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OPTIMALITY CONDITIONS OF FRACTIONAL DIFFUSION EQUATIONS WITH WEAK CAPUTO DERIVATIVES AND VARIATIONAL FORMULATION

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Abstract. In this paper we start by using a new definition of weak Caputo derivative in the sense of distributions, and we give a variational formulation to a fractional diffusion equation with Caputo derivative. We first prove the existence of the solution to this weak formulation and use it to obtain a result on distributed and boundary Fractional Optimal Control Problem (FOCP). Then we show that the considered optimal control problem has a unique solution. The performance index of a (FOCP) is considered as a function of both state and control variables, and the dynamic constraints are expressed by a Partial Fractional Differential Equation (PFDE). The time horizon is fixed. We impose some constraints on the boundary control. Interpreting the Euler-?Lagrange first order optimality condition with an adjoint problem defined by means of right fractional weak Caputo derivative, we obtain an optimality system for the optimal control. Finally we discuss the controllability of the fractional distributed Dirichlet problem with weak Caputo fractional derivatives. Some examples are analyzed in details.

1. Introduction

The study of fractional calculus (noninteger order) is gaining more and more attention. Compared with classical integer-order models, fractional-order models can describe reality more accurately, which has been shown recently in a variety of fields such as physics, chemistry, biology, economics, signal and image processing, control, porous media, aerodynamics, and so on see([1]-[8]).

In this paper, we study fractional diffusion equations with controls by the method of an abstract variational formulation. There has been a large and fast increasing literature on diffusion equations with time fractional derivatives [23].

An important obstacle to study solutions in fractional Sobolev spaces is that the Caputo derivative was not clearly defined when the first order derivative does not

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exist in the strong sense. In the recent work of Gorenflo et al. [14], they gave a definition of the Caputo derivative in fractional Sobolev space. Based on this reason in this paper we attempt to use a new definition of the weak fractional Caputo derivative via distribution theory and an integration by parts formula. This definition makes it very natural to adopt the theory of operational differential equations ([19]) and gives an abstract variational formulation of the fractional diffusion equation.

We will study this weak formulation with the classic Lax-Milgarm method and the integration by parts technique. We note that the integration by parts technique has been developed and extensively used in the theory of fractional calculus of variations, of which we refer to the monograph of Malinowska and Torres [18].

We obtain a solution in the functional setting of a fractional Sobolev space due to the following fact: the L^2- fractional derivative is the fractional power of the realization of a derivative in L^2?space [12]. For a detailed analysis and characterization of the fractional power of differential operator in the setting of Sobolev space, we refer to [12]-[13].

Using the fractional integration by parts formula, we can also construct the adjoint system to our variational (weak) formulation. By a classic result of convex analysis we can characterize the optimal control of a system of partial differential equations and inequalities, which can be applied to concrete fractional diffusion equations.

This paper is organized as follows. In section 2, we introduce some basic definitions for the weak Caputo fractional operators. In section 3, we formulate the fractional distributed Dirichlet problem. In section 4, we show that our fractional optimal control problem holds and gives the optimality system for the optimal control. In section 5, we formulate the fractional boundary Neumann problem. In section 6, the minimization problem is formulated and we state some illustrative examples. In section 7, we discuss the controllability of the fractional distributed Dirichlet problem with weak Caputo fractional derivatives. In section 8, we give the conclusion in the final section.

2. Preliminaries

Fractional differential equations have been studied by many investigators in recent years. The notion of fractional order (non-integer order) operator is much more recently improved. Different authors have presented different definitions of fractional order differential operators. The object of this section is to give the definition of some fractional integrals and fractional derivatives of function in the Riemann-Liouville sense and Caputo sense see [5]-[8]. Also we give a definition of fractional Hilbert spaces.

Let $n \in N^*$ where $(N^* = N \cup \{0\}, N$ is a set of natural numbers) and Ω be a bounded open subset of R^n with a smooth boundary Γ of class C^2. For a time $T > 0$, we set $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$.

Definition 1 Let $0 < \beta \leq 1$, $0 < t < T$, $f \in L^1(0, T)$, and Γ be the Gamma function. i.e.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Then

$$(J^\beta f)(t) = {}_0I_t^\beta f(t) = \int_0^t \frac{1}{\Gamma(\beta)} (t-s)^{\beta-1} f(s) ds,$$

is called the left Riemann-Liouville integral of fractional order β, while

$$ {}_tI_T^\beta f(t) = \int_t^T \frac{1}{\Gamma(\beta)}(s-t)^{\beta-1} f(s)ds, $$

is called the right Riemann-Liouville integral of fractional order β.

Definition 2 Let $0 < \beta < 1$, $0 < t < T$, ${}_0I_t^{1-\beta} f \in AC(0,T)$ (Admissible cone to the set $(0,T)$). Then the left Riemann-Liouville fractional derivative of fractional order β of f is defined by:

$$ {}_0D_t^\beta f(t) = \frac{\partial}{\partial t}\, {}_0I_t^{1-\beta} f(t) = \frac{\partial}{\partial t} \int_0^t \frac{1}{\Gamma(1-\beta)}(t-s)^{-\beta} f(s)ds, $$

while the right Riemann-Liouville fractional derivative of fractional order β of f such that ${}_tI_T^{1-\beta} f \in AC(0,T)$ is defined by:

$$ {}_tD_T^\beta f(t) = -\frac{\partial}{\partial t}\, {}_tI_T^{1-\beta} f(t) = \frac{\partial}{\partial t} \int_t^T \frac{-1}{\Gamma(1-\beta)}(s-t)^{-\beta} f(s)ds. $$

Definition 3 Let $0 < \beta < 1$, $0 < t < T$, ${}_0I_t^{1-\beta} f \in AC(0,T)$. Then the left Caputo fractional derivative of fractional order β of f is defined by:

$$ \partial_t^\beta f(t) = {}_0^CD_t^\beta f(t) = {}_0I_t^{1-\beta} \frac{\partial}{\partial t} f(t) = \int_0^t \frac{1}{\Gamma(1-\beta)}(t-s)^{-\beta} \frac{\partial}{\partial s} f(s)ds, $$

while the right Caputo fractional derivative of fractional order β of f such that ${}_tI_T^{1-\beta} f \in AC(0,T)$ is defined by:

$$ {}_t^CD_T^\beta f(t) = -{}_tI_T^{1-\beta} \frac{\partial}{\partial t} f(t) = \int_t^T \frac{-1}{\Gamma(1-\beta)}(s-t)^{-\beta} \frac{\partial}{\partial s} f(s)ds. $$

The Caputo fractional derivative is a sort of regularization in the time origin for the Riemann-Liouville fractional derivative.

From this definition we see that for ?$\partial_t^\beta f(t)$ to be well defined, ?$\frac{\partial}{\partial t} f(t)$ must be well defined. This is quite restrictive in applications, hence motivating our study of the weak Caputo derivative.

Definition 4 Let $0 < \beta < 1$. Let $g \in L^p(0,T), 1 \leq p \leq \infty$ and $\phi :]0,T] \mapsto R+$ be the function defined by: $\phi(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$.

Then for almost every $t \in [0,T]$, the function $s \to \phi(t-s)g(s)$ is integrable on $[0,T]$. Set

$$ \phi * g(t) = ?\int_0^t \phi(t-s)g(s)ds $$

.

