Journal of Fractional Calculus and Applications
Vol. 9(1) Jan. 2018, pp. 1-7.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

FEKETE-SZEGÖ PROBLEM FOR CERTAIN ANALYTIC FUNCTIONS DEFINED BY HYPERGEOMETRIC FUNCTIONS AND JACOBI POLYNOMIALS

J. M. JAHANGIRI, C. RAMACHANDRAN, S. ANNAMALAI

Abstract. In this paper we study the relationships between classes of Jacobi polynomials, hypergeometric and analytic univalent functions and obtain bounds for their respected *Fekete-Szegö* body of coefficients.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$. For complex numbers α_i $(i = 1, 2, \dots, p)$ and β_j $(j = 1, 2, \dots, q)$ where $\beta_j \neq 0, -1, -2, \dots; j = 1, 2, \dots, q$, the generalized hypergeometric function ${}_pF_q(z)$ is defined by

$${}_pF_q(z) = {}_p F_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n}{(\beta_1)_n \dots (\beta_q)_n} . \frac{z^n}{n!} \tag{2}$$

where $p \leq q+1$, $(\lambda)_0 = 1$ and $(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(n)} = \lambda(\lambda+1) \dots (\lambda+n-1)$ if $n = 1, 2, \dots$ The series given by (2) converges absolutely for $|z| < \infty$ if $p < q + 1$ and for z in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ if $p = q + 1$. For suitable values α_i and β_j the class of hypergeometric functions ${}_pF_q$ is closely related to classes of analytic and univalent functions. It is well-known that hypergeometric and univalent functions play important roles in a large variety of problems encountered in applied mathematics, probability and statistics, operations research, signal theory, moment problems, and other areas of science (e.g. see Exton [3, 4], Miller and Mocanu [11] and Rönning [12]). In this paper we introduce a new approach for studying the relationships between classes of hypergeometric and analytic univalent functions and

---

2010 *Mathematics Subject Classification.* Primary 30C45; 33C50; Secondary 30C80.

Key words and phrases. Analytic and univalent functions, hypergeometric functions, Jacobi polynomials.

Submitted Dec. 12, 2016.

will derive some new bounds for their respected *Fekete-Szegö* body of coefficients. We hope this new approach can motivate further research in this direction.

2. PRELIMINARIES

For $p = q + 1 = 2$, the series defined by (2) gives rise to the Gaussian hypergeometric series ${}_2F_1(a, b; c; z)$. This reduces to the elementary Gaussian geometric series $1+z+z^2+\ldots$ if (i) $a = c$ and $b = 1$ or (ii) $a = 1$ and $b = c$. For $\Re c > \Re b > 0$, we obtain

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} dt. $$

As a special case, we observe that

$$ {}_2F_1(1, 1; a; z) = (a-1) \int_0^1 \frac{t^{b-1}(1-t)^{a-2}}{1-tz} dt $$

and

$$ {}_2F_1(a, 1; 1; z) = \frac{1}{(1-z)^a} $$

so that

$$ {}_2F_1(a, 1; 1; z) *_2 F_1(a, 1; 1; z) = \frac{1}{1-z} =_2 F_1(1, 1; 1; z). $$

Here, the operator $*$ stands for the Hadamard product or convolution of two power series $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=0}^{\infty} b_n z^n$, that is

$$ (f * g)(z) = f(z) * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n. $$

If f and g are analytic in $\mathbb{U}$ then their Hadamard product $f * g$ is also analytic in $\mathbb{U}$. An alternative representation for the Hadamard product is the convolution integral

$$ (f * g)(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \zeta^{-1} f(\frac{z}{\zeta}) g(\zeta)\, d\zeta, \quad |z| < 1. $$

We shall need the following three definitions for stating and proving our theorems in the next section.

