

EXISTENCE OF SOLUTIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS WITH THREE-POINT BOUNDARY CONDITIONS AT RESONANCE WITH GENERAL CONDITIONS

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ABSTRACT. This paper presents a new technique to investigate the existence of solutions to fractional three-point boundary value problems at resonance in a Hilbert space. Based on the proposed method, the restricted conditions $A^2\xi^{2\alpha-2} = A\xi^{\alpha-1}$ and $A^2\xi^{2\alpha-2} = I$ on the operator A , which have been used in [18], are removed. It is shown that the system under consideration admits at least one solution by applying coincidence degree theory. Finally, an illustrative example is presented.

1. INTRODUCTION

In this article, we consider the problem of the existence of solutions for the following fractional three-point boundary value problems(BVPs) at resonance

$$\begin{aligned} D_{0+}^{\alpha}x(t) &= f(t, x(t), D_{0+}^{\alpha-1}x(t)), \quad 1 < \alpha \leq 2, \quad t \in [0, 1], \\ I_{0+}^{2-\alpha}x(t)|_{t=0} &= \theta, \quad x(1) = Ax(\xi), \end{aligned} \tag{1}$$

where D_{0+}^{α} and I_{0+}^{α} represents the Riemann-Liouville differentiation and integration, respectively; θ is the zero vector in $l^2 := \{x = (x_1, x_2, \dots, \cdot)^{\top} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$; $A : l^2 \rightarrow l^2$ is a bounded linear operator satisfying $1 \leq \dim \ker(I - A\xi^{\alpha-1}) < \infty$; $\xi \in (0, 1)$ is a fixed constant; $f : [0, 1] \times l^2 \times l^2 \rightarrow l^2$ is a Carathéodory function, that is,

- (i) for each $(u, v) \in l^2 \times l^2$, $t \mapsto f(t, u, v)$ is measurable on $[0, 1]$;
- (ii) for a.e. $t \in [0, 1]$, $(u, v) \mapsto f(t, u, v)$ is continuous on $l^2 \times l^2$;
- (iii) for every bounded set $\Omega \subseteq l^2 \times l^2$, $\{f(t, u, v) : (u, v) \in \Omega\}$ is a relatively compact set in l^2 . Moreover,

$$\varphi_{\Omega}(t) = \sup\{\|f(t, u, v)\|_{l^2} : (u, v) \in \Omega\} \in L^1[0, 1],$$

where $\|x\|_{l^2} = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$ is the norm of $x = (x_1, x_2, \dots, \cdot)^{\top}$ in l^2 .

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System (1) is said to be at resonance in l^2 if $\dim \ker(I - A\xi^{\alpha-1}) \geq 1$, otherwise, it is said to be non-resonant. The requirement $1 \leq \dim \ker(I - A\xi^{\alpha-1})$ is to make the problem to be resonant and the requirement $\dim \ker(I - A\xi^{\alpha-1}) < \infty$ is to make the kernel operator to be a Fredholm operator which is a basic requirement in applying the coincidence degree theory introduced by Mawhin.

In a recent paper [18], the authors studied the three-point BVPs (1) at resonance in infinite dimension space by assuming one of the following conditions holds

(A1) $A\xi^{\alpha-1}$ is idempotent, that is, $A^2\xi^{2\alpha-2} = A\xi^{\alpha-1}$;

(A2) $A^2\xi^{2\alpha-2} = I$, where I stands for the identity operator in l^2 .

The assumptions (A1) and (A2) are important in constructing the operator Q in [18] which plays a key role in the process of the proof. Our objective in this paper is to remove the restricted conditions (A1) and (A2) to study the existence of solutions for BVPs (1). It deserves to point out that the problem is new even when $\alpha = 2$, that is, system (1) is a second order differential system with resonant boundary conditions.

In the past three decades, the existence of solutions for the fractional differential equations with boundary value conditions have attained a great deal of attentions from many researchers, for instance, see [1, 2, 7, 6, 9, 10, 15, 17]. However, all results derived in these papers are for one equation with $\dim \ker L = 1$ or for two equations with $\dim \ker L = 2$. The case of problems where the $\dim \ker L$ can take any value in \mathbb{N} have treated with little attention.

Recently, the authors in [14, 13] investigated the following second order differential system

$$\begin{aligned} u''(t) &= f(t, u(t), u'(t)), \quad 0 < t < 1, \\ u'(0) &= \theta, u(1) = Au(\eta) \end{aligned} \tag{2}$$

where $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Carathéodory function and the square matrix A satisfies certain conditions. These results for second order differential equations in [14] and [13] were generalized to fractional order case $\alpha \in (1, 2]$ in [4] and [18]. It should be highlighted that, in [13], the authors successfully removed the constricted conditions used in [14] by making use of the property of Moore-Penrose pseudo-inverse matrix technique. Inspired by this, in this paper, we use the generalized inverse of the bounded linear operator in infinite dimensional space [12] to remove the restricted conditions (A1) and (A2), so that we can derive the existence of the solution for three-point BVPs (1).

We proceed as follows: in Section 2, we give some necessary background and some preparations for our consideration. The proof of the main results is presented in Section 3 by applying the coincidence degree theory of Mawhin. In Section 4, an illustrative example is included.

2. PRELIMINARIES

In this section, we present some necessary definitions and lemmas which will be used later. These definitions and lemmas can be found in [3, 8, 11, 12] and the references therein.

Definition 2.1 ([8]). *The fractional integral of order $\alpha > 0$ of a function $x : (0, \infty) \rightarrow \mathbb{R}$ is given by*

$$I_{0+}^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}x(s)ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Remark 2.1. *The notation $I_{0+}^{\alpha}x(t)|_{t=0}$ means that the limit is taken at almost all points of the right-hand side neighborhood $(0, \varepsilon)$ ($\varepsilon > 0$) of 0 as follows:*

$$I_{0+}^{\alpha}x(t)|_{t=0} = \lim_{t \rightarrow 0^+} I_{0+}^{\alpha}x(t).$$

Generally, $I_{0+}^{\alpha}x(t)|_{t=0}$ is not necessarily to be zero. For instance, let $\alpha \in (0, 1)$, $x(t) = t^{-\alpha}$. Then

$$I_{0+}^{\alpha}t^{-\alpha}|_{t=0} = \lim_{t \rightarrow 0^+} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}s^{-\alpha}ds = \lim_{t \rightarrow 0^+} \Gamma(1-\alpha) = \Gamma(1-\alpha).$$

Definition 2.2 ([8]). *The fractional derivative of order $\alpha > 0$ of a function $x : (0, \infty) \rightarrow \mathbb{R}$ in Riemann-Liouville sense is given by*

$$D_{0+}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{x(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n = [\alpha] + 1$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1 ([8]). *Assume that $x \in C(0, +\infty) \cap L_{\text{loc}}(0, +\infty)$ with a fractional derivative of order $\alpha > 0$ belonging to $C(0, +\infty) \cap L_{\text{loc}}(0, +\infty)$. Then*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}x(t) = x(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \dots + c_nt^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, \dots, n$, where $n = [\alpha] + 1$.

For any $x(t) = (x_1(t), x_2(t), \dots)^{\top} \in l^2$, the fractional derivative of order $\alpha > 0$ of x is defined by

$$D_{0+}^{\alpha}x(t) = (D_{0+}^{\alpha}x_1(t), D_{0+}^{\alpha}x_2(t), \dots)^{\top} \in l^2.$$

The following definitions and the coincidence degree theory are fundamental in the proof of our main result. We refer the readers to [3, 11].

Definition 2.3. *Let X and Y be normed spaces. A linear operator $L : \text{dom}(L) \subset X \rightarrow Y$ is said to be a Fredholm operator of index zero provided that*

- (i) $\text{im } L$ is a closed subset of Y ;
- (ii) $\dim \ker L = \text{codim im } L < +\infty$.