Then $\phi * g(t) \in L^p(0,T)$ and

$$ ||\phi * g||_{L^p(0,T)} \leq ||\phi||_{L^1(0,T)}.||g||_{L^p(0,T)}. $$

When $f \in L^1(0,T)$, the left Riemann-Liouville integral can also be defined via a convolution [25]:

$$ (J^\beta f)(t) = (g^\beta * f)(t), $$

where

$$g^{\beta}(t) = \begin{cases} \frac{1}{\Gamma(\beta)} t^{\beta-1}, & t > 0; \\ 0, & t \leq 0. \end{cases}$$

We denote by $H^{\beta}(0,T)$ the fractional Sobolev space of order β on $(?0,T)$ and

$${}_0H^{\beta}(0,T) = \{y \in H^{\beta}(0,T) : y(0) = 0\}.$$

For details of these definitions we refer to ([19].

It has been verified in previous literature that the Riemann-Liouville integral operator is injective ([16], hence we can define a new operator $J^{-\beta}$ as the inverse operator of J^{β}. By definition, $D(J^{-\beta}) = R(J^{\beta})$.

Lemma 1 Let $T > 0, u \in C^m([0,T]), p \in (m-1,m), m \in N$ and $v \in C^1([0,T])$. Then for $t \in [0,T]$, the following properties hold

$${}_0D_t^p v(t) = \frac{d}{dt}\, {}_0I_t^{1-p} v(t), \quad m = 1,$$

$${}_0D_t^p\, {}_0I_t^p v(t) = v(t);$$

$${}_0I_t^p\, {}_0D_t^p u(t) = u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0);$$

$$\lim_{t\to 0^+} {}_0^C D_t^p u(t) = \lim_{t \to 0^+} {}_0I_t^p u(t) = 0.$$

Note also that when $T = +\infty$, ${}_0^C D_t^{\beta} f(t)$ is the Weyl fractional integral of order β of f'.

We use the following result regarding a solution to a fractional differential equation [16]:

$$\partial^{\beta} y = f, \quad f \in L^2(0,T)$$

is given by $y = J^{\beta} f$ with the range of this operator J^{β} being

$$R(J^{\beta}) = \begin{cases} H^{\beta}(0,T), & 0 \leq \beta < \frac{1}{2}, \\ {}_0H^{\beta}(0,T), & \frac{1}{2} < \beta \leq 1, \\ \{y \in H^{\frac{1}{2}}(0,T) : \int_0^T t^{-1}|y(t)|^2 dt < \infty\}, & \beta = \frac{1}{2}. \end{cases} \tag{1}$$

In the literature of interpolation theory, one sometimes denotes

$$[{}_0H^1(0,T), H^0(0,T)]_{\frac{1}{2}} = {}_0H_0^{\frac{1}{2}}(0,T) = \{y \in H^{\frac{1}{2}}(0,T) : t^{\frac{1}{2}} y \in L^2(0,T)\}$$

With the above definitions we have the following result.

Lemma 2 ([25] The norms $||J^{\beta}y||_{L^2(0,T)}$ and $||y||_{H^{\beta}(0,T)}$ are equivalent for $y \in R(J^{\beta})$.

We denote by $H^{\beta}(0,T;V,V')$ the vector valued fractional Sobolev space. $W^{\beta,p}(0,T;V,V')$ is the restriction on $(0,T)$ of $W^{\beta,p}(-\infty,\infty;V,V')$ given by the Fourier transform and

$$H^{\beta}(0,T;V,V') = W^{\beta,2}(0,T;V,V').$$

Next we give a lemma from [25], which gives the embedding between fractional Sobolev spaces and spaces of continuous functions, for vector spaces on the interval $(0,T)$.

Lemma 3 [25] Denote by $C_u(0,t;H)$ the space of uniformly continuous functions from $(0,T)$ into H. Suppose $\beta > 1/p \quad (?0 < \beta < 1, 1 < p \leq \infty)$. Then

$$W^{\beta,p}(0,T;V,V') \hookrightarrow C_u(0,t;H) \quad \text{with compact embedding.}$$

Lemma 4 [25] Let β, p and q satisfy
if $\beta > 1/p$ then $p \leq q \leq \infty$,
if $\beta = 1/p$ then $p \leq q < \infty$,
if $\beta < 1/p$ then $p \leq q \leq p^*$, where $\beta - 1/p = -1/p^*$, that is $p^* = p/(1-\beta p)(0 < \beta < 1, 1 \leq p \leq q \leq \infty$. Then

$$W^{\beta,p}(0,T;V,V') \hookrightarrow L^q(0,t;H) \quad \text{with compact embedding.}$$

In order to use the theory of operational differential equations, we need to interpret the weak Caputo derivative in the sense of distributions, through fractional integration by parts in the formula

$$\int_0^T (\partial_t^\beta y(t), \psi(t))dt = \int_0^T (y(t), {}_t^C D_T^\beta \psi(t))dt + [{}_t I_T^{1-\beta} \psi(t).y(t)]_0^T, \ \psi(t) \in \mathcal{D}(]0,T[). \tag{2}$$

For $y(0) = 0, \psi(T) = 0$, we have

$$[{}_t I_T^{1-\beta} \psi(t).y(t)]_0^T = 0.$$

Hence we can proceed to the construction of a weak Caputo derivative in the sense of distributions, note that if a distribution function is infinitely differentiable then its Caputo fractional derivative must also exist. It greatly simplifies the situation since we have the initial condition $y(0) = y_0 = 0$. We denote by $\mathcal{D}(]0,T[)$ the space of infinitely differentiable functions in $]0,T[$ with compact support.We call every continuous linear mapping of $\mathcal{D}(]0,T[)$ into E a vectorial distribution over $]0,T[$ with values in a Banach space E, and we denote

$$\mathcal{D}'(]0,T[;E) = \mathcal{L}(\mathcal{D}(]0,T[);E).$$

Definition 5 [25] Define the test function $\phi \in \mathcal{D}(]0,T[)$ for the function y such that $y(0) = 0$, we call $\partial_t^\beta y$ a distributional weak Caputo derivative if it is a linear functional on $\mathcal{D}(]0,T[)$ that sends ϕ into $\int_0^T (y, {}_t^C D_T^\beta \phi(t))dt$ i.e.

$$(\partial_t^\beta y)\phi(t) = \int_0^T (y, {}_t^C D_T^\beta \phi(t))dt$$

Our new definition of a weak Caputo derivative generalizes the (left) Caputo derivative (Definition 3) since it is well defined even when $\partial y/\partial s$ does not exist in the strong sense. It coincides with the Caputo derivative if $\partial y/\partial s$ does exist.

Lemma 5 ?[25] $\partial_t^\beta (y(.), v) =< \partial_t^\beta y(.), v >$ in $\mathcal{D}(]0,T[)$, for $y \in {}_0H^\beta(0,T;V,V')$, $v \in V$. Here $(,)$ denotes duality in H, $<,>$ denotes a duality pairing of V and V'. Moreover, the weak Caputo derivative $\partial_t^\beta y = J^{-\beta} y$ in $L^2(0,T)$? for $y \in \mathcal{R}(J^\beta)$.

