{k}-\delta\right)(\mu_1+\mu_2)\left(t_2^{-1}+t_2\right), \\
\epsilon_2(\mu_1,\mu_2,t_2) &= 4(\delta-p)^2\, t_2^{(\mu_1+\mu_2)} \\
&\quad +4\left(\frac{\gamma+pk}{k}-\delta\right)\left[\left(\frac{\gamma+pk}{k}-\delta\right)+2(\delta-p)\, t_2^{\frac{(\mu_1+\mu_2)}{2}}\cos\left(\frac{\pi}{2}\mu_2\right)\right] \\
&\quad +(\delta-p)(\mu_1+\mu_2)\left(t_2^{-1}+t_2\right) t_2^{\frac{(\mu_1+\mu_2)}{2}}\sin\left(\frac{\pi}{2}\mu_2\right),
\end{aligned}$$

and

$$l\geq\frac{1-|a|}{1+|a|}.$$

Similarly, replacing $z=z_1$ in (23) and using the same procedure as above, we obtain that

$$\arg\left\{\frac{z_1\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z_1)\right)'}{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z_1)}+\delta\right\}\leq-\frac{\pi}{2}\mu_1, \qquad (29)$$

which contradicts the condition $f(z)\in\mathcal{S}_p^{\gamma+1,k}(\alpha,\beta,\mu_1,\mu_2,\delta)$. Thus the function $q(z)$ defined by (16) yields

$$-\frac{\pi}{2}\mu_1<\arg q(z)<\frac{\pi}{2}\mu_2,$$

which implies that

$$-\frac{\pi}{2}\mu_1<\arg\left\{\frac{z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)\right)'}{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}+\delta\right\}<\frac{\pi}{2}\mu_2.$$

Thus $f(z)\in\mathcal{S}_p^{\gamma,k}(\alpha,\beta,\mu_1,\mu_2,\delta)$. This completes the proof of Theorem 1. □

Theorem 2. The following inclusion relation holds:

$$\mathcal{S}_p^{\gamma,k}(\alpha,\beta,\mu_1,\mu_2,\delta)\subset\mathcal{S}_p^{\gamma,k}(\alpha,\beta+1,\mu_1,\mu_2,\delta). \qquad (30)$$

Proof. The proof of Theorem 2 is the same as the proof of Theorem 1 by using (11) instead of (10). □

Theorem 3. The following inclusion relation holds:

$$\mathcal{N}_p^{\gamma+1,k}(\alpha,\beta,\mu_1,\mu_2,\delta) \subset \mathcal{N}_p^{\gamma,k}(\alpha,\beta,\mu_1,\mu_2,\delta). \tag{31}$$

Proof.

$$\begin{aligned}
f(z) &\in \mathcal{N}_p^{\gamma+1,k}(\alpha,\beta,\mu_1,\mu_2,\delta) \Leftrightarrow \mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z) \in \mathcal{C}_p(\mu_1,\mu_2,\delta) \Leftrightarrow \\
-\frac{z}{p}\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)\right)' &\in \mathcal{S}_p^*(\mu_1,\mu_2,\delta) \Leftrightarrow \mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)\left(-\frac{z}{p}f'(z)\right) \in \mathcal{S}_p^*(\mu_1,\mu_2,\delta) \Leftrightarrow \\
-\frac{z}{p}f'(z) &\in \mathcal{S}_p^{\gamma+1,k}(\alpha,\beta,\mu_1,\mu_2,\delta) \Rightarrow -\frac{z}{p}f'(z) \in \mathcal{S}_p^{\gamma,k}(\alpha,\beta,\mu_1,\mu_2,\delta) \Leftrightarrow \\
\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)\left(-\frac{z}{p}f'(z)\right) &\in \mathcal{S}_p^*(\mu_1,\mu_2,\delta) \Leftrightarrow -\frac{z}{p}\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)\right)' \in \mathcal{S}_p^*(\mu_1,\mu_2,\delta) \Leftrightarrow \\
\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z) &\in \mathcal{C}_p(\mu_1,\mu_2,\delta) \Leftrightarrow f(z) \in \mathcal{N}_p^{\gamma,k}(\alpha,\beta,\mu_1,\mu_2,\delta),
\end{aligned}$$

the proof of Theorem 3 is completed. □

Similarly, we can prove the following result.