Definition 1. *For $t > -\frac{1}{2}$, $k > -\frac{1}{2}$ and $\mid x \mid \leq 1$ define $F(t, k, x)$ by*

$$ \begin{aligned} R_n^{(t,k)}(x) \equiv F(t, k, x) &= \sum_{n=0}^{\infty} \frac{P_n^{(t,k)}(x)}{P_n^{(t,k)}(1)} z^{n+1}, \\ &= \sum_{n=0}^{\infty} {}_2F_1\left(-n, t+k+n+1; t+1; \frac{1-x}{2}\right) z^{n+1} \\ &= \sum_{n=0}^{\infty} F_n z^{n+1} \end{aligned} $$

where $F_n =_2 F_1\left(-n, t+k+n+1; t+1; \frac{1-x}{2}\right)$, $z \in \mathbb{U}$, and $P_n^{(t,k)}(x)$ is (also see Lewis [9]) the Jacobi polynomial

$$ P_n^{(t,k)}(x) = \frac{(1+t)_n}{n!} {}_2F_1\left(-n, t+k+n+1; t+1; \frac{1-x}{2}\right). $$

To note the significance of the class $P_n^{(t,k)}(x) \equiv F(t,k,x)$, we list the following six special cases of the Jacobi polynomials

(1) $C_i^t(x) = R_i^{(t-\frac{1}{2},k-\frac{1}{2})}(x)$, called the ultra spherical polynomial,
(2) $T_i(x) = R_i^{(-\frac{1}{2},-\frac{1}{2})}(x)$, called the Chebyshev first polynomial,
(3) $U_i(x) = (i+1)R_i^{(\frac{1}{2},\frac{1}{2})}(x)$, called the Chebyshev second polynomial,
(4) $V_i(x) = R_i^{(-\frac{1}{2},\frac{1}{2})}(x)$, called the Chebyshev third polynomial,
(5) $W_i(x) = (2i+1)R_i^{(\frac{1}{2},-\frac{1}{2})}(x)$, called the Chebyshev fourth polynomial,
(6) $P_i(x) = R_i^{(0,0)}(x)$, called the Legendre polynomial.

Using the convolution operator $*$, we define

$$\mathcal{F} := \left\{ F : F(z) = (f * F(t,k,x))(z) = z + \sum_{n=2}^{\infty} F_n a_n z^n, f \in \mathcal{A} \right\}.$$

Let $\mho$ be the class of analytic functions w, normalized by $w(0) = 0$, satisfying the condition $|w(z)| < 1$. For analytic functions f and g, we say that f is subordinate to g in $\mathbb{U}$, denoted by $f \prec g$, if there exists a function $w \in \mho$ so that $f(z) = g(w(z))$ in $\mathbb{U}$. In particular, if g is univalent in $\mathbb{U}$, then $f \prec g \Leftrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

For $0 < q < 1$, the Jackson's *q-derivative* ([5, 6]) of a function $f \in \mathcal{A}$ is given by

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & for \quad z \neq 0, \\ f'(0) & for \quad z = 0, \end{cases} \tag{3}$$

where $D_q^2 f(z) = D_q(D_q f(z))$. It follows from (3) that

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad \text{where} \quad [n]_q = \frac{1-q^n}{1-q}$$

is sometimes called *the basic number n*. If $q \to 1^-$ then $[n]_q \to n$.

Moreover, as a consequence of (3), for $F \in \mathcal{F}$ we obtain

$$D_q F(z) = 1 + \sum_{n=2}^{\infty} [n]_q F_n a_n z^{n-1}.$$

Definition 2. *Let $\mathcal{P}$ denote the well known class of Caratheodory functions with positive real part in $\mathbb{U}$. We let $\mathcal{P}(p_k)$ $(0 \leq k < \infty)$ denote the family of functions p, such that $p \in \mathcal{P}$, and $p \prec p_k$ in $\mathbb{U}$, where the function p_k maps the unit disk conformally onto the region Ω_k such that $1 \in \Omega_k$ and*

$$\partial\Omega_k = \{u + iv : u^2 = k^2(u-1)^2 + k^2 v^2\}.$$

We remark that, the domain Ω_k is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, parabolic for $k = 1$ and covers the right half plane when $k = 0$. We note that the class $\mathcal{P}(p_k)$ and their extremal functions were presented and investigated by Kanas ([7], [8]). Evidently, for $k = 0$ we have