Proof Denote the function $\phi \in \mathcal{D}(]0,T[)$. For all $t, \phi(t)$ is a scalar. We can write $v(t) = \phi(t)v$. Observe $y(t), v \in V \subset H$ and the duality $<,>$ is compatible with the identification of H with its dual. This implies

$$< v, y(t) >= (v, y(t)) = (y(t), v).$$

From Definition 5 and (2) we obtain

$$\int_0^T < \partial_t^\beta y(t), v > \phi(t)dt = \int_0^T < v, y(t) > {}_t^C D_T^\beta \phi(t)dt = \int_0^T (y(t), v) {}_t^C D_T^\beta \phi(t)dt$$
$$= \int_0^T \partial_t^\beta (y(t), v)\phi(t)dt,$$

hence ?$\partial_t^\beta (y(t), v) =< \partial_t^\beta y(t), v >$ in $\mathcal{D}(]0, T[)$.

Since the Sobolev space ${}_0H^2(0, T)$ is dense in $\mathcal{R}(J^\beta)$, for each $y \in \mathcal{R}(J^\beta)$ we can construct an approximating sequence ϕ_n such that

$$\lim_{n\to\infty} ?\phi_n = y, \ \phi_n \in {}_0H^2(0, T).$$

By the Hahn-Banach theorem we can uniquely extend the domain of linear operator $y \rightarrow ?\partial_t^\beta y$ from ${}_0H^2(0, T)$ to $\mathcal{R}(J^\beta)$. From [16] (Lemma 3.1) we know ?

$$\partial_t^\beta \phi_n = J^{-\beta}\phi_n, \quad \phi_n \in_0 H^2(0, T),$$

hence we obtain

$$\int_0^T (\partial_t^\beta y(t), \phi(t))dt = \int_0^T (J^{-\beta} y(t), \phi(t))dt, \quad y \in \mathcal{R}(J^\beta).$$

From Definition (5) and the fact that $D(]., T[) \subset L^2(0, T)$ we obtain the weak Caputo derivative ?$\partial_t^\beta y(t) = J^{-\beta}y(t)$ in $L^2(0, T)$ for $y \in \mathcal{R}(J^\beta)$.

From Lemma (2) and Lemma (5)?we obtain the following: suppose we have a sequence of approximating solutions $y_m \in \mathcal{R}(J^\beta)$; if we have a priori estimates independent of m, such that $y_m(t) \in L^2?(?0, T; V)$ and $\partial_t^\beta y_m(t) \in L^2(0, T; V')$, then we have $y_m(t) \in H^\beta(0, T; V, V')$.

Lemma 6 (Green's Theorem for fractional operators).

Let $0 < \beta \leq 1 - \frac{1}{n}$, $n \in N$. Then for any $\phi \in C^\infty(\overline{Q})$ we have

$$\int_0^T \int_\Omega (\partial_t^\beta y(x, t) + \mathcal{A}y(x, t))\phi(x, t)dxdt = \int_\Gamma y(x, T){}_tI_T^{1-\beta}\phi(x, T)d\Gamma -$$
$$\int_\Gamma y(x, 0){}_tI_T^{1-\beta}\phi(x, 0)d\Gamma - \int_0^T \int_\Gamma y(x, t)\frac{\partial \phi(x, t)}{\partial \nu_\mathcal{A}}d\Gamma dt \tag{3}$$
$$+ \int_0^T \int_\Gamma \frac{\partial y(x, t)}{\partial \nu_\mathcal{A}}\phi d\Gamma dt + \int_0^T \int_\Omega y(x, t)({}_t^C D_T^\beta \phi(x, t) + \mathcal{A}^*\phi(x, t))dxdt.$$

where $\mathcal{A}$ is a given operator which is defined by

$$\mathcal{A}y = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}\right) + a_0(x)y, \tag{4}$$

where $a_{ij}, i, j = 1, 2, ..., n$, be given function on Ω with the properties

$$a_0(x), a_{ij}(x) \in L^\infty(\Omega) \quad \text{(with real values)},$$
$$a_0(x) \geq \alpha > 0, \quad \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \alpha(\xi_1^2 + ... + \xi_n^2), \quad \forall \xi \in R^n,$$

almost everywhere on Ω and

$$\frac{\partial y}{\partial \nu_\mathcal{A}} = \sum_{i,j=1}^n a_{ij}\frac{\partial y}{\partial x_j}cos(n, x_j) \quad \text{on}\, \Gamma, \tag{5}$$

$cos(n, x_j)$ is the i-th direction cosine of n,n being the normal at Γ exterior to Ω.
Proof Let $\phi \in C^\infty(\overline{Q})$, we have

$$\int_a^b \int_\Omega ({}_a^C D_t^\beta y(x,t) + \mathcal{A}y(x,t))\phi(x,t)dxdt = \int_a^b \int_\Omega ({}_a^C D_t^\beta y(x,t))\phi(x,t)dxdt + \int_a^b \int_\Omega \mathcal{A}y(x,t)\phi(x,t)dxdt \tag{6}$$

we have:

$$\int_a^b \int_\Omega \mathcal{A}y(x,t)\phi(x,t)dxdt = -\int_a^b \int_\Gamma y(x,t)\frac{\partial \phi(x,t)}{\partial \nu_{\mathcal{A}}}d\Gamma dt + \int_a^b \int_\Gamma \frac{\partial y(x,t)}{\partial \nu_{\mathcal{A}}}\phi(x,t)d\Gamma dt + \int_a^b \int_\Omega y(x,t)\mathcal{A}^*\phi(x,t)dxdt. \tag{7}$$

$$\begin{aligned} \int_a^b \int_\Omega ({}_a^C D_t^\beta y(x,t))\phi(x,t)dxdt &= \int_a^b \int_\Omega y(x,t)({}_t D_b^\beta \phi(x,t))dxdt + \\ \int_a^b \int_\Gamma y(x,t)({}_t I_b^{1-\beta}\phi(x,t))d\Gamma dt &= \int_a^b \int_\Omega y(x,t)({}_t D_b^\beta \phi(x,t))dxdt \\ &+ \int_\Gamma y(x,b){}_t I_b^{1-\beta}\phi(x,b)d\Gamma - \int_\Gamma y(x,a){}_t I_b^{1-\beta}\phi(x,a)d\Gamma \end{aligned} \tag{8}$$

substitute from (8) and (7) into (6) we deduce (3), which completes the proof.
Definition 6 We also introduce the space

$$\mathcal{W}(0,T) := \{y : y \in L^2(0,T;\, {}_0H^\beta(\Omega)), \partial_t^\beta y(x,t) \in L^2(0,T;\, {}_0H^{-\beta}(\Omega))\}$$

in which a solution of a differential systems is contained. The spaces considered in this paper are assumed to be real.

3. Fractional Dirichlet problem with weak Caputo derivative

Let us consider the fractional partial differential system:

$$\partial_t^\beta y(x,t) + \mathcal{A}y(t) = f(t), \quad t \in [0,T], \tag{9}$$

$$y(x,0) = y_0, \quad x \in \Omega, \tag{10}$$

$$y(x,t) = 0, \quad x \in \Gamma, t \in (0,T), \tag{11}$$

where $\frac{1}{n} < \beta < 1$, $n \in N$, $y_0 \in H^2(\Omega) \bigcap H_0^1(\Omega)$, the function f belongs to $L^2(Q)$. The fractional derivative $\partial_t^\beta y(t)$ is understood here in the weak Caputo sense (Definition 5), Ω has the same properties as in section 2. The operator $\mathcal{A}$ in the state equation (9) is a second order operator given by (4) and $\mathcal{A} \in \mathcal{L}\bigg(H_0^1(\Omega), H_0^{-1}(\Omega)\bigg)$.