It follows from Definition 2.3 that, if L is a Fredholm operator of index zero, then there exist continuous projectors $P : X \rightarrow X$ and $Q : Y \rightarrow Y$ such that

$$\text{im } P = \ker L, \quad \ker Q = \text{im } L, \quad X = \ker L \oplus \ker P, \quad Y = \text{im } L \oplus \text{im } Q,$$

and the mapping $L|_{\text{dom } L \cap \ker P} : \text{dom } L \cap \ker P \rightarrow \text{im } L$ is invertible. We denote the inverse of $L|_{\text{dom } L \cap \ker P}$ by $K_P : \text{im } L \rightarrow \text{dom } L \cap \ker P$. The generalized inverse of L denoted by $K_{P,Q} : Y \rightarrow \text{dom } L \cap \ker P$ is defined by $K_{P,Q} = K_P(I - Q)$. Furthermore, for every isomorphism $J : \text{im } Q \rightarrow \ker L$, we can obtain that the mapping $K_{P,Q} + JQ : Y \rightarrow \text{dom } L$ is also an isomorphism and for all $x \in \text{dom } L$, we have

$$(K_{P,Q} + JQ)^{-1}x = (L + J^{-1}P)x. \quad (3)$$

Definition 2.4. Let L be a Fredholm operator of index zero, $\Omega \subseteq X$ be a bounded subset and $\text{dom } L \cap \Omega \neq \emptyset$. Then the operator $N : \overline{\Omega} \rightarrow Y$ is called to be L -compact in $\overline{\Omega}$ if

- (i) the mapping $QN : \overline{\Omega} \rightarrow Y$ is continuous and $QN(\overline{\Omega}) \subseteq Y$ is bounded;
- (ii) the mapping $K_{P,Q}N : \overline{\Omega} \rightarrow X$ is completely continuous.

The following lemma is the main tool in this paper.

Lemma 2.2 ([11]). Let $\Omega \subset X$ be a bounded subset, L be a Fredholm mapping of index zero and N be L -compact in $\overline{\Omega}$. Suppose that the following conditions are satisfied:

- (i) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in ((\text{dom } L \setminus \ker L) \cap \partial\Omega) \times (0, 1)$;
- (ii) $Nx \notin \text{im } L$ for every $x \in \ker L \cap \partial\Omega$;
- (iii) $\text{deg}(JQN|_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, 0) \neq 0$, with $Q : Y \rightarrow Y$ a continuous projector such that $\ker Q = \text{im } L$ and $J : \text{im } Q \rightarrow \ker L$ is an isomorphism.

Then the equation $Lx = Nx$ has at least one solution in $\text{dom } L \cap \overline{\Omega}$.

In this paper, we use spaces \mathbb{X}, \mathbb{Y} introduced as

$$\mathbb{X} = \{x(t) \in l^2 : x(t) = I_{0+}^{\alpha-1}u(t), u \in C([0, 1]; l^2), t \in [0, 1]\}$$

with the norm $\|x\|_{\mathbb{X}} = \max\{\|x\|_{C([0,1];l^2)}, \|D_{0+}^{\alpha-1}x\|_{C([0,1];l^2)}\}$ and $\mathbb{Y} = L^1([0, 1]; l^2)$ with the norm $\|y\|_{L^1([0,1];l^2)} = \int_0^1 \|y(s)\|_{l^2} ds$, respectively, where $\|x\|_{C([0,1];l^2)} = \sup_{t \in [0,1]} \|x(t)\|_{l^2}$.

Lemma 2.3. $F \subset \mathbb{X}$ is a sequentially compact set if and only if $F(t)$ is relatively compact and equicontinuous which are understood in the following sense:

- (1) for any $t \in [0, 1]$, $F(t) := \{x(t) | x \in F\}$ is a relatively compact set in l^2 ;
- (2) for any given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in F$,

$$\|x(t_1) - x(t_2)\|_{l^2} < \varepsilon, \|D_{0+}^{\alpha-1}x(t_1) - D_{0+}^{\alpha-1}x(t_2)\|_{l^2} < \varepsilon,$$

for $t_1, t_2 \in [0, 1], |t_1 - t_2| < \delta$.

In order to use Lemma 2.2, we define the linear operator $L : \text{dom } L \subseteq \mathbb{X} \rightarrow \mathbb{Y}$ by

$$Lx := D_{0+}^{\alpha}x, \tag{4}$$

where $\text{dom } L = \{x \in X : D_{0+}^{\alpha}x \in Y, I_{0+}^{2-\alpha}x(0) = \theta, x(1) = Ax(\xi)\}$ and define $N : X \rightarrow Y$ by

$$Nx(t) := f(t, x(t), D_{0+}^{\alpha-1}x(t)), \quad t \in [0, 1]. \tag{5}$$

Then the problem (1) can be equivalently rewritten as $Lx = Nx$.

Now we define operator \mathcal{M} as:

$$\mathcal{M} = I - A\xi^{\alpha-1}, \tag{6}$$

and define a continuous linear operator $h : \mathbb{Y} \rightarrow l^2$ by

$$h(y) := \frac{A}{\Gamma(\alpha)} \int_0^{\xi} (\xi - s)^{\alpha-1}y(s)ds - \frac{I}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1}y(s)ds. \tag{7}$$

In order to remove the restricted conditions (A1) and (A2), we will employ the following lemma on the property of bounded linear operator in general Hilbert space.

Lemma 2.4. [12] *If \mathcal{M} is a bounded linear transformation from Hilbert space H_1 to Hilbert space H_2 with closed rang $R(\mathcal{M})$, then the generalized inverse \mathcal{M}^+ of A is characterized as the unique solution X of the following equivalent equations:*

- (I) $X\mathcal{M}X = X$, $(X\mathcal{M})^* = X\mathcal{M}$, $\mathcal{M}X\mathcal{M} = \mathcal{M}$, $(\mathcal{M}X)^* = \mathcal{M}X$;
- (II) $\mathcal{M}X = P_{R(\mathcal{M})}$, $N(X^*) = N(\mathcal{M})$ where $P_{R(\mathcal{M})}$ denotes the orthogonal projection of H_2 onto $R(\mathcal{M})$;
- (III) $\mathcal{M}X = P_{R(\mathcal{M})}$, $X\mathcal{M} = P_{R(\mathcal{M}^*)}$, $X\mathcal{M}X = X$;
- (IV) $X\mathcal{M}\mathcal{M}^* = \mathcal{M}^*$, $XX^*\mathcal{M}^* = X$;
- (V) $X\mathcal{M}x = x$ for all $x \in R(\mathcal{M}^*)$ and $Xy = 0$ for all $y \in N(\mathcal{M}^*)$;
- (VI) $X\mathcal{M} = P_{R(\mathcal{M}^*)}$, $N(X) = N(\mathcal{M}^*)$;
- (VII) $\mathcal{M}X = P_{R(\mathcal{M})}$, $X\mathcal{M} = P_{R(\mathcal{M})}$.

Remark 2.2. *By the definition of \mathcal{M} given in (6), since A is a bounded linear operator, we know that \mathcal{M} satisfies the condition in Lemma 2.4. Thus, for such \mathcal{M} , there exists unique \mathcal{M}^+ satisfies the equations in Lemma 2.4.*

The next lemma plays a vital role in estimating the boundedness of some sets.

Lemma 2.5. [18] *Let $z_1, z_2 \geq 0$, $\gamma_1, \gamma_2 \in [0, 1)$ and $\lambda_i, \mu_i \geq 0, i = 1, 2, 3$, and the following two inequalities hold,*

$$\begin{aligned} z_1 &\leq \lambda_1 z_1^{\gamma_1} + \lambda_2 z_2 + \lambda_3, \\ z_2 &\leq \mu_1 z_1 + \mu_2 z_2^{\gamma_2} + \mu_3. \end{aligned} \tag{8}$$

Then z_1, z_2 is bounded if $\lambda_2 \mu_1 < 1$.