Theorem 4. The following inclusion relation holds:

$$\mathcal{N}_p^{\gamma,k}(\alpha,\beta,\mu_1,\mu_2,\delta) \subset \mathcal{N}_p^{\gamma,k}(\alpha,\beta+1,\mu_1,\mu_2,\delta). \tag{32}$$

Theorem 5. Let The following inclusion relation holds:

$$\mathcal{V}_p^{\gamma,k}(\alpha,\beta,\lambda,\mu_1,\mu_2,\delta) \subset \mathcal{V}_p^{\gamma,k}(\alpha,\beta,0,\mu_1,\mu_2,\delta). \tag{33}$$

Proof. We observe that, if $\lambda = 0$, the result (33) is obvious. If, for $\lambda > 0$, let $f(z) \in \mathcal{V}_p^{\gamma,k}(\alpha,\beta,\lambda,\mu_1,\mu_2,\delta)$, that is

$$-\frac{\pi}{2}\mu_1 < \arg\left[(1-\lambda)\frac{z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)\right)'}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)} + \lambda\left(1+\frac{z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)\right)''}{\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)\right)'}\right)+\delta\right] < \frac{\pi}{2}\mu_2 \tag{34}$$

and let $q(z)$ be given by (16). From (21), we get

$$z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)\right)' = \left[(\delta-p)\,q(z)-\delta\right]\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z). \tag{35}$$

Now, differentiating (35), we get

$$\begin{aligned}
\left(1+\frac{z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)\right)''}{\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)\right)'}\right) &= \left[(\delta-p)\,q(z)-\delta\right] \\
&+\frac{(\delta-p)\,zq'(z)}{\left[(\delta-p)\,q(z)-\delta\right]}.
\end{aligned} \tag{36}$$

From (35) and (36), we have

$$\begin{aligned}
&(1-\lambda)\frac{z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)\right)'}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)} + \lambda\left(1+\frac{z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)\right)''}{\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)\right)'}\right)+\delta \\
= \ &(\delta-p)\,q(z) + \frac{\lambda\,(\delta-p)\,zq'(z)}{\left[(\delta-p)\,q(z)-\delta\right]}.
\end{aligned} \tag{37}$$

Suppose that there exists two point $z_1, z_2 \in \mathbb{U}$ such that

$$-\frac{\pi}{2}\mu_1 = \arg q(z_1) < \arg q(z) < \arg q(z_2) = \frac{\pi}{2}\mu_2 \quad (|z| < |z_1| = |z_2|).$$

Then, applying Lemma 1, we can write that

$$\frac{z_1 q'(z_1)}{q(z_1)} = -\frac{il(\mu_1+\mu_2)\left(1+t_1^2\right)}{4t_1} \tag{38}$$

and

$$\frac{z_2 q'(z_2)}{q(z_2)} = \frac{il(\mu_1+\mu_2)\left(1+t_2^2\right)}{4t_2}, \tag{39}$$

where

$$q(z_1) = (-it_1)^{\frac{(\mu_1+\mu_2)}{2}} \exp\left\{i\frac{\pi}{4}(\mu_2-\mu_1)\right\} \quad t_1 > 0, \tag{40}$$

$$q(z_2) = (it_2)^{\frac{(\mu_1+\mu_2)}{2}} \exp\left\{i\frac{\pi}{4}(\mu_2-\mu_1)\right\} \quad t_2 > 0 \tag{41}$$