For this operator we define the bilinear form as follows:
Definition 7 For each $t \in]0,T[$, we define a family of bilinear forms $\pi(t; y, \phi)$ on $H_0^1(\Omega)$ by:

$$\pi(t; y, \phi) = (\mathcal{A}y, \phi)_{L^2(\Omega)}, \quad y,\, \phi \in H_0^1(\Omega), \tag{12}$$

where $\mathcal{A}$ maps $H_0^1(\Omega)$ onto $H_0^{-1}(\Omega)$ and takes the form (6). Then

$$\begin{aligned}\pi(t;y,\phi) &= \Big(\mathcal{A}y,\phi\Big)_{L^2(\Omega)}\\ &= \Big(-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial y}{\partial x_j}\Big)+a_0(x)y,\phi(x)\Big)_{L^2(\Omega)}\\ &= \int_\Omega\sum_{i,j=1}^{n}a_{ij}\frac{\partial}{\partial x_i}y(x)\frac{\partial}{\partial x_j}\phi(x)dx+\int_\Omega a_0(x)y(x)\phi(x)dx.\end{aligned}$$

Lemma 7 The bilinear form $\pi(t;y,\phi)$ is coercive on $H_0^1(\Omega)$ that is

$$\pi(t;y,y)\geq\lambda\,||y||^2_{H_0^1(\Omega)},\quad \lambda>0. \tag{13}$$

Proof It is well known that the ellipticity of $\mathcal{A}$ is sufficient for the coerciveness of $\pi(t;y,\phi)$ on $H_0^1(\Omega)$.

Since

$$\pi(t;y,\phi)=\int_\Omega\sum_{i,j=1}^{n}a_{ij}\frac{\partial}{\partial x_i}y(x)\frac{\partial}{\partial x_j}\phi(x)dx+\int_\Omega a_0(x)y(x)\phi(x)dx,$$

then we get

$$\begin{aligned}\pi(t;y,y) &= \int_\Omega\sum_{i,j=1}^{n}a_{ij}\frac{\partial}{\partial x_i}y(x)\frac{\partial}{\partial x_j}y(x)dx+\int_\Omega a_0(x)y(x)y(x)dx\\ &= \sum_{i,j=1}^{n}a_{ij}||\frac{\partial}{\partial x_i}y(x)||^2_{L^2(\Omega)}+||y(x)||^2_{L^2(\Omega)}\\ &\geq\lambda||y||^2_{H_0^1(\Omega)},\ \lambda>0.\end{aligned}$$

Lemma 8 Also we assume that $\forall y,\phi\in H_0^1(\Omega)$ the function $t\to\pi(t;y,\phi)$ is continuously differentiable in $]0,T[$ and the bilinear form $\pi(t;y,\phi)$ is symmetric,

$$\pi(t;y,\phi)=\pi(t;\phi,y)\qquad\forall y,\phi\in H_0^1(\Omega). \tag{14}$$

The equations (9)-(11) constitute a fractional Dirichlet problem. First by using the Lax-Milgram lemma, we prove sufficient conditions for the existence of a unique solution of the mixed initial-boundary value problem (9)-(11).

Lemma 9 (see [6]-[8]) (Fractional Green's formula). Let y be the solution of system (9)-(11). Then for any $\phi\in C^\infty(\overline{Q})$ such that $\phi(x,T)=0$ in Ω and $\phi=0$ on Σ, we have

$$\begin{aligned}\int_0^T\int_\Omega(\partial_t^\beta y(x,t)+\mathcal{A}y(x,t))\phi(x,t)dxdt &= -\int_\Gamma y(x,0)\,{}_tI_T^{1-\beta}\phi(x,0)d\Gamma\\ &-\int_0^T\int_\Gamma y(x,t)\frac{\partial\phi(x,t)}{\partial\nu_{\mathcal{A}}}d\Gamma dt+\int_0^T\int_\Gamma\frac{\partial y(x,t)}{\partial\nu_{\mathcal{A}}}\phi d\Gamma dt\\ &+\int_0^T\int_\Omega y(x,t)({}_t^CD_T^\beta\phi(x,t)+\mathcal{A}^*\phi(x,t))dxdt.\end{aligned} \tag{15}$$

Definition 8 ?We define the variational fractional equation (we also call it a fractional operational differential equation). Suppose $f \in L^2(?0,T;V')$,

$$\begin{cases} \partial_t^\beta(y(t),\phi)+\pi(t;y(t),\phi)=L(\phi), & \text{in } \mathcal{D}'(]?0,T[), t\in(0,T], \forall\phi\in V, \\ y_0=0. \end{cases} \tag{16}$$

Here $\partial_t^\beta y(t)$ is defined in the weak sense (Definition 5).

From Lemma ?(5) we see that the first equation of (3) is equivalent to

$$\partial_t^\beta y+A(t)y=f \quad \text{in the sense of } L^2?(0,T;V'), t\in(?0,T]. \tag{17}$$

Definition 9 y is a (distributional) weak solution to system (9)-(11); it satisfies (3) with

$$y\in\begin{cases} H^\beta(0,T;V,V'), & 0\le\beta<\frac{1}{2}, \\ {}_0H^\beta(0,T;V,V'), & \frac{1}{2}<\beta\le 1, \\ \{y\in H^{\frac{1}{2}}(0,T;V,V'): \int_0^T t^{-1}|y(t)|^2dt<\infty\}, & \beta=\frac{1}{2}. \end{cases} \tag{18}$$

and $V=H_0^1(\Omega)$.

Lemma 10 If and hold, then the problem admits a unique solution $y\in\mathcal{W}(0,T)$.

Proof Uniqueness. Suppose there exist two different solutions y_1 and $y?_2, y_3 = y_1-y_2$, then

$$?\partial_t^\beta y_3+A(t)y_3=0,$$

$$\int_0^T(\partial_t^\beta y_3(t),y_3(t))dt+\int_0^T\pi(t;y_3(t),y_3(t))dt=0$$

Suppose $y_3(t)\in V\subset H$, we know the following inequality (inequality (3.1)([25]);

$$\int_0^T(\frac{d}{dt}(g^{1-\beta}*y_3(t)),y_3(t))_H dt\ge g^{1-\beta}(T)\int_0^T||y_3(t)||_H^2dt.$$

Since y is a weak solution hence by Definition (3) and Lemma (5) we have ?

$$\partial_t^\beta y_3(t)=\frac{d}{dt}(J^{1-\beta}y_3(t))=\frac{d}{dt}(g^{1-\beta}*y_3(t)).$$

Hence we have

$$\int_0^T(\partial_t^\beta y_3(t),y_3(t))dt\ge g^{1-\beta}(T)\int_0^T||y_3(t)||_H^2dt$$

and from condition (A3) we obtain

$$g^{1-\beta}(T)\int_0^T||y_3(t)||_H^2dt+\theta\int_0^T||y_3(t)||_H^2dt\le 0,$$

hence $||y_3(t)||_V=0$ and the solution to is unique.