Lemma 2.6. *The operator L , defined by (4), is a Fredholm operator of index zero.*

Proof

For any $x \in \text{dom } L$, by Lemma 2.1 and $I_{0+}^{2-\alpha} x(0) = \theta$, we have

$$x(t) = I_{0+}^{\alpha} Lx(t) + ct^{\alpha-1}, \quad c \in l^2, \quad t \in [0, 1], \tag{9}$$

which, together with $x(1) = Ax(\xi)$, yields

$$\begin{aligned} \ker L &= \{x \in \mathbb{X} : x(t) = ct^{\alpha-1}, t \in [0, 1], c \in \ker \\ &\simeq \ker \mathcal{M}t^{\alpha-1}. \end{aligned} \tag{10}$$

Now we claim that

$$\text{im } L = \{y \in Y : h(y) \in \text{im } \mathcal{M}\}. \tag{11}$$

Actually, for any $y \in \text{im } L$, there exists a function $x \in \text{dom } L$ such that $y = Lx$. It follows from (9) that $x(t) = I_{0+}^{\alpha} y(t) + ct^{\alpha-1}$, this jointly with $x(1) = Ax(\xi)$, follows

$$\frac{A}{\Gamma(\alpha)} \int_0^{\xi} (\xi - s)^{\alpha-1} y(s) ds - \frac{I}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} y(s) ds = \mathcal{M}c, \quad c \in l^2,$$

which means that $h(y) \in \text{im } \mathcal{M}$.

On the other hand, for any $y \in \mathbb{Y}$ satisfying $h(y) \in \text{im } \mathcal{M}$, there exists a constant c^* such that $h(y) = \mathcal{M}c^*$. Let $x^*(t) = I_{0+}^{\alpha} y(t) + c^*t^{\alpha-1}$, a straightforward computation shows that $x^*(0) = \theta$ and $x^*(1) = Ax^*(\xi)$. Hence, $x^* \in \text{dom } L$ and $y(t) = D_{0+}^{\alpha} x^*(t)$, which implies that $y \in \text{im } L$.

Furthermore, notice that if $y = ct^{\alpha-1}$, $c \in l^2$, then

$$\begin{aligned} h(y) &= \frac{A}{\Gamma(\alpha)} \int_0^\xi (\xi - s)^{\alpha-1} y(s) ds - \frac{I}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} y(s) ds \\ &= \frac{(\xi^{2\alpha-1} A - I)c}{\Gamma(\alpha)\Gamma(2\alpha)}. \end{aligned} \quad (12)$$

Also, the relation

$$(I - \mathcal{M}\mathcal{M}^+)(\xi^{2\alpha-1} A - I) = (\xi^\alpha - 1)(I - \mathcal{M}\mathcal{M}^+) \quad (13)$$

holds. This is deduced from

$$(I - \mathcal{M}\mathcal{M}^+)(I - A\xi^{\alpha-1}) = (I - \mathcal{M}\mathcal{M}^+)\mathcal{M} = 0,$$

which is equivalent to

$$\begin{aligned} (I - \mathcal{M}\mathcal{M}^+)A\xi^{\alpha-1} &= (I - \mathcal{M}\mathcal{M}^+) \\ \Leftrightarrow (I - \mathcal{M}\mathcal{M}^+)\xi^{2\alpha-1}A &= \xi^\alpha(I - \mathcal{M}\mathcal{M}^+) \\ \Leftrightarrow (I - \mathcal{M}\mathcal{M}^+)(\xi^{2\alpha-1}A - I) &= (\xi^\alpha - 1)(I - \mathcal{M}\mathcal{M}^+). \end{aligned} \quad (14)$$

Define the continuous linear mapping $Q : \mathbb{Y} \rightarrow \mathbb{Y}$ by

$$Qy(t) := \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1} (I - \mathcal{M}\mathcal{M}^+)h(y)t^{\alpha-1}, \quad t \in [0, 1], \quad y \in \mathbb{Y}. \quad (15)$$

Then it follows from (7), (11), (13) and Lemma 2.4 that

$$\begin{aligned} Q^2y(t) &= \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1} (I - \mathcal{M}\mathcal{M}^+)h(Qy(t))t^{\alpha-1} \\ &= \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1} (I - \mathcal{M}\mathcal{M}^+) \frac{(\xi^{2\alpha-1}A - I)}{\Gamma(\alpha)\Gamma(2\alpha)} \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1} (I - \mathcal{M}\mathcal{M}^+)h(y)t^{\alpha-1} \\ &= \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1} (I - \mathcal{M}\mathcal{M}^+)^2 h(y)t^{\alpha-1} \\ &= \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1} (I - \mathcal{M}\mathcal{M}^+)h(y)t^{\alpha-1} = Qy(t), \end{aligned}$$

and

$$y \in \ker Q \Leftrightarrow h(y) \in \ker(I - \mathcal{M}\mathcal{M}^+) \Leftrightarrow h(y) \in \operatorname{im} \mathcal{M}\mathcal{M}^+ \Leftrightarrow h(y) \in \operatorname{im} \mathcal{M} \Leftrightarrow y \in \operatorname{im} L,$$

which implies that Q is a projection operator with $\ker Q = \operatorname{im} L$. Therefore, $\mathbb{Y} = \ker Q \oplus \operatorname{im} Q = \operatorname{im} L \oplus \operatorname{im} Q$.

Finally, we shall prove that $\operatorname{im} Q = \ker L$. Indeed, for any $z \in \operatorname{im} Q$, let $z = Qy$, $y \in \mathbb{Y}$. we have

$$(\mathcal{M}\mathcal{M}^+)z(t) = (\mathcal{M}\mathcal{M}^+)Qy(t) = \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1} \mathcal{M}\mathcal{M}^+(I - \mathcal{M}\mathcal{M}^+)g(y)t^{\alpha-1} = \theta,$$

which implies $z \in \ker L$. Conversely, for each $z \in \ker L$, there exists a constant $c^* \in \ker(\mathcal{M})$ such that $z = c^*t^{\alpha-1}$ for $t \in [0, 1]$. By (12) and (14), we derive

$$Qz(t) = \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1} (I - \mathcal{M}\mathcal{M}^+)h(c^*t^{\alpha-1})t^{\alpha-1} = c^*t^{\alpha-1} = z(t), \quad t \in [0, 1],$$

which implies that $z \in \operatorname{im} Q$. Hence we know that $\operatorname{im} Q = \ker L$. By assumption that $\dim \ker(I - A\xi^{\alpha-1}) < \infty$, the operator L is a Fredholm operator of index zero. The proof is completed.

Now to establish the generalized inverse of L , we define the operator $P : \mathbb{X} \rightarrow \mathbb{X}$ by

$$Px(t) = \frac{1}{\Gamma(\alpha)}(I - \mathcal{M}^+\mathcal{M})D_{0+}^{\alpha-1}x(0)t^{\alpha-1}. \quad (16)$$

Lemma 2.7. *The following assertions hold:*

(1) *The mapping $P : \mathbb{X} \rightarrow \mathbb{X}$ defined by (16), is a continuous projector satisfying*

$$\text{im } P = \ker L, \quad \mathbb{X} = \ker L \oplus \ker P;$$

(2) *The linear operator $K_P : \text{im } L \rightarrow \text{dom } L \cap \ker P$, which is the inverse of $L|_{\text{dom } L \cap \ker P}$, can be written as*

$$K_P y(t) = \mathcal{M}^+h(y)t^{\alpha-1} + I_{0+}^\alpha y(t), \quad (17)$$

moreover, K_P satisfies

$$\|K_P y\|_{\mathbb{X}} \leq C\|y\|_{L^1([0,1];l^2)},$$

where $C = 1 + \|\mathcal{M}^+\mathcal{M}\|(1 + \|A\|)$.