and

$$l \geq \frac{1-|a|}{1+|a|}.$$

Replacing z by z_2 in (37) and using (39) and (41), we get

$$\begin{aligned}
& (1-\lambda)\frac{z_2\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z_2)\right)'}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z_2)} + \lambda\left(1+\frac{z_2\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z_2)\right)''}{\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z_2)\right)'}\right) + \delta \\
= \; & (\delta-p)\,q(z_2)\left[1+\frac{\lambda\frac{z_2 q'(z_2)}{q(z_2)}}{(\delta-p)q(z_2)-\delta}\right] \\
= \; & (\delta-p)\,t_2^{\frac{(\mu_1+\mu_2)}{2}} \exp\left(i\frac{\pi}{2}\mu_2\right)\left\{1+\frac{i\lambda l(\mu_1+\mu_2)\left(1+t_2^2\right)}{4t_2\left[(\delta-p)t_2^{\frac{(\mu_1+\mu_2)}{2}}\exp\left(i\frac{\pi}{2}\mu_2\right)-\delta\right]}\right\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
& \arg\left\{(1-\lambda)\frac{z_2\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z_2)\right)'}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z_2)} + \lambda\left(1+\frac{z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z_2)\right)''}{\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z_2)\right)'}\right) + \delta\right\} \\
= \; & \frac{\pi}{2}\mu_2 + \arg\left\{1+\frac{i\lambda l(\mu_1+\mu_2)\left(t_2^{-1}+t_2\right)}{4\left[(\delta-p)t_2^{\frac{(\mu_1+\mu_2)}{2}}\exp\left(i\frac{\pi}{2}\mu_2\right)-\delta\right]}\right\} \\
= \; & \frac{\pi}{2}\mu_2 + \tan^{-1}\left\{\frac{\epsilon_1(\mu_1,\mu_2,t_2)}{\epsilon_2(\mu_1,\mu_2,t_2)}\right\} \geq \frac{\pi}{2}\mu_2,
\end{aligned} \tag{42}$$

where

$$\begin{aligned}
\epsilon_1(\mu_1,\mu_2,t_2) = \; & \lambda l(\mu_1+\mu_2)\left(t_2^{-1}+t_2\right)(\delta-p)\,t_2^{\frac{(\mu_1+\mu_2)}{2}}\cos\left(\frac{\pi}{2}\mu_2\right) \\
& -\delta\lambda l(\mu_1+\mu_2)\left(t_2^{-1}+t_2\right), \\
\epsilon_2(\mu_1,\mu_2,t_2) = \; & 4(\delta-p)^2\,t_2^{(\mu_1+\mu_2)} \\
& +4\delta\left[\delta-2(\delta-p)\,t_2^{\frac{(\mu_1+\mu_2)}{2}}\cos\left(\frac{\pi}{2}\mu_2\right)\right] \\
& +\lambda l(\delta-p)(\mu_1+\mu_2)\left(t_2^{-1}+t_2\right)t_2^{\frac{(\mu_1+\mu_2)}{2}}\sin\left(\frac{\pi}{2}\mu_2\right),
\end{aligned}$$

and

$$l \geq \frac{1-|a|}{1+|a|}.$$

Similarly, replacing $z = z_1$ in (3.18) and using the same procedure as above, we obtain that

$$\arg\left\{(1-\lambda)\frac{z_2\left(\mathcal{H}^{\gamma,k}_{p,\alpha,\beta}f(z_2)\right)'}{\mathcal{H}^{\gamma,k}_{p,\alpha,\beta}f(z_2)} + \lambda\left(1+\frac{z_2\left(\mathcal{H}^{\gamma,k}_{p,\alpha,\beta}f(z_2)\right)''}{\left(\mathcal{H}^{\gamma,k}_{p,\alpha,\beta}f(z_2)\right)'}\right)+\delta\right\} \leq -\frac{\pi}{2}\mu_1, \quad (3.24)$$

which contradicts the condition $f(z) \in \mathcal{V}^{\gamma,k}_p(\alpha,\beta,\lambda,\mu_1,\mu_2,\delta)$. Thus the function $q(z)$ defined by (16) yields

$$-\frac{\pi}{2}\mu_1 < \arg q(z) < \frac{\pi}{2}\mu_2,$$

which implies that

$$-\frac{\pi}{2}\mu_1 < \arg\left\{\frac{z\left(\mathcal{H}^{\gamma,k}_{p,\alpha,\beta}f(z)\right)'}{\mathcal{H}^{\gamma,k}_{p,\alpha,\beta}f(z)} + \delta\right\} < \frac{\pi}{2}\mu_2.$$

Thus $f(z) \in \mathcal{V}^{\gamma,k}_p(\alpha,\beta,0,\mu_1,\mu_2,\delta)$. This completes the proof of Theorem 5. □

Remarks.

(i) Putting $\lambda = 0$ in Theorem 5, we get the corresponding result for Theorems 1and 2;

(ii) Putting $\lambda = 1$ in Theorem 5, we get the corresponding result for Theorems 3 and 4.

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