Existence. From the coerciveness condition and using the Lax-Milgram lemma, there exists a unique element $y(t)\in H_0^1(\Omega)$ such that

$$(\partial_t^\beta y(t),\phi)_{L^2(Q)}+\pi(t;y,\phi)=L(\phi) \quad \text{for all} \quad \phi\in H_0^1(\Omega), \tag{19}$$

which is equivalent to there exists a unique solution $y(t)\in H_0^1(\Omega)$ for

$$(\partial_t^\beta y(t),\phi)_{L^2(Q)}+(\mathcal{A}y(t),\phi)_{L^2(Q)}=L(\phi) \quad \text{for all} \quad \phi\in H_0^1(\Omega),$$

i.e. for

$$(\partial_t^\beta y(t) + \mathcal{A}y(t), \phi(x))_{L^2(Q)} = L(\phi),$$

which can be written as

$$\int_Q (\partial_t^\beta y(t) + \mathcal{A}y(t))\phi(x)dxdt = L(\phi) \quad \text{for all} \quad \phi \in H_0^1(\Omega). \tag{20}$$

This is know as the variational fractional Dirichlet problem, where $L(\phi)$ is a continuous linear form on $H_0^1(\Omega)$ and takes the form

$$L(\phi) = \int_Q f\phi\, dxdt - \int_\Gamma y_0\, {}_tI_T^{1-\beta}\phi(x,0)d\Gamma, \quad f \in L^2(Q), y_0 \in L^2(\Omega). \tag{21}$$

Then equation (20) is equivalent to

$$\int_Q (\partial_t^\beta y(t) + \mathcal{A}y(t))\phi(x)dxdt = \int_Q f\phi\, dxdt - \int_\Gamma y_0\, {}_tI_T^{1-\beta}\phi(x,0)d\Gamma \quad \text{for all} \quad \phi \in H_0^1(\Omega), \tag{22}$$

that is, the FPDE

$$\partial_t^\beta y(t) + \mathcal{A}y(t) = f,$$

"tested" against $\phi(x)$.

Applying Green's formula (Lemma 9) to equation (22), we have

$$-\int_\Gamma y(x,0)_t I_T^{1-\beta}\phi(x,0)d\Gamma - \int_0^T \int_{\partial\Omega} y\frac{\partial\phi}{\partial\nu}d\Gamma dt + \int_0^T \int_{\partial\Omega} \frac{\partial y}{\partial\nu}\phi d\Gamma dt$$
$$+ \int_0^T \int_\Omega y(x,t)({}_t^C D_T^\beta\phi(x,t) + \mathcal{A}^*\phi(x,t))dxdt = \int_Q f\phi\, dxdt - \int_\Gamma y_0\, {}_tI_T^{1-\beta}\phi(x,0)d\Gamma$$
$$\int_\Gamma y(x,0)_t I_T^{1-\beta}\phi(x,0)d\Gamma = \int_\Gamma y_0\, {}_tI_T^{1-\beta}\phi(x,0)d\Gamma$$

Then for any $\phi \in C^\infty(\overline{Q})$ such that $\phi(x,T) = 0$ in Ω and $\phi = 0$ on Σ, we deduce (10) and (11).

4. Fractional optimal control problem of variational formulation

For a control $u \in L^2(Q)$ the state $y(u)$ of the system is given by

$$\partial_t^\beta y(u) + \mathcal{A}y(u) = u, \quad (x,t) \in Q \tag{23}$$
$$y(u)|_\Sigma = 0, \tag{24}$$
$$y(x,0;u) = y_0(x), \quad x \in \Omega. \tag{25}$$

The observation equation is given by

$$z(u) = y(u), \tag{26}$$

The cost function $J(v)$ is given by

$$J(v) = \int_Q (y(v) - z_d)^2 dxdt + (Nv, v)_{L^2(Q)}$$

where z_d is a given element in $L^2(Q)$ and $N \in \mathcal{L}(L^2(Q), L^2(Q))$ is hermitian positive definite operator:

$$(Nu, u) \geq c||u||^2_{L^2(Q)},\, c > 0. \tag{27}$$

Control Constraints: We define U_{ad}(set of admissible controls) as closed, convex subset of $U = L^2(Q)$. **Control Problem:** We want to minimize J over U_{ad} i.e. find u such that

$$J(u) = \inf_{v \in U_{ad}} J(v). \tag{28}$$

Under the given considerations we have the following theorem:

Theorem 1 The problem (28) admits a unique solution given by (23)-(25) and

$$\int_Q (p(u) + Nu)(v - u)\, dxdt \geq 0, \tag{29}$$

where $p(u)$ is the adjoint state.

Proof Since the control $u \in U_{ad}$ is optimal if and only if

$$J'(u)(v-u) \geq 0 \quad \text{for all}\, v \in U_{ad}$$

The above condition, when explicitly calculated for this case, gives

$$(y(u) - z_d, y(v) - y(u))_{L^2(Q)} + (Nu, v-u)_{L^2(Q)} \geq 0$$

i.e.

$$\int_Q (y(u) - z_d)(y(v) - y(u))dxdt + (Nu, v-u)_{L^2(Q)} \geq 0. \tag{30}$$

For the control $u \in L^2(Q)$ the adjoint state $p(u) \in L^2(Q)$ is defined by

$${}_t^C D_T^\beta p(u) + \mathcal{A}^* p(u) = y(u) - z_d, \quad \text{in}\, Q, \tag{31}$$

$$p(u) = 0, \qquad \text{on}\, \Sigma, \tag{32}$$

$$p(x, T; u) = 0, \qquad \text{in}\, \Omega, \tag{33}$$

where $\mathcal{A}^*$ is the adjoint operator for the operator $\mathcal{A}$, which is given by

$$\mathcal{A}^* p = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial p}{\partial x_i}\right) + a_0(x)p.$$

Now, multiplying the equation (31) by $(y(v) - y(u))$ and applying Green's formula,

$$\begin{aligned}\int_0^T \int_\Omega y(x,t)({}_t^C D_T^\beta \phi(x,t) + \mathcal{A}^*\phi(x,t))dxdt = &\int_\Gamma y(x,0)\,{}_t I_T^{1-\beta}\phi(x,0)d\Gamma \\ &+ \int_0^T \int_\Gamma y(x,t)\frac{\partial \phi(x,t)}{\partial \nu_{\mathcal{A}}} d\Gamma dt \\ &- \int_0^T \int_\Gamma \frac{\partial y(x,t)}{\partial \nu_{\mathcal{A}}} \phi d\Gamma dt \\ &+ \int_0^T \int_\Omega ({}_0^C D_t^\beta y(x,t) + \mathcal{A}y(x,t))\phi(x,t)dxdt,\end{aligned} \tag{34}$$

we obtain

$$\begin{aligned}\int_Q (y(u)-z_d)(y(v)-y(u))dxdt &= \int_Q ({}_t^C D_T^\beta p(u) + \mathcal{A}^* p(u))(y(v)-y(u))\,dxdt \\ &= \int_\Gamma (y(v;x,0)-y(u;x,0))_t I_T^{1-\beta} p(x,0) d\Gamma \\ &- \int_\Sigma p(u)(\frac{\partial y(v)}{\partial \nu_{\mathcal{A}}} - \frac{\partial y(u)}{\partial \nu_{\mathcal{A}}})\,d\Sigma \\ &+ \int_\Sigma \frac{\partial p(u)}{\partial \nu_{\mathcal{A}}}(y(v)-y(u))\,d\Sigma \\ &+ \int_Q p(u)({}_0^C D_t^\beta + \mathcal{A})(y(v)-y(u))\,dxdt.\end{aligned}$$