Proof

(1) By Lemma 2.4, $I - \mathcal{M}^+\mathcal{M}$ is a projection on $\ker \mathcal{M} \subset l^2$. It follows from (16) that P is a continuous projection. If $v \in \text{im } P$, there exists $x \in \mathbb{X}$ such that $v = Px$, then

$$v = \frac{1}{\Gamma(\alpha)}(I - \mathcal{M}^+\mathcal{M})D_{0+}^{\alpha-1}x(0)t^{\alpha-1}.$$

By (10) and Lemma 2.4, we have $\mathcal{M}(I - \mathcal{M}^+\mathcal{M})D_{0+}^{\alpha-1}x(0) = 0$, which gives $v \in \ker L$. Conversely, if $v \in \ker L$, then $v(t) = c_*t^{\alpha-1}$ for some $c_* \in \ker \mathcal{M} = \text{im}(I - \mathcal{M}^+\mathcal{M})$, that is, $c_* = (I - \mathcal{M}^+\mathcal{M})\bar{c}_*$ for $\bar{c}_* \in l^2$. Thus, we deduce that

$$\begin{aligned} Pv(t) &= \frac{1}{\Gamma(\alpha)}(I - \mathcal{M}^+\mathcal{M})D_{0+}^{\alpha-1}v(0)t^{\alpha-1} = (I - \mathcal{M}^+\mathcal{M})c_*t^{\alpha-1} \\ &= (I - \mathcal{M}^+\mathcal{M})^2\bar{c}_*t^{\alpha-1} = (I - \mathcal{M}^+\mathcal{M})\bar{c}_*t^{\alpha-1} \\ &= v(t), \quad t \in [0, 1], \end{aligned}$$

which gives $v \in \text{im } P$. Thus, we get that $\ker L = \text{im } P$ and consequently $\mathbb{X} = \ker L \oplus \ker P$.

(2) Let $y \in \text{im } L$. There exists $x \in \text{dom } L$ such that $y = Lx$ and $h(y) \in \text{im } \mathcal{M}$. By the definitions of P and K_P , we obtain that

$$\begin{aligned} PK_P y(t) &= \frac{1}{\Gamma(\alpha)}(I - \mathcal{M}^+\mathcal{M})D_{0+}^{\alpha-1}(K_P y(0))t^{\alpha-1} \\ &= \frac{1}{\Gamma(\alpha)}(I - \mathcal{M}^+\mathcal{M})[\mathcal{M}^+D_{0+}^{\alpha-1}h(y(0)) + I_{0+}^\alpha y(0)]t^{\alpha-1} \\ &= 0, \end{aligned}$$

and

$$\mathcal{M}(K_P y(0)) = \mathcal{M}[\mathcal{M}^+h(y(0)) + I_{0+}^\alpha y(0)] = h(y).$$

Thus, $K_P y \in \ker P \cap \text{dom } L$, K_P is well defined.

On the other hand, if $x \in \ker P \cap \text{dom } L$, then $x(t) = I_{0+}^\alpha Lx(t) + ct^{\alpha-1}$, and

$$\mathcal{M}c = h(Lx), \quad c \in \ker(I - \mathcal{M}^+\mathcal{M}).$$

Hence

$$\begin{aligned} K_P L_P x(t) &= \mathcal{M}^+ h(Lx) t^{\alpha-1} + I_{0+}^\alpha Lx(t) \\ &= \mathcal{M}^+ h(Lx) \\ &= \mathcal{M}^+ \mathcal{M} c t^{\alpha-1} + I_{0+}^\alpha Lx(t) \\ &= c t^{\alpha-1} + I_{0+}^\alpha Lx(t) = x(t) \end{aligned}$$

and $L_P K_P x(t) = x(t)$, $t \in [0, 1]$ for all $x \in \text{im } L$, then $K_P = L_P^{-1}$.

Finally, by the definition of K_P , we have

$$(D_{0+}^{\alpha-1} K_P y)(t) = \Gamma(\alpha) \mathcal{M}^+ h(y) + I_{0+}^1 y(t). \quad (18)$$

It follows from (7), (17) and (18) that

$$\begin{aligned} \|D_{0+}^{\alpha-1} K_P y\|_{C([0,1];l^2)} &= \Gamma(\alpha) \|\mathcal{M}^+\| \|h(y)\|_{C([0,1];l^2)} + \left\| \int_0^\cdot y(s) ds \right\|_{C([0,1];l^2)} \\ &\leq \Gamma(\alpha) \|\mathcal{M}^+\| \|h(y)\|_{C([0,1];l^2)} + \|y\|_{L^1([0,1];l^2)}, \end{aligned}$$

$$\begin{aligned} \|K_P y\|_{C([0,1];l^2)} &= \|\mathcal{M}^+\| \|h(y)\|_{C([0,1];l^2)} + \left\| \frac{1}{\Gamma(\alpha)} \int_0^\cdot (\cdot - s)^{\alpha-1} y(s) ds \right\|_{C([0,1];l^2)} \\ &\leq \|\mathcal{M}^+\| \|h(y)\|_{C([0,1];l^2)} + \frac{1}{\Gamma(\alpha)} \|y\|_{L^1([0,1];l^2)}, \end{aligned}$$

and

$$\|h(y)\|_{C([0,1];l^2)} \leq \frac{1}{\Gamma(\alpha)} (1 + \|A\|) \|y\|_{L^1([0,1];l^2)}.$$

This show that

$$\begin{aligned} \|K_P y\|_{\mathbb{X}} &= \max\{\|D_{0+}^{\alpha-1} K_P y\|_{C([0,1];l^2)}, \|K_P y\|_{C([0,1];l^2)}\} \\ &\leq [1 + \|\mathcal{M}^+ \mathcal{M}\| (1 + \|A\|)] \|y\|_{L^1}. \end{aligned}$$

This completes of the proof.

Lemma 2.8. *Let f be a Carathéodory function. Then N , defined by (5), is L -compact.*

Proof

Let Ω be a bounded subset in \mathbb{X} . By hypothesis (iii) on the function f , there exists a function $\varphi_\Omega(t) \in L^1[0, 1]$ such that for all $x \in \Omega$,

$$\|f(t, x(t), D_{0+}^{\alpha-1} x(t))\|_{l^2} \leq \varphi_\Omega(t), \quad \text{a.e. } t \in [0, 1], \quad (19)$$

which, along with (7), implies

$$\begin{aligned} \|h(Nx(t))\|_{l^2} &= \left\| \frac{A}{\Gamma(\alpha)} \int_0^\xi (\xi - s)^{\alpha-1} f(s, x(s), D_{0+}^{\alpha-1} x(s)) ds \right. \\ &\quad \left. - \frac{I}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} f(s, x(s), D_{0+}^{\alpha-1} x(s)) ds \right\|_{l^2} \\ &\leq \frac{\|A\| + 1}{\Gamma(\alpha)} \|\varphi_\Omega\|_{L^1[0,1]}. \end{aligned} \quad (20)$$

Thus, from (15) and (20), it follows that

$$\begin{aligned} \|QNx\|_{L^1([0,1];l^2)} &= \left\| \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1} (I - \mathcal{M}^+\mathcal{M})h(Nx) \right\|_{l^2} \int_0^1 s^{\alpha-1} ds \\ &\leq \frac{\Gamma(2\alpha)(\|A\| + 1)\|I - \mathcal{M}^+\mathcal{M}\|}{|1 - \xi^\alpha|} \|\varphi_\Omega\|_{L^1[0,1]} < \infty. \end{aligned} \quad (21)$$

This shows that $QN(\bar{\Omega}) \subseteq \mathbb{Y}$ is bounded. The continuity of QN follows from the hypothesis on f and the Lebesgue dominated convergence theorem.