$$p_0(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + 2z^3 + 2z^4 + \dots\ ,$$

for $k=1$ *we have*

$$\begin{aligned} p_1(z) &= 1+\frac{2}{\pi^2}\log^2\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right) \\ &= 1+\frac{8}{\pi^2}z+\frac{16}{3\pi^2}z^2+\frac{184}{45\pi^2}z^3+\dots, \end{aligned}$$

for $0<k<1$ *and* $A=A(k)=(2/\pi)\arccos k$ *we obtain*

$$\begin{aligned} p_k(z) &= 1+\frac{2}{1-k^2}\sinh^2\left(A(k)\quad arc\tanh\sqrt{z}\right) \\ &= \frac{1}{1-k^2}\cos\left\{A(k)i\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\}-\frac{k^2}{1-k^2}. \\ &= 1+\frac{1}{1-k^2}\sum_{n=1}^{\infty}\left[\sum_{l=1}^{2n}2^l\binom{A}{l}\binom{2n-1}{2n-l}\right]z^n \\ &= 1+\frac{2A^2}{1-k^2}z+\frac{4A^2+2A^4}{3(1-k^2)}z^2+\frac{\frac{46A^2}{15}+\frac{8A^4}{3}+\frac{4A^6}{15}}{3(1-k^2)}z^3+\dots \end{aligned}$$

and for $k>1$ *and* $u(z)=\dfrac{z-\sqrt{\kappa}}{1-\sqrt{\kappa}z}$ *we have*

$$\begin{aligned} p_k(z) &= \frac{1}{k^2-1}\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{\frac{u(z)}{\sqrt{k}}}\frac{dt}{\sqrt{1-t^2}\sqrt{1-\kappa^2t^2}}\right) \\ &= 1+\frac{\pi^2}{4\sqrt{(\kappa)}(k^2-1)K^2(\kappa)(1+\kappa)}\left\{z+\frac{4K^2(\kappa)(\kappa^2+6\kappa+1)-\pi^2}{4\sqrt{(\kappa)}K^2(\kappa)(1+\kappa)}z^2+\dots\right\} \end{aligned}$$

where $K(\kappa)$ *denotes the Legendre's complete elliptic integral of the first kind, and* $K'(\kappa)$ *is the complementary integrand of* $K(\kappa)$ *with* $k\in(0,1)$ *is chosen such that* $k=\cosh\left[(\pi K'(\kappa))\,/\,(4K(\kappa))\right]$. *By virtue of*

$$p(z)=\frac{zf'(z)}{f(z)}\prec p_k(z)\ \text{or}\ p(z)=1+\frac{zf''(z)}{f'(z)}\prec p_k(z)$$

and the properties of the domains, we have

$$\Re(p(z))>\Re(p_k(z))>\frac{k}{k+1}$$

Definition 3. *For the real numbers* $0\le k<\infty$, $0\le\alpha<1$, $0<q<1$ *and* $b\neq 0$ *and for* $p_k(z)$ *as in the Definition 2, we say that a function* $f\in\mathcal{A}$ *is in the class* $\mathcal{FS}_q^b(p_k)$ *if*

$$1+\frac{1}{b}\left(\frac{zD_qF(z)}{F(z)}-1\right)\prec p_k(z)\qquad(z\in\mathbb{U})$$

and is in the class $\mathcal{FC}_q^b(p_k)$ *if*

$$1+\frac{1}{b}\left(\frac{D_q(zD_qF(z))}{D_q(F(z))}\right)\prec p_k(z)\qquad(z\in\mathbb{U}).$$

Finally, prior to the start of the next section, we state the following lemma, which can be found in [1] *or* [2] *and is a reformulation of the corresponding result for functions with positive real part due to Ma and Minda* [10].

