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FABER POLYNOMIAL COEFFICIENT BOUNDS FOR ANALYTIC BI-CLOSE-TO-CONVEX FUNCTIONS DEFINED BY FRACTIONAL CALCULUS

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Abstract. In this study, we obtain coefficient expansions for analytic bi-close-to convex functions defined by fractional calculus and determine coefficients for such functions using the Faber Polynomials. Among other results, the general coefficient bound $|a_n|$ and the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ are found in our investigation. Furthermore, we show the coefficient bound for $|a_2^2 - a_3|$. We also show that our class is generalization class of them for some special cases.

1. Introduction

We know that a function is *univalent* if it never takes the same value twice. Also we know that a function is *bi-univalent* if both it and its inverse are univalent.

Let $\mathcal{A}$ denote the class of functions f which are *analytic* in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

For α; $0 \leq \alpha < 1$, we let $S^*(\alpha)$ denote the class of function $g \in S$ that are starlike of order α in $\mathbb{U}$, namely, $Re\left\{\frac{zg'(z)}{g(z)}\right\} > \alpha$ in $\mathbb{U}$ and $C(\alpha)$ indicate the class of functions $f \in S$ that are close-to-convex of order α in $\mathbb{U}$, namely, if a function g is in $S^*(0) = S^*$ so that $Re\left\{\frac{zf'(z)}{g(z)}\right\} > \alpha$ in $\mathbb{U}$ (see [11] and [7]). We note that $S^*(\alpha) \subset C(\alpha) \subset S$ and that $|a_n| \leq n$ for $f \in S$ by Bieberbach Conjecture (see [4] and [7]).

The Koebe 1/4 Theorem [7] asserts that the image of $\mathbb{U}$ under each univalent function $f \in \mathcal{A}$ contains the disk of radius 1/4. According to this, if $F = f^{-1}$ is the inverse of a function $f \in S$, then F has a Maclaurin series expansion in some disk about the origin. So every function $f \in S$ has an inverse f^{-1} which satisfies $f^{-1}(f(z)) = z$ for $z \in \mathbb{U}$ and $f(f^{-1}(w)) = w$ for $|w| < 1/4$.

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A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both f and $F = f^{-1}$ are univalent in $\mathbb{U}$. Similarly, a function $f \in \mathcal{A}$ is said to be bi-close-to-convex of order α if both f and $F = f^{-1}$ are bi-close-to-convex of order α in $\mathbb{U}$. Let Σ define the class of all bi-univalent functions in $\mathbb{U}$ represented by the Taylor-Maclaurin series expansion (1). For a short history and examples of functions in the class Σ, see [16] (see also [6],[18],[12],[14]).

Faber polynomials, which is used by us in this paper, play a considerable act in geometric function theory which was introduced by Faber [8].

Firstly, Lewin [12] considered the class of bi-univalent functions, obtaining the estimate $|a_2| \leq 1.51$. Subsequently, Brannan and Clunie [5] developed Lewin's result to $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Accordingly, Netanyahu [14] showed that $|a_2| \leq \frac{4}{3}$. Brannan and Taha [6] defined certain subclasses of bi-univalent function class Σ similar to the usual subclasses. In fact, the aforementioned work of Srivastava et al. [16] essentially revived the investigation of various subclasses of bi-univalent function class Σ in recent years. Lately, many mathematicians found bounds for several subclasses of bi-univalent functions (see [16],[10],[20]). Only few papers determine general coefficient bounds $|a_n|$ for the analytic bi-close-to-convex functions in the associated documents. Especially, in [9] Hamidi and Jahangiri introduced the class of bi-close-to-convex functions and determined estimates for the general coefficient $|a_n|$ of bi-close-to-convex function under certain gap series condition by using Faber polynomials.

A detailed operation is given in the books, which have the applications of the fractional calculus, [15] by Oldham and Spanier, and [13] by Miller and Ross . For the comprehensive concept of the fractional calculus, one can be seen to [17] .