Next, we shall show that $K_{P,Q}N$ is completely continuous. For any $x \in \Omega$, we have

$$\begin{aligned} K_{P,Q}Nx(t) &= K_P(I - Q)Nx(t) = K_PNx(t) - K_PQ Nx(t) \\ &= \mathcal{M}^+h(Nx)t^{\alpha-1} + I_{0+}^\alpha Nx(t) \\ &\quad - \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1} (I - \mathcal{M}^+\mathcal{M})h(Nx(t))I_{0+}^\alpha t^{\alpha-1}, \end{aligned} \quad (22)$$

and

$$\begin{aligned} D_{0+}^{\alpha-1}K_{P,Q}Nx(t) &= \Gamma(\alpha)\mathcal{M}^+h(Nx) + I_{0+}^1 Nx(t) \\ &\quad - \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1} (I - \mathcal{M}^+\mathcal{M})h(Nx(t))I_{0+}^1 t^{\alpha-1}. \end{aligned} \quad (23)$$

By the hypothesis on f and the Lebesgue dominated convergence theorem, it is easy to see that $K_{P,Q}N$ is continuous. Since f is a Carathéodory function, for every bounded set $\Omega_0 \subseteq l^2 \times l^2$, the set $\{f(t, u, v) : (u, v) \in \Omega_0\}$ is relatively compact set in l^2 . Therefore, for almost all $t \in [0, 1]$, $\{K_{P,Q}Nx(t) : x \in \Omega\}$ and $\{D_{0+}^{\alpha-1}K_{P,Q}Nx(t) : x \in \Omega\}$ are relatively compact in l^2 .

From (20), (22) and (23), we derive that

$$\begin{aligned} &\|K_{P,Q}Nx\|_{C([0,1];l^2)} \\ &= \left\| \mathcal{M}^+h(Nx)t^{\alpha-1} + I_{0+}^\alpha Nx(t) - \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1} (I - \mathcal{M}^+\mathcal{M})h(Nx(t))I_{0+}^\alpha t^{\alpha-1} \right\|_{C([0,1];l^2)} \\ &\leq \|\mathcal{M}^+\| \frac{\|A\| + 1}{\alpha\Gamma(\alpha)} \|\varphi_\Omega\|_{L^1[0,1]} + \frac{1}{\Gamma(\alpha)} \|\varphi_\Omega\|_{L^1(0,1)} + \frac{\Gamma(2\alpha)\|I - \mathcal{M}^+\mathcal{M}\|}{|\xi^\alpha - 1|} \|h(Nx(t))\|_{l^2} \\ &\leq \|\mathcal{M}^+\| \frac{\|A\| + 1}{\alpha\Gamma(\alpha)} \|\varphi_\Omega\|_{L^1[0,1]} + \frac{1}{\Gamma(\alpha)} \|\varphi_\Omega\|_{L^1(0,1)} + \frac{\Gamma(2\alpha)\|I - \mathcal{M}^+\mathcal{M}\|(\|A\| + 1)}{\Gamma(\alpha)|\xi^\alpha - 1|} \|\varphi_\Omega\|_{L^1(0,1)} \\ &< \infty, \end{aligned}$$

and

$$\begin{aligned} &\|D_{0+}^{\alpha-1}K_{P,Q}Nx\|_{C([0,1];l^2)} \\ &\leq \frac{\Gamma(\alpha)}{\alpha} \|\mathcal{M}^+\| \|\varphi_\Omega\|_{L^1} + \|I_{0+}^1 Nx(t) - \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1} (I - \mathcal{M}^+\mathcal{M})h(Nx(t))I_{0+}^1 t^{\alpha-1}\|_{C([0,1];l^2)} \\ &\leq \frac{\Gamma(\alpha)}{\alpha} \|\mathcal{M}^+\| \|\varphi_\Omega\|_{L^1} + \|\varphi_\Omega\|_{L^1(0,1)} + \frac{\Gamma(2\alpha)\|I - \mathcal{M}^+\mathcal{M}\|}{|\xi^\alpha - 1|} \|h(Nx(t))\|_{l^2} \\ &\leq \frac{\Gamma(\alpha)}{\alpha} \|\mathcal{M}^+\| \|\varphi_\Omega\|_{L^1} + \|\varphi_\Omega\|_{L^1(0,1)} + \frac{\Gamma(2\alpha)\|I - \mathcal{M}^+\mathcal{M}\|(\|A\| + 1)}{\Gamma(\alpha)|\xi^\alpha - 1|} \|\varphi_\Omega\|_{L^1(0,1)} \\ &< \infty, \end{aligned}$$

which shows that $K_{P,Q}N\bar{\Omega}$ is uniformly bounded in \mathbb{X} . Noting that

$$b^p - a^p \leq (b - a)^p \quad \text{for any } b \geq a > 0, 0 < p \leq 1. \quad (24)$$

for any $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$, we shall see that

$$\begin{aligned} & \|K_{P,Q}Nx(t_2) - K_{P,Q}Nx(t_1)\|_{l^2} \\ & \leq \frac{1}{\Gamma(\alpha)} \left\| \Gamma(\alpha)\mathcal{M}^+h(Nx)(t_2^{\alpha-1} - t_1^{\alpha-1}) + \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]Nx(s)ds \right. \\ & \quad \left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}Nx(s)ds - \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1}(I - \mathcal{M}^+\mathcal{M})h(Nx(t))[I_{0^+}^\alpha t_2^{\alpha-1} - I_{0^+}^\alpha t_1^{\alpha-1}] \right\|_{l^2} \\ & \leq \|\mathcal{M}^+h(Nx)\|_{l^2}(t_2 - t_1)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_2 - t_1)^{\alpha-1}\varphi_\Omega(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \varphi_\Omega(s)ds \\ & \quad + \frac{\Gamma^2(\alpha)\|I - \mathcal{M}^+\mathcal{M}\|(\|A\| + 1)}{|\xi^\alpha - 1|} \|\varphi_\Omega\|_{L^1(0,1)}|t_2^{2\alpha-1} - t_1^{2\alpha-1}| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_2 - t_1)^{\alpha-1}\varphi_\Omega(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \varphi_\Omega(s)ds \\ & \quad + \|\mathcal{M}^+\| \frac{\|A\| + 1}{\Gamma(\alpha)} \|\varphi_\Omega\|_{L^1}(t_2 - t_1)^{\alpha-1} \\ & \quad + \frac{\Gamma^2(\alpha)\|I - \mathcal{M}^+\mathcal{M}\|(\|A\| + 1)}{|\xi^\alpha - 1|} \|\varphi_\Omega\|_{L^1}|t_2^{2\alpha-1} - t_1^{2\alpha-1}| \rightarrow 0, \quad \text{as } t_2 \rightarrow t_1 \end{aligned}$$

and

$$\begin{aligned} & \|D_{0^+}^{\alpha-1}K_{P,Q}Nx(t_2) - D_{0^+}^{\alpha-1}K_{P,Q}Nx(t_1)\|_{l^2} \\ & = \left\| \int_{t_1}^{t_2} Nx(s)ds \right\|_{l^2} + \left\| \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1}(I - \mathcal{M}^+\mathcal{M})h(Nx(t)) \int_{t_1}^{t_2} s^{\alpha-1}ds \right\|_{l^2} \\ & \leq \int_{t_1}^{t_2} \varphi_\Omega(s)ds + \frac{\Gamma(2\alpha)\|I - \rho_A\|(\|A\| + 1)}{|\xi^\alpha - 1|} \|\varphi_\Omega\|_{L^1(0,1)}|t_2^\alpha - t_1^\alpha| \rightarrow 0, \quad \text{as } t_2 \rightarrow t_1. \end{aligned}$$

Then $K_{P,Q}N\bar{\Omega}$ is equicontinuous in \mathbb{X} . By Lemma 2.3, $K_{P,Q}N\bar{\Omega} \subseteq \mathbb{X}$ is relatively compact. Thus we can conclude that the operator N is L -compact in $\bar{\Omega}$. The proof is completed.

3. MAIN RESULTS

Theorem 3.1. *Let f be a Carathéodory function and the following conditions hold:*

- (H1) *There exist five nonnegative functions $a_1, a_2, b_1, b_2, c \in L^1[0, 1]$ and two constants $\gamma_1, \gamma_2 \in [0, 1]$ such that for all $t \in [0, 1]$, $u, v \in l^2$,*

$$\|f(t, u, v)\|_{l^2} \leq a_1(t)\|u\|_{l^2} + b_1(t)\|v\|_{l^2} + a_2(t)\|u\|_{l^2}^{\gamma_1} + b_2(t)\|v\|_{l^2}^{\gamma_2} + c(t)$$

holds.