Lemma 1. *Let $w(z) = w_1 z + w_2 z^2 + ... \in \mho$ be so that $|w(z)| < 1$ in $\mathbb{U}$. If t is a complex number, then*

$$\left|w_2 + t w_1^2\right| \leq max\{1, |t|\}.$$

The inequality is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

3. The Main Results

In this section we determine the Fekete-Szegö functional related to the conical domains.

Theorem 1. *Let $0 \leq k < \infty,\ 0 \leq \alpha < 1,\ 0 < q < 1,\ b \neq 0$ and let $p_k(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be defined as in the Definition 2. If f given by (1) belongs to $\mathcal{FS}_q^b(p_k)$ then we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b| p_1}{\left([3]_q - 1\right) F_3} \max\left\{1, \left|\frac{p_2}{p_1} + \frac{p_1 b\left([2]_q - 1\right) F_2^2 - \mu p_1 b\left([3]_q - 1\right) F_3}{\left([2]_q - 1\right)^2 F_2^2}\right|\right\}. \tag{4}$$

Actually, (4) holds for any complex number μ.

Proof. If $f \in \mathcal{FS}_q^b(p_k)$, then there is a Schwarz function $w = w_1 z + w_2 z^2 + \cdots \in \mho$ such that

$$1 + \frac{1}{b}\left(\frac{z D_q F(z)}{F(z)} - 1\right) = p_k(w(z)). \tag{5}$$

We note that

$$\frac{z D_q F(z)}{F(z)} = 1 + ([2]_q - 1) a_2 F_2 z + \left(([3]_q - 1) a_3 F_3 - ([2]_q - 1) F_2^2 a_2^2\right) z^2 + \ldots \tag{6}$$

and

$$p_k(w(z)) = 1 + p_1 w_1 z + (p_1 w_2 + p_2 w_1^2) z^2 + (p_1 w_3 + 2 p_2 w_1 w_2 + p_3 w_1^3) z^3 + \cdots . \tag{7}$$

Applying (5), (6) and (7), we obtain

$$a_2 = \frac{b p_1 w_1}{([2]_q - 1) F_2}, \tag{8}$$

and

$$a_3 = \frac{b p_1 w_2}{([3]_q - 1) F_3} + \frac{w_1^2 p_2 b}{F_3([3]_q - 1)} + \frac{p_1^2 w_1^2 b^2}{([2]_q - 1)([3]_q - 1) F_3}. \tag{9}$$

Hence, by (8), (9), we get the following

$$a_3 - \mu a_2^2 = \frac{b p_1}{\left([3]_q - 1\right) F_3}\left(w_2 + t w_1^2\right),$$

where

$$t = \frac{p_2}{p_1} + \left[\frac{p_1 b\left([2]_q - 1\right) F_2^2 - \mu p_1 b\left([3]_q - 1\right) F_3}{([2]_q - 1)^2 F_2^2}\right]. \tag{10}$$

The result (4) now follows by an application of Lemma 1 to the equation (10). $\square$

For the class of functions $\mathcal{FC}_{q,b}^{\beta}(p_k)$ we can prove the following

Theorem 2. *Let* $0 \leq k < \infty$, $0 \leq \alpha < 1$, $0 < q < 1$, $b \neq 0$, *and let* $p_k(z) = 1 + p_1 z + p_2 z^2 + \cdots$ *be defined as in Definition 2. If* f *given by* (1) *belongs to* $\mathcal{FC}^{\beta}_{q,b}(p_k)$, *then we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|p_1}{[2]_q[3]_qF_3} \max\left\{1, \left|\frac{p_2}{p_1} + \frac{\left(p_1 b[2]_q F_2^2 - \mu p_1 b[3]_q F_3\right)}{[2]_q F_2^2}\right|\right\} \tag{11}$$

Actually, (11) *holds for any complex number* μ.