Since from (23), (24) we have

$$(\partial_t^\beta + \mathcal{A})(y(v)-y(u)) = v-u, \quad y(u)|_\Sigma = 0, \quad p(u)|_\Sigma = 0.$$

Then we obtain

$$\int_Q (y(u)-z_d)(y(v)-y(u))dxdt = \int_Q p(u)(v-u)dxdt,$$

and hence (30) is equivalent to

$$\int_Q p(u)(v-u)\,dxdt + (Nu, v-u)_{L^2(Q)} \geq 0$$

i.e.

$$\int_Q (p(u)+Nu)(v-u)\,dxdt \geq 0$$

which completes the proof.

Example 1 In the case of no constraints on the control ($\mathcal{U}_{ad} = \mathcal{U}$). Then (29) reduces to

$$p + Nu = 0 \quad \text{in}\, Q$$

The optimal control is obtained by the simultaneous solution of the following system of fractional partial differential equations:

$$\begin{gathered}\partial_t^\beta y + \mathcal{A} y = f - N^{-1} p, \quad {}_t^C D_T^\beta p + \mathcal{A}^* p = y - z_d \quad \text{in}\, Q, \\ y = 0, \quad p(u) = 0 \quad \text{on}\, \Sigma, \\ y(x,0) = y_0(x), \quad p(x,T) = 0 \quad x \in \Omega,\end{gathered}$$

further

$$u = -N^{-1} p \quad in\, Q.$$

Example 2 We consider the fractional diffusion equation with weak Caputo fractional derivatives:

$$\partial_t^\beta y(t) - \Delta y(t) = v, \quad t \in [0,T] \tag{35}$$

$$y(0) = y_0, \quad x \in \Omega, \tag{36}$$

$$y(x,t) = 0, \quad x \in \Gamma, t \in (0,T), \tag{37}$$

where $y_0 \in H^2(\Omega) \cap H_0^1(\Omega), \Delta$ is the Laplace operator and the control v belongs to $L^2(Q)$. We can minimize

$$J(v) = ||y(v) - z_d||^2_{L^2(Q)} + N||v||^2_{L^2(Q)}, \quad z_d \in L^2(Q), N > 0 \tag{38}$$

subject to system (35)-(37) and the optimal control v will be characterized by system (35)-(37) with the adjoint system

$$ {}_t^C D_T^{\beta} p(t) - \Delta p(t) = y - z_d, \quad t \in \ [0, T] \tag{39}$$

$$ p(x,t) = 0, \quad x \in \Omega, t \in (0,T) \tag{40}$$

$$ p(x,T) = 0, \quad x \in \Gamma, \tag{41}$$

and with the optimality condition

$$ u = -N^{-1} p \quad in\, Q. \tag{42}$$

Example 3 We can also consider the fractional diffusion equation with weak Riemann-Liouville fractional derivatives:

$$ \partial_t^{\beta} y(t) - \Delta y(t) = v, \quad t \in \ [0, T] \tag{43}$$

$$ {}_0 I_T^{1-\beta(t)} y(0^+) = y_0, \quad x \in \Omega, \tag{44}$$

$$ y(x,t) = 0, \quad x \in \Gamma, t \in (0,T), \tag{45}$$

where ${}_0 I_T^{1-\beta(t)} y(0^+) = \lim_{t \to 0^+} {}_0 I_T^{1-\beta(t)} y(t)$, the control v belongs to $L^2(Q)$. We can minimize (3.16) subject to system (43)-(45) and the optimal control v will be characterized by system (43)-(45) with the adjoint system (39)-(41) and with the optimality condition (42).

5. Fractional Neumann control problem

Since $H_0^1(\Omega) \subset H^1(\Omega)$ we can show that the bilinear form (12) is coercive in $H^1(\Omega)$ that is

$$ \pi(y,y) \geq c||y||^2_{H^1(\Omega)}, \quad c > 0 \quad \text{for all} \quad y \in H^1(\Omega). \tag{46}$$

From the above coerciveness condition (46) and using the Lax-Milgram lemma we have the following lemma which define the fractional Neumann problem for the operator $\mathcal{A}$ with $\mathcal{A} \in \mathcal{L}(H^1(\Omega), H^{-1}(\Omega))$ and enables us to obtain the state of our control problem.

Lemma 11 If (46) is satisfied then there exists a unique element $y \in H^1(\Omega)$ satisfying the fractional Neumann problem with β is a constant that is non-integer.

$$ \partial_t^{\beta} y + \mathcal{A} y = f \quad \text{in}\, Q, \tag{47}$$

$$ \frac{\partial y}{\partial \nu_{\mathcal{A}}} = h \qquad \text{on}\, \Sigma, \tag{48}$$

$$ y(x,0) = y_0(x), x \in \Gamma, \tag{49}$$

Proof From the coerciveness condition (46) and using the Lax-Milgram lemma, there exists a unique element $y \in H^1(\Omega)$ such that

$$ \int_Q y(x,t) ({}_t^C D_T^{\beta} \psi(x,t) + \mathcal{A}^* \psi(x,t)) dxdt = M(\psi) \quad \text{for all} \quad \psi \in H^1(\Omega). \tag{50}$$

This know as the fractional Neumann problem, where $M(\psi)$ is a continuous linear form on $H^1(\Omega)$ and takes the form

$$M(\psi) = \int_Q f\psi \, dxdt + \int_\Gamma y_0(x) \, {}_tI_T^{1-\beta}\psi(x,0)d\Gamma - \int_\Sigma h\frac{\partial \psi}{\partial \nu_{\mathcal{A}^*}}d\Sigma, \tag{51}$$

$$f \in L^2(Q), y_0 \in L^2(\Omega), h \in H^1(\Sigma).$$

The equation (50) is equivalent to

$$\partial_t^\beta y(x,t) + \mathcal{A}y(x,t) = f \quad \text{on}\, Q. \tag{52}$$

Let us multiply both sides in (52) by $\psi(x,t)$ such that $\frac{\partial \psi(x,t)}{\partial \nu_{\mathcal{A}}} = 0$ on Γ, and applying Green's formula, we have

$$\begin{aligned}\int_0^T \int_\Omega (\partial_t^\beta y(x,t) + \mathcal{A}y(x,t))\psi(x,t)dxdt = &- \int_\Gamma y(x,0)_tI_T^{1-\beta}\psi(x,0)d\Gamma \\ &+ \int_0^T \int_\Gamma \frac{\partial y(x,t)}{\partial \nu_{\mathcal{A}}}\psi d\Gamma dt \\ &+ \int_0^T \int_\Omega y(x,t)({}_t^C D_T^\beta \psi(x,t) + \mathcal{A}^*\psi(x,t))dxdt \\ &= \int_Q f\psi(x,t)dxdt,\end{aligned} \tag{53}$$

whence comparing with (50), (51)

$$\int_\Gamma y(x,0)_tI_T^{1-\beta}\psi(x,0)d\Gamma - \int_0^T \int_\Gamma \frac{\partial y(x,t)}{\partial \nu_{\mathcal{A}}}\psi d\Gamma dt = \int_\Gamma y_0(x)_tI_T^{1-\beta}\psi(x,0)d\Gamma - \int_0^T \int_{\partial\Omega} h\psi d\Gamma dt. \tag{54}$$

From this we deduce (48) and (49).