- (H2) *There exists a constant $A_1 > 0$ such that for $x \in \text{dom } L$, if $\|D_{0^+}^{\alpha-1}x(t)\|_{l^2} > A_1$ for all $t \in [0, 1]$, then*

$$\begin{aligned} & \frac{A}{\Gamma(\alpha)} \int_0^\xi (\xi - s)^{\alpha-1} f(s, x(s), D_{0^+}^{\alpha-1}x(s))ds \\ & - \frac{I}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} f(s, x(s), D_{0^+}^{\alpha-1}x(s))ds \notin \text{im } \mathcal{M}. \end{aligned}$$

(H3) *There exists a constant $A_2 > 0$ and an isomorphism $J : \text{im } Q \rightarrow \ker L$ such that for any $e = \{(e_i)\} \in l^2$ satisfying $e = \xi^{\alpha-1} A e$ and $\|e\|_{l^2} > A_2$, either*

$$\langle e, JQNe \rangle_{l^2} \leq 0 \quad \text{or} \quad \langle e, JQNe \rangle_{l^2} \geq 0 \quad \text{holds,}$$

where $\langle \cdot, \cdot \rangle_{l^2}$ is the inner product in l^2 .

Then (1) has at least one solution in space \mathbb{X} provided that

$$\begin{aligned} \Gamma(\alpha) > \max \{ (\|I - \mathcal{M}^+ \mathcal{M}\| + 1) \|a_1\|_{L^1(0,1)}, (\|I - \mathcal{M}^+ \mathcal{M}\| + 1) \|b_1\|_{L^1(0,1)} \}, \\ \frac{(\|I - \mathcal{M}^+ \mathcal{M}\| + 1)^2 \|a_1\|_{L^1(0,1)} \|b_1\|_{L^1(0,1)}}{(\Gamma(\alpha) - (\|I - \mathcal{M}^+ \mathcal{M}\| + 1) \|a_1\|_{L^1(0,1)}) (\Gamma(\alpha) - (\|I - \mathcal{M}^+ \mathcal{M}\| + 1) \|b_1\|_{L^1(0,1)})} < 1. \end{aligned} \quad (25)$$

To prove the above theorem, we need the following auxiliary lemmas.

Lemma 3.1. *The set $\Omega_1 = \{x \in \text{dom } L \setminus \ker L : Lx = \lambda Nx \text{ for some } \lambda \in [0, 1]\}$ is bounded in \mathbb{X} .*

Proof

For any $x \in \Omega_1$, $x \notin \ker L$, we have $\lambda \neq 0$. Since $Nx \in \text{im } L = \ker Q$, by (11), we have $h(Nx) \in \text{im } \mathcal{M}$, where

$$\begin{aligned} h(Nx) &= \frac{A}{\Gamma(\alpha)} \int_0^\xi (\xi - s)^{\alpha-1} f(s, x(s), D_{0+}^{\alpha-1} x(s)) ds \\ &\quad - \frac{I}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} f(s, x(s), D_{0+}^{\alpha-1} x(s)) ds. \end{aligned} \quad (26)$$

From (H2), there exists $t_0 \in [0, 1]$ such that $\|D_{0+}^{\alpha-1} x(t_0)\|_{l^2} \leq A_1$. Then from the equality $D_{0+}^{\alpha-1} x(0) = D_{0+}^{\alpha-1} x(t_0) - \int_0^{t_0} D_{0+}^\alpha x(s) ds$, we deduce that

$$\|D_{0+}^{\alpha-1} x(0)\|_{l^2} \leq A_1 + \|D_{0+}^\alpha x\|_{L^1(0,1;l^2)} = A_1 + \|Lx\|_1 \leq A_1 + \|Nx\|_{L^1(0,1;l^2)},$$

which implies

$$\|Px\|_{\mathbb{X}} = \left\| \frac{1}{\Gamma(\alpha)} (I - \mathcal{M}^+ \mathcal{M}) D_{0+}^{\alpha-1} x(0) t^{\alpha-1} \right\|_{\mathbb{X}} \leq \frac{\|I - \mathcal{M}^+ \mathcal{M}\|}{\Gamma(\alpha)} (A_1 + \|Nx\|_{L^1(0,1;l^2)}). \quad (27)$$

Further, for $x \in \Omega_1$, since $\text{im } P = \ker L$, $\mathbb{X} = \ker L \oplus \ker P$, we have $(I - P)x \in \text{dom } L \cap \ker P$ and $LPx = \theta$. Then

$$\begin{aligned} \|(I - P)x\|_{\mathbb{X}} &= \|K_P L(I - P)x\|_{\mathbb{X}} \leq \|K_P Lx\|_{\mathbb{X}} \\ &\leq \frac{1}{\Gamma(\alpha)} \|Lx\|_{L^1(0,1;l^2)} \leq \frac{1}{\Gamma(\alpha)} \|Nx\|_{L^1(0,1;l^2)}. \end{aligned} \quad (28)$$

From (27) and (28), we conclude that

$$\begin{aligned} \|x\|_{\mathbb{X}} &= \|Px + (I - P)x\|_{\mathbb{X}} \leq \|Px\|_{\mathbb{X}} + \|(I - P)x\|_{\mathbb{X}} \\ &\leq \frac{\|I - \mathcal{M}^+ \mathcal{M}\|}{\Gamma(\alpha)} A_1 + \frac{\|I - \mathcal{M}^+ \mathcal{M}\| + 1}{\Gamma(\alpha)} \|Nx\|_{L^1(0,1;l^2)}. \end{aligned} \quad (29)$$

Moreover, by the definition of N and (H1), we derive

$$\begin{aligned}
& \|Nx\|_{L^1(0,1;l^2)} \\
&= \int_0^1 \|f(s, x(s), D_{0+}^{\alpha-1}x(s))\|_{l^2} dt \\
&\leq \|a_1\|_{L^1(0,1)} \|x\|_{C([0,1];l^2)} + \|b_1\|_{L^1(0,1)} \|D_{0+}^{\alpha-1}x\|_{C([0,1];l^2)} \\
&\quad + \|a_2\|_{L^1(0,1)} \|x\|_{C([0,1];l^2)}^{\gamma_1} + \|b_2\|_{L^1(0,1)} \|D_{0+}^{\alpha-1}x\|_{C([0,1];l^2)}^{\gamma_2} + \|c\|_{L^1(0,1)}.
\end{aligned} \tag{30}$$

Thus,

$$\begin{aligned}
\|x\|_{\mathbb{X}} &\leq \frac{\|I - \mathcal{M}^+\mathcal{M}\|}{\Gamma(\alpha)} A_1 + \frac{\|I - \mathcal{M}^+\mathcal{M}\| + 1}{\Gamma(\alpha)} \left(\|a_1\|_{L^1(0,1)} \|x\|_{C([0,1];l^2)} \right. \\
&\quad \left. + \|b_1\|_{L^1(0,1)} \|D_{0+}^{\alpha-1}x\|_{C([0,1];l^2)} \right) + \frac{\|I - \mathcal{M}^+\mathcal{M}\| + 1}{\Gamma(\alpha)} \\
&\quad \times \left(\|a_2\|_{L^1(0,1)} \|x\|_{C([0,1];l^2)}^{\gamma_1} + \|b_2\|_{L^1(0,1)} \|D_{0+}^{\alpha-1}x\|_{C([0,1];l^2)}^{\gamma_2} \right. \\
&\quad \left. + \|c\|_{L^1(0,1)} \right).
\end{aligned} \tag{31}$$

It follows from (25), (31), $\|x\|_{C([0,1];l^2)} \leq \|x\|_{\mathbb{X}}$, $\|D_{0+}^{\alpha-1}x\|_{C([0,1];l^2)} \leq \|x\|_{\mathbb{X}}$ and Lemma 2.5 that there exists $M_0 > 0$ such that

$$\max\{\|x\|_{C([0,1];l^2)}, \|D_{0+}^{\alpha-1}x\|_{C([0,1];l^2)}\} \leq M_0,$$

which means that Ω_1 is bounded in \mathbb{X} .