Proof. If $f \in \mathcal{FC}^{\beta}_{q,b}(p_k)$, then there is a Schwarz function $w = w_1 + w_2 + \cdots \in \mho$ such that

$$1 + \frac{1}{b}\left(\frac{D_q(zD_qF(z))}{D_qF(z)}\right) = p_k(w(z)). \tag{12}$$

We note that

$$\frac{D_q(zD_qF(z))}{D_q(F(z))} = [2]_q a_2 F_2 z + \left([2]_q[3]_q a_3 F_3 - [2]_q^2 F_2^2 a_2^2\right) z^2 + \ldots \tag{13}$$

and

$$p_k(w(z)) = 1 + p_1 w_1 z + (p_1 w_2 + p_2 w_1^2) z^2 + (p_1 w_3 + 2p_2 w_1 w_2 + p_3 w_1^3) z^3 + \cdots. \tag{14}$$

Applying (12), (13) and (14), we obtain

$$a_2 = \frac{bp_1 w_1}{F_2[2]_q}, \tag{15}$$

and

$$a_3 = \frac{bp_1 w_2}{[2]_q[3]_qF_3} + \frac{w_1^2 p_2 b}{[2]_q[3]_qF_3} + \frac{p_1^2 w_1^2 b^2}{[2]_q[3]_qF_3}. \tag{16}$$

Hence, by (15), (16), we get the following

$$a_3 - \mu a_2^2 = \frac{bp_1}{[2]_q F_3 [3]_q}\left(w_2 + t w_1^2\right),$$

where

$$t = \frac{p_2}{p_1} + \left[\frac{p_1 b F_2^2 [2]_q - \mu p_1 b F_3 [3]_q}{F_2^2 [2]_q}\right]. \tag{17}$$

The result (11) now follows by an application of Lemma 1 to the equation (17). □

References

[1] R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramanian, *The Fekete-Szegö Coefficient functional for transforms of analytic funtions*, Bull. . Iran. Math. Soc. vol. 35, N0. **2**, pp 119–142, 2009.

[2] R. M. Ali, V. Ravichandran and N. Seenivasagan, *Coefficient bounds for p-valent functions*, Appl. Math. Comput. **187**, no. 1, 35–46, 2007.

[3] H. Exton, *Multiple hypergeometric functions and applications*, Ellis Horwood Ltd. (Chichester), 1976.

[4] H. Exton, *Handbook of hypergeometric integrals: theory, applications, tables, computer programs*, Ellis Horwood Ltd. (Chichester), (ISBN: 0-85312-122-2), 1978.

[5] F. H. Jackson, *On q-definite integrals*, Quart. J. Pure Appl. Math., **41**, 193–203, 1910.

[6] F. H. Jackson, *On q-functions and a certain difference operator*, Transactions of the Royal Society of Edinburgh, **46**, 253–281, 1908.

[7] S. Kanas and A. Wiśniowska, *Conic regions and k-uniform convexity*, J. Comput. Appl. Math. **105**, no. 1-2, 327–336, 1999.

[8] S. Kanas and A. Wiśniowska, *Conic domains and starlike function*, Rev. Roumanie Math. Pures Appl., **45**, no. 4, 647–657, 2000.

[9] J. L. Lewis, *Applications of a convolution theorem to Jacobi polynomials*, SIAM J. Math. Anal. **10**, no. 6, 1110–1120, 1979.

[10] W. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA, 1994.

[11] S. S. Miller and P. T. Mocanu, *Univalence of Gaussian and confluent hypergeometric functions*, Proc. Amer. Math. Soc. 110, no. 2, 333–342, 1990.

[12] F. Rønning, *PC-fractions and Szego polynomials associated with starlike univalent functions*, Numerical Algorithms **3**, no. 1-4, 383–391, 1982.

J. M. Jahangiri

Mathematical Sciences, Kent State University, Burton, Ohio, 44021-9500, U. S. A

E-mail address: jjahangi@kent.edu

C. Ramachandran,

Department of Mathematics, University College of Engineering, Villupuram, Anna University ,Villupurm 605 103, Tamil Nadu, India

E-mail address: crjsp2004@yahoo.com

S. Annamalai

Department of Mathematics, University College of Engineering, Villupuram, Anna University ,Villupurm 605 103, Tamil Nadu, India

E-mail address: annagopika02@gmail.com