6. Fractional boundary control problem

We consider the space $U = L^2(\Sigma)$ (the space of controls), for every control $u \in U$, the state of the system $y(u) \in H^1(\Omega)$ is given by the solution of

$$\partial_t^\beta y(u) + \mathcal{A}y(u) = f \quad \text{in}\, Q, \tag{55}$$

$$\frac{\partial y(u)}{\partial \nu_{\mathcal{A}}} = u \quad \text{on}\, \Sigma, \tag{56}$$

$$y(x,0;u) = y_0(x), x \in \Omega. \tag{57}$$

For the observation, we consider the following two cases:

(i)

$$z(u) = y(u) \tag{58}$$

(ii) observation of final state

$$z(u) = y(x,T;u) \tag{59}$$

Case (i)

The cost function is given by

$$J(v) = \int_Q (y(v) - z_d)^2 \, dxdt + (Nv, v)_{L^2(\Sigma)}, \quad z_d \in L^2(Q), \tag{60}$$

where $N \in \mathcal{L}(L^2(\Sigma), L^2(\Sigma)), N$ is hermitian positive definite

$$(Nu, u)_{L^2(\Sigma)} \geq c||u||^2_{L^2(\Sigma)}, c > 0. \tag{61}$$

Control Constraints: We define U_{ad}(set of admissible controls) as closed, convex subset of $U = L^2(\Sigma)$. **Control Problem:** We wish to find

$$\inf_{v \in U_{ad}} J(v). \tag{62}$$

Under the given considerations we have the following theorem.
Theorem 2 Assume that (61) holds and the cost function being given by (60). The optimal control u is characterized by (55), (56), and (57) together with

$${}_t^C D_T^\beta p(u) + \mathcal{A}^* p(u) = y(u) - z_d \quad \text{in}\, Q, \tag{63}$$

$$\frac{\partial p(u)}{\partial \nu_{\mathcal{A}^*}} = 0 \qquad \text{on}\, \Sigma, \tag{64}$$

$$p(x, T; u) = 0, \quad x \in \Omega, \tag{65}$$

and the optimality condition is

$$\int_\Sigma (p(u) + Nu)(v - u) d\Sigma \geq 0 \quad \forall\, v \in U_{ad} \tag{66}$$

where $p(u)$ is the adjoint state.
Proof Since the control $u \in U_{ad}$ is optimal if and only if

$$J'(u)(v - u) \geq 0 \quad \forall v \in U_{ad} \tag{67}$$

that is

$$\Big(y(u) - z_d, y(v) - y(u)\Big)_{L^2(Q)} + (Nu, v - u)_U \geq 0. \tag{68}$$

The adjoint state is given by the solution of the adjoint Neumann problem (63), (64) and (65). Now, multiplying the equation in (63) by $y(v) - y(u)$ and applying Green's formula, with taking into account the conditions in (55), (56), we obtain

$$\begin{aligned}
\int_Q (y(u) - z_d)(y(v) - y(u))\, dxdt &= \int_Q ({}_t^C D_T^\beta p(u) + \mathcal{A}^* p(u))(y(v) - y(u))\, dxdt \\
&= -\int_\Omega p(x, 0)_t I_T^{1-\beta}(y(v; x, 0) - y(u; x, 0)) dx \\
&+ \int_\Sigma p(u)(\frac{\partial}{\partial \nu_{\mathcal{A}}} y(v) - \frac{\partial}{\partial \nu_{\mathcal{A}}} y(u))\, d\Sigma \\
&- \int_\Sigma \frac{\partial}{\partial \nu_{\mathcal{A}^*}} p(u)(y(v) - y(u))\, d\Sigma \\
&+ \int_Q p(u)((\partial_t^\beta + \mathcal{A})(y(v) - y(u))\, dxdt \\
&= \int_\Sigma p(u)(v - u)\, d\Sigma.
\end{aligned} \tag{69}$$

Hence we substitute from (69) in (68), to get

$$\int_{\Sigma} p(u)(v-u)\,d\Sigma + (Nu, v-u)_{L^2(\Sigma)} \geq 0$$

i.e.

$$\int_{\Sigma} (p(u) + Nu)(v-u) d\Sigma \geq 0 \quad \forall\, v \in U_{ad}$$

which completes the proof.

Example 4

In the case of no constraints on the control ($\mathcal{U}_{ad} = \mathcal{U}$). Then (66) reduces to

$$p + Nu = 0 \quad \text{on}\, \Sigma.$$

The optimal control is obtained by the simultaneous solution of the following system of fractional partial differential equations:

$$\begin{gathered} \partial_t^{\beta} y + \mathcal{A}y = f, \quad {}_t^C D_T^{\beta} p + \mathcal{A}^* p = y - z_d \quad \text{in}\, Q, \\ \frac{\partial y}{\partial \nu_{\mathcal{A}}}|_{\Sigma} + N^{-1} p|_{\Sigma} = 0, \quad \frac{\partial p}{\partial \nu_{\mathcal{A}^*}} = 0 \quad \text{on}\, \Sigma, \\ y(x,0) = y_0(x), \quad p(x,T) = 0 \quad x \in \Omega, \end{gathered}$$

further

$$u = -N^{-1}(P|_{\Sigma}).$$

Example 5 If we take

$$\mathcal{U}_{ad} = \left\{ u | u \in L^2(\Sigma), u \geq 0 \quad \text{almost everywhere on} \quad \Sigma \right\}.$$

The optimal control is obtained by the solution of the fractional problem

$$\begin{gathered} \partial_t^{\beta} y + \mathcal{A}y = f, \quad {}_t^C D_T^{\beta} p + \mathcal{A}^* p = y - z_d \quad \text{in}\, Q, \\ \frac{\partial y}{\partial \nu_{\mathcal{A}}} \geq 0, \quad \frac{\partial p}{\partial \nu_{\mathcal{A}}^*} = 0 \quad \text{on}\, \Sigma, \\ p + N\frac{\partial y}{\partial \nu_{\mathcal{A}}} \geq 0, \quad \frac{\partial y}{\partial \nu_{\mathcal{A}}}[p + N\frac{\partial y}{\partial \nu_{\mathcal{A}}}] = 0 \quad \text{on}\, \Sigma, \\ y(x,0) = y_0(x), \quad p(x,T) = 0 \quad x \in \Omega, \end{gathered}$$

hence

$$u = \frac{\partial y}{\partial \nu_{\mathcal{A}}}|_{\Sigma}.$$

Case (ii) observation of final state

$$z(u) = y(x,T;u).$$

The cost function is given by

$$J(v) = \int_{\Omega} (y(x,T;v) - z_d)^2\, dx + (Nv, v)_{L^2(\Sigma)}, \quad z_d \in L^2(\Omega).$$

The adjoint state is defined by

$$\begin{aligned} {}_t^C D_T^{\beta} p(u) + \mathcal{A}^* p(u) = 0 \quad &\text{in}\, Q, \\ \frac{\partial p(u)}{\partial \nu_{\mathcal{A}^*}} = 0 \qquad\qquad &\text{on}\, \Sigma, \end{aligned}$$

$$p(x,T;u) = y(x,T;u) - z_d(x), \quad x \in \Omega,$$

and the optimality condition is

$$\int_\Sigma (p + Nu)(v - u) d\Sigma \geq 0 \quad \forall\, v \in U_{ad}, \tag{70}$$

where $p(u)$ is the adjoint state.

