Journal of Fractional Calculus and Applications
Vol. 9(2) July 2018, pp. 127-132.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

ON CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH GEGENBAUER POLYNOMIALS

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Abstract. In this work, the authors considered a new subclass $T\mathcal{G}_{\lambda,t}(\alpha,\beta)$ consisting of analytic univalent functions with negative coefficients define by Gegenbauer polynomials. Coefficient inequalities, extreme points and integral means inequalities for the class $T\mathcal{G}_{\lambda,t}(\alpha,\beta)$ were determined.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}) \tag{1}$$

which are analytic in the unit disk $\mathbb{U} = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$ in $\mathbb{U}$. Recall that, S denote the subclass of $\mathcal{A}$ consisting of functions that are univalent. Also, denote by T a subclass of $\mathcal{A}$ consisting functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0 \quad (z \in \mathbb{U}) \tag{2}$$

introduced and studied by Silverman [5].
The class $\mathcal{T}(\lambda), \lambda \geq 0$ were introduced and investigated by Szynal [8] as the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = \int_{-1}^{1} k(z,t) d\mu(t), \tag{3}$$

where

$$k(z,t) = \frac{z}{(1 - 2tz + z^2)^{\lambda}} \quad (z \in \mathbb{U}), \quad t \in [-1,1] \tag{4}$$

and μ is a probability measure on the interval $[-1,1]$. The collection of such measures on $[a,b]$ is denoted by $P_{[a,b]}$.
The Taylor series expansion of the function in (4) gives

$$k(z,t) = z + c_1^{(\lambda)}(t) z^2 + c_2^{(\lambda)}(t) z^3 + ... \tag{5}$$

2010 *Mathematics Subject Classification.* 30C45, 30C50.
Key words and phrases. analytic function, univalent function, integral means inequalities, convolution, subordination, Gegenbauer polynomials.
Submitted March 21, 2017.

and the coefficients for (5) were given below:

$$c_0^{(\lambda)}(t)=1, c_1^{(\lambda)}(t)=2\lambda t, c_2^{(\lambda)}(t)=2\lambda(\lambda+1)t^2-\lambda, c_3^{(\lambda)}(t)=\frac{4}{3}\lambda(\lambda+1)(\lambda+2)t^3-2\lambda(\lambda+1)t, ... \tag{6}$$

where $c_n^{(\lambda)}(t)$ denotes the Gegenbauer polynomial of degree n. Varying the parameter λ in (5), we obtain the class of typically real functions studied by [1], [3],[4] and [6].
For $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$, the Hadamard product of f and g is defined by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n \quad (z\in\mathbb{U}).$$

Also, for two analytic functions g and h with $g(0)=h(0)$, g is said to be subordinate to h, denoted by $g\prec h$, if there exists an analytic function ω such that $\omega(0)=0, |\omega(z)|<1$ and $g(z)=h(\omega(z))$, for all $(z\in\mathbb{U})$.
Let $\mathcal{G}_{\lambda,t}:A\longrightarrow A$ defined in terms of the convolution by

$$\mathcal{G}_{\lambda,t}f(z)=k(z,t)*f(z),$$

we have

$$\mathcal{G}_{\lambda,t}f(z)=z+\sum_{n=2}^{\infty}c_{n-1}^{\lambda}(t)a_nz^n. \tag{7}$$

A class $UCD(\alpha), \alpha\leq 0$ consisting of functions $f\in A$ satisfying

$$Re[f'(z)]\geq\alpha|f''(z)|, (z\in\mathbb{U})$$

was introduced and investigated in [10].
A related class $SD(\alpha)$ have been introduced and studied in [7] and [9]. A function f of the form (1) is said to be in the class $SD(\alpha)$ if

$$Re\left\{\frac{f(z)}{z}\right\}\geq\alpha\left|f'(z)-\frac{f(z)}{z}\right|, for \quad \alpha\geq 0$$

Recently, [11] extended the class of functions studied by [7] and [9] by making use of Hurwitz-Lerch Zeta Function, the coefficient inequalities, extreme points, integral means inequalities and subordination results for the class $T\mathcal{J}_{\mu,b}(\alpha,\beta)$ were obtained in which

$$Re\left\{\frac{\mathcal{J}_{\mu,b}f(z)}{z}\right\}\geq\alpha\left|(\mathcal{J}_{\mu,b}f(z))'-\frac{\mathcal{J}_{\mu,b}f(z)}{z}\right|+\beta, for \alpha\geq 0.$$

For $\alpha\geq 0, \beta\in[0,1), \lambda>0, t\in[-1,1]$, we let $\mathcal{G}_{\lambda,t}(\alpha,\beta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1) and its geometrical condition satisfy

$$Re\left\{\frac{\mathcal{G}_{\lambda,t}f(z)}{z}\right\}\geq\alpha\left|(\mathcal{G}_{\lambda,t}f(z))'-\frac{\mathcal{G}_{\mu,b}f(z)}{z}\right|+\beta, \tag{8}$$

where $\mathcal{G}_{\lambda,t}f(z)$ is given by (7).
Motivated by earlier works of [11] and [12], in this paper, we investigate the coefficient inequalities, extreme points and the integral means inequalities for the class $T\mathcal{G}_{\lambda,t}(\alpha,\beta)$.