Lemma 3.2. *The set $\Omega_2 = \{x \in \ker L : Nx \in \text{im } L\}$ is bounded in \mathbb{X} .*

Proof

For any $x \in \Omega_2$, it follows from $x \in \ker L$ that $x = et^{\alpha-1}$ for some $e \in \ker \mathcal{M} \subset l^2$, and it follows from $Nx \in \text{im } L$ that $h(Nx) \in \text{im } \mathcal{M}$, where $h(Nx)$ is defined by (26). By hypothesis (H₂), we arrive at $\|D_{0+}^{\alpha-1}x(t_0)\|_{l^2} = \|e\|_{l^2} \Gamma(\alpha) \leq A_1$. Thus, we obtain $\|x\| \leq \|e\|_{l^2} \Gamma(\alpha) \leq A_1$. i.e., Ω_2 is bounded in \mathbb{X} .

Lemma 3.3. *Let $\Omega_3 = \{x \in \ker L : -\lambda x + (1 - \lambda)JQNx = \theta, \lambda \in [0, 1]\}$ if the first part of (H₃) holds, and $\Omega_3 = \{x \in \ker L : \lambda x + (1 - \lambda)JQNx = \theta, \lambda \in [0, 1]\}$ if the other part of (H₃) holds. Then, the set Ω_3 is bounded in \mathbb{X} .*

Proof

If the first part of (H₃) holds, that is, $\langle e, JQNe \rangle_{l^2} \leq 0$,

then for any $x \in \Omega_3$, we know that

$$x = et^{\alpha-1} \quad \text{with } e \in \ker \mathcal{M} \text{ and } \lambda x = (1 - \lambda)JQNx.$$

If $\lambda = 0$, we have $Nx \in \ker Q = \text{im } L$, then $x \in \Omega_2$, by the argument above, we get that $\|x\| \leq A_1$. Moreover, if $\lambda \in (0, 1]$ and if $\|e\|_{l^2} > A_2$, by (H₃), we deduce that

$$0 < \lambda \|e\|_{l^2}^2 = \lambda \langle e, e \rangle_{l^2} = (1 - \lambda) \langle e, JQNe \rangle_{l^2} \leq 0,$$

which is a contradiction. Then $\|x\|_{\mathbb{X}} = \|et^{\alpha-1}\|_{\mathbb{X}} \leq \max\{\|e\|_{l^2}, \Gamma(\alpha)\|e\|_{l^2}\}$. That is to say, Ω_3 is bounded.

For the case of the second part of (H₃) holds, we can obtain the result that Ω_3 is bounded by a similar method as above, so we omit it.

Proof of Theorem 3.1: We first construct an open bounded subset Ω in \mathbb{X} such that $\cup_{i=1}^3 \bar{\Omega}_i \subseteq \Omega$. By Lemmas 2.6 and 2.8, we know that L is a Fredholm operator of index zero and N is L -compact on $\bar{\Omega}$. Thus, it follows from Lemmas 3.1, 3.2 and

3.3 that conditions (i) and (ii) of Lemma 2.2 hold. By the construction of Ω and the argument above, to complete the theorem, it suffices to prove that condition (iii) of Lemma 2.2 is satisfied. To this end, let

$$H(x, \lambda) = \pm\lambda x + (1 - \lambda)JQNx, \quad (32)$$

here we let the isomorphism $J : \text{im } Q \rightarrow \ker L$ be the identical operator. Since $\Omega_3 \subseteq \Omega$, $H(x, \lambda) \neq 0$ for $(x, \lambda) \in \ker L \cap \partial\Omega \times [0, 1]$, then by homotopy property of degree, we obtain

$$\begin{aligned} \deg(JQN|_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, \theta) &= \deg(H(\cdot, 0), \Omega \cap \ker L, \theta) \\ &= \deg(H(\cdot, 1), \Omega \cap \ker L, \theta) \\ &= \deg(\pm Id, \Omega \cap \ker L, \theta) = \pm 1 \neq 0. \end{aligned}$$

Thus (H3) of Lemma 2.2 is fulfilled and Theorem 3.1 is proved.

4. EXAMPLE

In this section, we shall present an example to illustrate our main result in l^2 . Consider the following system with $\dim \ker L = k$, $k = 1, 2, 3, \dots$ in l^2 .

$$\begin{aligned} & D_{0+}^{3/2} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ \vdots \end{pmatrix} \\ &= \frac{1}{10} \begin{pmatrix} \begin{cases} 1, & \text{if } \|D_{0+}^{1/2}x(t)\|_{l^2} < 1 \\ D_{0+}^{1/2}x_1(t) + [D_{0+}^{1/2}x_1(t)]^{-1} - 1, & \text{if } \|D_{0+}^{1/2}x(t)\|_{l^2} \geq 1 \end{cases} \\ (x_2(t) + D_{0+}^{1/2}x_2(t))/2 \\ (x_3(t) + D_{0+}^{1/2}x_3(t))/2^2 \\ (x_4(t) + D_{0+}^{1/2}x_4(t))/2^3 \\ \vdots \end{pmatrix} \quad (33) \\ & I_{0+}^{\frac{1}{2}}x_i(0) = 0, \quad i = 1, 2, \dots \\ & x(1) = Ax(1/4). \end{aligned}$$

For all $t \in [0, 1]$, let $u = (x_1, x_2, x_3, \dots)^\top$, $v = (y_1, y_2, y_3, \dots)^\top \in l^2$ and $f = (f_1, f_2, \dots)^\top$ with

$$f_1(t, u, v) = \begin{cases} 1/10, & \text{if } \|v\|_{l^2} < 1, \\ (y_1 + y_1^{-1} - 1)/10, & \text{if } \|v\|_{l^2} \geq 1, \end{cases}$$

$f_i(t, u, v) = \frac{1}{5} \frac{x_i + y_i}{2^i}$, $i = 2, 3, 4, \dots$. Moreover,

$$A = \begin{bmatrix} B_1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & B_2 & 0 & 0 & 0 & 0 & \dots \\ \vdots & & \ddots & & & & \vdots \\ 0 & 0 & 0 & B_k & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & & & & & & \ddots \end{bmatrix} \quad \text{with} \quad B_i = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{7}{4} & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad (34)$$

and we denote

$$\mathcal{M}_i = I - \xi^{\frac{1}{2}} B_i = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then

$$\mathcal{M}_i^+ = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$i = 1, 2, \dots, k$, $k \in \mathbb{N}$. Obviously, $\dim \ker(I_3 - \xi^{\alpha-1} B_i) = \dim \ker(I_3 - B_i/2) = 1$, $i = 1, 2, \dots$, where I_3 is the 3×3 identity matrix. Then $\dim \ker(I - A\xi^{\alpha-1}) = k$, $k \in \mathbb{N}$ and the problem (33), with A and f defined above, has one solution if and only if problem (1) admits one solution.

Checking (H1) of Theorem 3.1: For some $r \in \mathbb{R}$, $\Omega = \{(u, v) \in l^2 \times l^2 : \|u\|_{l^2} \leq r, \|v\|_{l^2} \leq r\}$, let $\varphi_\Omega(t) = \frac{1}{10} [(r + 1/r + 1)^2 + \frac{4r^2}{3}]^{1/2} \in L^1[0, 1]$. Letting

$$a_1(t) = b_1(t) = \frac{1}{5\sqrt{3}}, \quad a_2(t) = b_2(t) = 0, \quad c(t) = \frac{r + 1/r + 1}{10}, \quad (35)$$

condition (H1) is satisfied.