Example 6 In the case of no constraints on the control ($\mathcal{U}_{ad} = \mathcal{U}$). Then (70) reduces to

$$p + Nu = 0 \quad \text{on}\,\Sigma.$$

The optimal control is obtained by the simultaneous solution of the following system of variable order fractional differential systems

$$\partial_t^\beta y + \mathcal{A} y = f, \quad {}_t^C D_T^\beta p + \mathcal{A}^* p = 0 \,\text{in}\, Q,$$
$$\frac{\partial y}{\partial \nu_{\mathcal{A}}}|_\Sigma + N^{-1} p|_\Sigma = 0, \quad \frac{\partial p}{\partial \nu_{\mathcal{A}}^*} = 0 \quad \text{on}\,\Sigma,$$
$$y(x,0) = y_0(x), \quad p(x,T) = y(x,T;u) - z_d(x) \quad x \in \Omega,$$

further

$$u = -N^{-1}(P|_\Sigma).$$

Example 7 If we take

$$\mathcal{U}_{ad} = \left\{ u | u \in L^2(\Sigma), u \geq 0 \quad \text{almost everywhere on} \quad \Sigma \right\}.$$

Then (70) is equivalent to

$$u \geq 0, \quad p(u) + Nu \geq 0, \quad u(p(u) + Nu) = 0 \quad \text{on } \Sigma.$$

Example 8 We consider the Neumann fractional diffusion equation with weak Caputo fractional derivatives:

$$\partial_t^\beta y(u) - \Delta y(u) = f, \quad t \in\, [0,T] \tag{71}$$

$$\frac{\partial y}{\partial \nu} y(u) = u, \quad x \in \Omega, \tag{72}$$

$$y(x,t;u) = 0, \quad x \in \Gamma, t \in (0,T), \tag{73}$$

We can minimize

$$J(v) = ||y(v) - z_d||^2_{L^2(Q)} + N||v||^2_{L^2(Q)}, \quad z_d \in L^2(Q), N > 0 \tag{74}$$

subject to system (71)-(73) and the optimal control v will be characterized by system (71)-(73) with the adjoint system

$${}_t^C D_T^\beta p(t) - \Delta p(t) = y - z_d, \quad t \in\, [0,T] \tag{75}$$

$$\frac{\partial y}{\partial \nu} p(x,t) = 0, \quad x \in \Omega, t \in (0,T) \tag{76}$$

$$p(x,T) = 0, \quad x \in \Gamma, \tag{77}$$

and with the optimality condition

$$u = -N^{-1} p \quad in\, Q. \tag{78}$$

Example 9 We can also consider the Neumann fractional diffusion equation with weak Riemann-Liouville fractional derivatives:

$$\partial_t^\beta y(t) - \Delta y(t) = v, \quad t \in [0,T] \tag{79}$$

$$_0I_T^{1-\beta(t)} y(0^+) = y_0, \quad x \in \Omega, \tag{80}$$

$$\frac{\partial y}{\partial \nu} y(x,t) = 0, \quad x \in \Gamma, t \in (0,T), \tag{81}$$

We can minimize the cost function (74) subject to system (79)-(81) and the optimal control v will be characterized by system (79)-(81) with the adjoint system (75)-(77) and with the optimality condition (78).

7. Controllability of fractional problem

This section is devoted to study the controllability of the fractional differential system (23),(24), and (25). We begin by the following definition.

Definition 10 [[19]] The system whose state is defined by (23),(24), and (25) is said to be controllable if as u is varied without any constraints, the observation $Cy(u)$ generates a dense (affine) subspace of the space of observations.

Let us consider the the case of section 4. Hence for a control $u \in L^2(Q)$ the state of the system $y(u)$ is given by

$$\partial_t^\beta y(u) + \mathcal{A} y(u) = u, \quad (x,t) \in Q$$

$$y(u)|_\Sigma = 0,$$

$$y(x,0;u) = y_0(x), \quad x \in \Omega.$$

The observation $y(y)$ is in $L^2(Q)$ and given by

$$z(u) = y(u).$$

As u ranges over $L^2(Q)$, $y(u)$ generates a dense (affine) subspace of $L^2(Q)$; hence the system is controllable.

To see this, let us first remark that by translation we may always reduce the problem to the case where $y_0(x) = 0$.

Let $\psi \in L^2(Q)$ be the orthogonal to the subspace generated by $y(u)$;

$$\int_Q y(u)\psi dxdt = 0 \quad \forall u.$$

We consider ξ as the solution of

$${}_t^C D_T^\beta \xi + \mathcal{A}^* \xi = \psi, \quad (x,t) \in Q$$

$$\xi|_\Sigma = 0,$$

$$\xi(x,T) = 0, \quad x \in \Omega.$$

Then

$$\begin{aligned}\int_Q \psi y(u)dxdt &= \int_Q ({}_t^C D_T^\beta \xi + \mathcal{A}^*\xi)y(u)dxdt \\ &= -\int_\Omega \xi(x,0)_t I_T^{1-\beta} y(u;x,0)dx + \int_\Sigma \xi \frac{\partial y(u)}{\partial \nu_{\mathcal{A}}} d\Sigma \\ &- \int_\Sigma \frac{\partial \xi}{\partial \nu_{\mathcal{A}}} y(u)\, d\Sigma + \int_Q \xi(\partial_t^\beta + \mathcal{A})y(u)\, dxdt \\ &= \int_Q \xi u dxdt = 0 \quad \forall u;\end{aligned}$$

hence $\xi = 0$ and hence $\psi = 0$.

Remark 1 We can also study by a similar manner the controllability of the system whose state is given by (55),(56), and (57).

Remark 2 If we take $\beta = 1$ in the previous sections we obtain the classical results in the optimal control with integer derivatives.

8. Conclusions

In this work we considered fractional diffusion equation with Dirichlet and Neumann boundary conditions with distributed and boundary control using the weak formulation. The fractional derivatives were defined in the weak Caputo sense. The analytical results were given in terms of Euler-Lagrange equations for the fractional optimal control problems. The formulation presented and the resulting equations are very similar to those for classical optimal control problems. The optimization problem presented in this paper constitutes a generalization of the optimal control problems of parabolic systems with Dirichlet and Neumann boundary conditions considered in [[19]] to fractional optimal control problems. Also the main result of the paper contains necessary and sufficient conditions of optimality for non-integer order fractional systems that give characterization of optimal control (Theorems 1 and 2).

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