2. Main Results

Theorem 2.1 A function $f(z)$ be the form (1) is in $\mathcal{G}_{\lambda,t}(\alpha,\beta)$ if

$$\sum_{n=2}^{\infty}\left(1+\alpha(n-1)\right)c_{n-1}^{\lambda}(t)\left|a_n\right|\leq 1-\beta \tag{9}$$

where $\alpha\geq 0, \beta\in[0,1), \lambda>0, t\in[-1,1]$.
Proof It suffices to show that

$$\alpha\left|(\mathcal{G}_{\lambda,t}f(z))'-\frac{\mathcal{G}_{\mu,b}f(z)}{z}\right|-Re\left\{\frac{\mathcal{G}_{\lambda,t}f(z)}{z}-1\right\}\leq 1-\beta.$$

We have

$$\begin{aligned}
&\alpha\left|(\mathcal{G}_{\lambda,t}f(z))'-\frac{\mathcal{G}_{\mu,b}f(z)}{z}\right|-Re\left\{\frac{\mathcal{G}_{\lambda,t}f(z)}{z}-1\right\}\\
&\leq\alpha\left|(\mathcal{G}_{\lambda,t}f(z))'-\frac{\mathcal{G}_{\mu,b}f(z)}{z}\right|-Re\left\{\frac{\mathcal{G}_{\lambda,t}f(z)}{z}-1\right\}\\
&\leq\alpha\left|\frac{\sum_{n=2}^{\infty}(n-1)c_{n-1}^{\lambda}(t)a_nz^n}{z}\right|+\left|\frac{\sum_{n=2}^{\infty}c_{n-1}^{\lambda}(t)a_nz^n}{z}\right|\\
&\leq\alpha\sum_{n=2}^{\infty}(n-1)c_{n-1}^{\lambda}(t)|a_n|+\sum_{n=2}^{\infty}c_{n-1}^{\lambda}(t)|a_n|\\
&=\sum_{n=2}^{\infty}\left(1+\alpha(n-1)\right)c_{n-1}^{\lambda}(t)|a_n|.
\end{aligned}$$

The last expression is bounded above by $(1-\beta)$ if

$$\sum_{n=2}^{\infty}\left(1+\alpha(n-1)\right)c_{n-1}^{\lambda}(t)|a_n|\leq 1-\beta$$

and this completes the proof.
For the next theorem, the necessary and sufficient conditions for the functions of the class $T\mathcal{G}_{\lambda,t}(\alpha,\beta)$
Theorem 2.1 A function $f(z)$ be the form (2) is in $T\mathcal{G}_{\lambda,t}(\alpha,\beta)$ if

$$\sum_{n=2}^{\infty}\left(1+\alpha(n-1)\right)c_{n-1}^{\lambda}(t)\left|a_n\right|\leq 1-\beta \tag{10}$$

where $\alpha\geq 0, \beta\in[0,1), \lambda>0, t\in[-1,1]$.
Proof Suppose $f(z)$ of the form (2) is in the class $T\mathcal{G}_{\lambda,t}(\alpha,\beta)$. Then

$$Re\left\{\frac{\mathcal{G}_{\lambda,t}f(z)}{z}\right\}-\alpha\left|(\mathcal{G}_{\lambda,t}f(z))'-\frac{\mathcal{G}_{\mu,b}f(z)}{z}\right|\geq\beta$$

Equivalently

$$Re\left[1-\sum_{n=2}^{\infty}c_{n-1}^{\lambda}(t)|a_n|z^{n-1}\right]-\alpha\left|\sum_{n=2}^{\infty}(n-1)c_{n-1}^{\lambda}(t)a_nz^{n-1}\right|\geq\beta$$

Letting z to take real values and as $|z|\longrightarrow 1$, we have

$$1-\sum_{n=2}^{\infty}c_{n-1}^{\lambda}(t)|a_n|-\alpha\sum_{n=2}^{\infty}(n-1)c_{n-1}^{\lambda}(t)|a_n|\geq\beta$$

which implies Theorem 2.2.

Corollary 2.3: A function $f(z)$ be the form (2) is in $T\mathcal{G}_{\lambda,t}(\alpha,\beta)$ if

$$|a_n| \leq \frac{1-\beta}{(1+\alpha(n-1))\, c_{n-1}^{\lambda}(t)}$$

where $\alpha \geq 0, \beta \in [0,1), \lambda > 0, t \in [-1,1]$.