Checking (H2) of Theorem 3.1: From the definition of f it follows that $f_1 > 1/10 > 0$ when $\|D_{0+}^{1/2} x(t)\|_{l^2} > 1$. This,

$$(B_1 \xi^\alpha - I) \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{bmatrix} -13/16 & 0 & 0 \\ 0 & -25/32 & 0 \\ 0 & 0 & -3/4 \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} -\frac{13}{16} f_1 \\ -\frac{25}{32} f_2 \\ -\frac{3}{4} f_3 \end{pmatrix},$$

and $\text{im}(\mathcal{M}) = \{(2\tau_1, \tau_1, 0, 2\tau_2, \tau_2, 0, 2\tau_3, \tau_3, 0, \dots)^\top : \tau_i \in \mathbb{R}, i = 1, 2, \dots\}$ implies that condition (H2) is satisfied.

Checking (H3) of Theorem 3.1: Since $\dim \ker(\mathcal{M}) = k$, $k \in \mathbb{N}$, for any $e \in l^2$ satisfying $e = \xi^{\alpha-1} A e$, e can be expressed as $e = e_1 + e_2 + \dots + e_k$, with

$$e_i = \sigma_i(\varepsilon_{3i}), \quad \sigma_i \in \mathbb{R}, \quad i = 1, 2, \dots, k, j = 1, 2,$$

where $\varepsilon_j = (0, 0, \dots, 0, 1_{j\text{-th}}, 0, 0, \dots)^\top \in l^2$ is a vector with all elements equaling to 0 except the j -th equaling to 1, $j = 1, 2, \dots$, that is

$$e = (0, 0, \sigma_1, 0, 0, \sigma_2, 0, 0, \sigma_3, \dots)^\top.$$

In addition, for any $y \in \mathbb{Y}$, by (15), we have

$$Qy(t) = \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1} (I - \mathcal{M}\mathcal{M}^+)h(y)t^{\alpha-1} = -\frac{8\sqrt{\pi}}{7} (I - \mathcal{M}\mathcal{M}^+)h(y)t^{\alpha-1}, \quad (36)$$

where

$$h(y) = \frac{A}{\Gamma(\alpha)} \int_0^{1/4} \left(\frac{1}{4} - s\right)^{\alpha-1} y(s) ds - \frac{I}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds. \quad (37)$$

By (5), let $d = t^{1/2} + \frac{\sqrt{\pi}}{2}$, we have

$$N(et^{1/2}) = \frac{1}{10} \begin{cases} \left(1, 0, \frac{d\sigma_1}{2^2}, 0, 0, \frac{d\sigma_2}{2^5}, 0, 0, \frac{d\sigma_3}{2^8}, \dots, 0, 0, \frac{d\sigma_i}{2^{3i-1}} \dots\right)^\top, \\ \text{if } |\sigma_{11}| < 1, 2 \leq i \leq k; \\ \left(-1, 0, \frac{d\sigma_1}{2^2}, 0, 0, \frac{d\sigma_2}{2^5}, 0, 0, \frac{d\sigma_3}{2^8}, \dots, 0, 0, \frac{d\sigma_i}{2^{3i-1}} \dots\right)^\top, \\ \text{if } |\sigma_{11}| \geq 1, 2 \leq i \leq k. \end{cases} \quad (38)$$

For $|\sigma_{11}| > 1$, let $\hat{d} = \frac{\pi}{128} + \frac{\sqrt{\pi}}{24}$, $\tilde{d} = \frac{\pi}{8} + \frac{\sqrt{\pi}}{3}$, and let $A_2 = 1$, we have

$$\int_0^{1/4} (1/4 - s)^{1/2} (Nes^{1/2}) ds = \frac{1}{10} \left(-\frac{1}{12}, 0, \frac{\hat{d}\sigma_1}{2^2}, 0, 0, \frac{\hat{d}\sigma_2}{2^5}, 0, 0, \frac{\hat{d}\sigma_3}{2^8}, 0, 0, \frac{\hat{d}\sigma_4}{2^{11}}, \dots\right)^\top,$$

and

$$\int_0^1 (1-s)^{1/2} (Nes^{1/2}) ds = \frac{1}{10} \left(-\frac{3}{2}, 0, \frac{\tilde{d}\sigma_1}{2^2}, 0, 0, \frac{\tilde{d}\sigma_2}{2^5}, 0, 0, \frac{\tilde{d}\sigma_3}{2^8}, 0, 0, \frac{\tilde{d}\sigma_4}{2^{11}}, \dots\right)^\top,$$

$$\begin{aligned} h(Net^{1/2}) &= \frac{A}{\Gamma(\alpha)} \int_0^{1/4} (1/4 - s)^{1/2} Nes^{1/2} ds - \frac{I}{\Gamma(\alpha)} \int_0^1 (1-s)^{1/2} Nes^{1/2} ds \\ &= \frac{1}{5\sqrt{\pi}} \left(\frac{11}{8}, 0, \frac{(2\hat{d} - \tilde{d})\sigma_1}{2^2}, 0, 0, \frac{(2\hat{d} - \tilde{d})\sigma_2}{2^5}, 0, 0, \frac{(2\hat{d} - \tilde{d})\sigma_3}{2^8}, 0, 0, \frac{(2\hat{d} - \tilde{d})\sigma_4}{2^{11}}, \dots\right)^\top. \end{aligned}$$

Then

$$\begin{aligned} Q(Net^{1/2}) &= -\frac{8\sqrt{\pi}t^{1/2}}{7} (I - \mathcal{M}\mathcal{M}^+) h(Net^{1/2}) \\ &= -\frac{8t^{1/2}}{35} \left(0, 0, \frac{(2\hat{d} - \tilde{d})\sigma_1}{2^2}, 0, 0, \frac{(2\hat{d} - \tilde{d})\sigma_2}{2^5}, 0, 0, \frac{(2\hat{d} - \tilde{d})\sigma_3}{2^8}, 0, 0, \frac{(2\hat{d} - \tilde{d})\sigma_4}{2^{11}}, \dots\right)^\top, \end{aligned}$$

and

$$\langle e, QNet^{1/2} \rangle = -\frac{8t^{1/2}}{35} \left(\frac{(2\hat{d} - \tilde{d})}{2^2}\sigma_1^2 + \frac{(2\hat{d} - \tilde{d})}{2^5}\sigma_2^2 + \frac{(2\hat{d} - \tilde{d})}{2^8}\sigma_3^2 + \frac{(2\hat{d} - \tilde{d})}{2^{11}}\sigma_4^2 + \dots\right) > 0.$$

Therefore, (33) admits at least one solution.

Remark 4.1. By a simply calculation of B , we can get

$$B^2 = \begin{pmatrix} \frac{9}{4} & 0 & 0 \\ 0 & \frac{49}{16} & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

from which, we can see that A does not satisfies the conditions (A1) and (A2), so the result in [18] is no longer applicable. Thus, our result is more general than the one in [18].

5. CONCLUDING REMARKS

In this paper, we consider the fractional BVPs at resonance in l^2 . The dimension of the kernel of the fractional differential operator with the boundary conditions be any positive integer. We remove the restricted conditions $A^2\xi^{2\alpha-2} = A\xi^{\alpha-1}$ and $A^2\xi^{2\alpha-2} = I$ on the operator A , which have been used in [18]. Our result can also be easily generalized to other fractional BVPs, for instance,

$$\begin{aligned} D_{0+}^{\alpha}x(t) &= f(t, x(t), D_{0+}^{\alpha-1}x(t)), \quad 1 < \alpha \leq 2, \quad t \in (0, 1), \\ x(0) &= \theta, D_{0+}^{\alpha-1}x(1) = AD_{0+}^{\alpha-1}x(\xi), \end{aligned} \quad (39)$$

where the bounded linear operator $A \in \mathcal{L}(l^2)$ satisfies $1 \leq \dim \ker(I - A) < \infty$ which leads this system is resonant. Moreover, notice that \mathbb{R}^n is the closed space of l^2 , taking $\alpha = 2$, the system (39) becomes the system of second order differential equations, which can be regarded as a generalization results in [14] and [13].

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