Theorem 2.4: Let $f_1(z) = z$ and $f_n(z) = z - \frac{1-\beta}{(1+\alpha(n-1))c_{n-1}^{\lambda}(t)} z^n, n \geq 2$ for where $\alpha \geq 0, \beta \in [0,1), \lambda > 0$ and $t \in [-1,1]$. Then $f(z)$ is in the class $T\mathcal{G}_{\lambda,t}(\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \psi_n f_n(z)$$

where $\psi \geq 0$ and $\sum_{n=1}^{\infty} \psi_n = 1$.

Proof: Let f(z) be expressible in the form $f(z) = \sum_{n=1}^{\infty} \psi_n f_n(z)$. Then

$$f(z) = \psi_1 f_1(z) + \sum_{n=2}^{\infty} \psi_n f_n(z) = \psi_1 z + \sum_{n=2}^{\infty} \psi_n \left[z - \frac{1-\beta}{(1+\alpha(n-1))\, c_{n-1}^{\lambda}(t)} z^n \right]$$
$$= z - \frac{1-\beta}{(1+\alpha(n-1))\, c_{n-1}^{\lambda}(t)} z^n.$$

Now

$$\sum_{n=2}^{\infty} \frac{(1+\alpha(n-1))\, c_{n-1}^{\lambda}(t)}{1-\beta} . \frac{1-\beta}{(1+\alpha(n-1))\, c_{n-1}^{\lambda}(t)} \psi_n = \sum_{n=1}^{\infty} \psi_n = 1 - \psi_1 \leq 1.$$

Thus $f \in T\mathcal{G}_{\lambda,t}(\alpha,\beta)$.

Conversely, suppose $f \in T\mathcal{G}_{\lambda,t}(\alpha,\beta)$. Then corollary 2.3 gives

$$a_n \leq \frac{1-\beta}{(1+\alpha(n-1))\, c_{n-1}^{\lambda}(t)}, n \geq 2$$

Set $\psi_n = \frac{(1+\alpha(n-1))c_{n-1}^{\lambda}(t)}{1-\beta} a_n, n \geq 2$, where $\psi_1 = 1 - \sum_{n=2}^{\infty} \psi_n$. Then $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$

$$z - \sum_{n=2}^{\infty} \psi_n \frac{1-\beta}{(1+\alpha(n-1))\, c_{n-1}^{\lambda}(t)} z^n$$
$$= z - \sum_{n=2}^{\infty} \psi_n z + \sum_{n=2}^{\infty} \psi_n f_n(z)$$
$$= z \left[1 - \sum_{n=2}^{\infty} \psi_n \right] + \sum_{n=2}^{\infty} \psi_n f_n(z)$$
$$= \psi_1 f_1 z + \sum_{n=2}^{\infty} \psi_n f_n(z)$$
$$= \sum_{n=1}^{\infty} \psi_n f_n(z)$$

Hence the proof.

For the purpose of the last theorem, the lemma below shall be necessary.

Lemma:[12]: If the functions $f(z)$and $g(z)$ are analytic in $(z \in \mathbb{U})$ with $g(z) \prec f(z)$, then $\int_0^{2\pi} |g(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^p d\theta, (0 \leq r < 1, p > 0)$.

Theorem 2.5 Suppose $f \in T\mathcal{G}_{\lambda,t}(\alpha,\beta), p>0, \alpha \geq 0, \lambda > 0, \beta \in [0,1), t \in [-1,1]$ and $f_2(z)$ is defined by $f_2(z) = z - \frac{1-\beta}{2\lambda t(1+\alpha)}z^2$. Then for $z = re^{i\theta}, 0 \leq r < 1$, we have

$$\int_0^{2\pi} |f(z)|^p d\theta \leq \int_0^{2\pi} |f_2(z)|^p d\theta. \tag{11}$$

Proof For $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$, (11) is equivalent to proving that

$$\int_0^{2\pi} |z - \sum_{n=2}^{\infty} |a_n| z^n|^p d\theta \leq \int_0^{2\pi} |z - \frac{1-\beta}{2\lambda t\,(1+\alpha)} z^2|^p d\theta \quad (p>0). \tag{12}$$

By applying Littlewood's subordination theorem, it will be sufficient to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{1-\beta}{2\lambda t\,(1+\alpha)} z \tag{13}$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{1-\beta}{2\lambda t\,(1+\alpha)} \omega(z), \tag{14}$$

we obtain $\omega(z) = \frac{2\lambda t(1+\alpha)}{1-\beta} \sum_{n=2}^{\infty} a_n z^{n-1}$ and $\omega(z)$ is analytic in $(z \in \mathbb{U})$ with $\omega(0) = 0$.

Moreover, it sufficies to prove that $\omega(z)$ satisfies $|\omega(z)| < 1, (z \in \mathbb{U})$. Now

$$|\omega(z)| = \left| \sum_{n=2}^{\infty} \frac{2\lambda t\,(1+\alpha)}{1-\beta} a_n z^{n-1} \right| \leq |z| \sum_{n=2}^{\infty} \frac{2\lambda t\,(1+\alpha)}{1-\beta} |a_n| \leq |z| < 1. \tag{15}$$

In view of the inequality (15) the subordination (13) follows, which proves the theorem.

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