λ-fractional operator was defined by Aydogan et al. in [3] as follows,

If $f(z)$ defined by as (1) then $D_z^\lambda f(z) = D_z^\lambda(z + a_2 z^2 + ... + a_n z^n + ...)$

$$D^\lambda f(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z) = z + \sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} z^n.$$

From the definition of $D^\lambda f(z)$ some properties can be written as follows,

i. $$D^1 f(z) = Df(z) = lim_{\lambda \to 1} D^\lambda f(z) = z f'(z) \qquad ,$$

ii. $$D^\lambda(D^\delta f(z)) = D^\delta(D^\lambda f(z)) = z + \sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\lambda)\Gamma(2-\delta)(\Gamma(n+1))^2}{\Gamma(n+1-\lambda)\Gamma(n+1-\delta)} z^n,$$

iii. $$D(D^\delta f(z)) = z + \sum_{n=2}^{\infty} n \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^n = z(D^\delta f(z))' = \Gamma(2-\lambda) z^\lambda (\lambda D_z^\delta + z D_z^{\lambda+1} f(z));$$

$$iv. \quad \frac{D(D^{\lambda}f(z))}{D^{\lambda}f(z)} = z\frac{f'(z)}{f(z)} \qquad for \quad \lambda = 0.$$
$$= 1 + z\frac{f''(z)}{f'(z)} \qquad for \quad \lambda = 1.$$

Now we start by giving the function class $K_{\Sigma}(\lambda)$ as follows:

Definition 1.1 Let $f(z)$ given by (1) be an element of S. Then $f(z)$ is said to be λ-fractional close-to-convex function in $\mathbb{U}$ if a function $g(z)$ is in S^* such that

$$\Re\left(\frac{D(D^{\lambda}f(z))}{g(z)}\right) > 0; \qquad for \quad all \quad z \in \mathbb{U}. \tag{2}$$

The class of these functions is represented by $K_{\Sigma}(\lambda)$.

It is trivial that $K_{\Sigma}(0) = K$.

Let consider the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (1). So, the coefficients of its inverse map $F = f^{-1}$ may be stated as, [1],

$$F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ..., a_n)w^n = w + \sum_{n=2}^{\infty} A_n w^n, \tag{3}$$

and

$$G(w) = g^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ..., a_n)w^n = w + \sum_{n=2}^{\infty} B_n w^n, \tag{4}$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4 \\ &+ \frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}[a_5 + (-n+2)a_3^2] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6 + (-2n+5)a_3a_4] + \sum_{j\geq 7} a_2^{n-j}V_j, \end{aligned} \tag{5}$$

where V_j is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$ (see [1]and [2]). Especially, the first few terms of K_{n-1}^{-n} are given below:

$$K_1^{-2} = -2a_2$$
$$K_2^{-3} = 3\left(2a_2^2 - a_3\right)$$

and

$$K_3^{-4} = -4\left(5a_2^3 - 5a_2a_3 + a_4\right).$$

Generally, for any $p \in \mathbb{N}$ an expansion of K_n^p is as, [1],

$$K_n^p = pa_n + \frac{p(p-1)}{2}E_n^2 + \frac{p!}{(p-3)!3!}E_n^3 + \cdots + \frac{p!}{(p-n)!n!}E_n^n, \quad (p \in \mathbb{Z}) \tag{6}$$

where $\mathbb{Z} = \{0, \mp 1, \mp 2, \cdots\}$ and $E_n^p = E_n^p(a_2, a_3, \ldots)$ and by [19],

$$E_n^m(a_1, a_2, \dots a_n) = \sum_{m=2}^{\infty} \frac{m!(a_2)^{\mu_1} ... (a_n)^{\mu_n}}{\mu_1! ... \mu_n!}, \quad for, \quad m \le n, \tag{7}$$

while $a_1 = 1$ and the sum is taken over all non-negative integers $\mu_1, \mu_2, ..., \mu_n$ satisfying

$$\begin{cases} \mu_1 + \mu_2 + ... + \mu_n = m \\ \\ \mu_1 + 2\mu_2 + ... + n\mu_n = n \end{cases}$$

(see, for details, [1] and [2]).
It is clearly that $E_n^n(a_1, a_2, ..., a_n) = a_1^n$.

In this paper, we firstly, obtain general coefficient expansions of $|a_n|$ for analytic bi-close to convex functions defined by fractional calculus using the Faber Polynomials. Also, determine the first coefficients $|a_2|$, $|a_3|$, and $|a_2^2 - a_3|$ for such functions. For some special cases, also we show that our class is generalization class of them. The bi-close to convex functions considered in this paper are largest subclass of bi-univalent functions and generalization of the results of the paper in [9].

2. Main Results

Our first theorem giving by Theorem 2.1 shows an upper bound for $|a_n|$ of analytic bi-univalent functions in the class $K_\Sigma(\lambda)$.

Theorem 2.1 Let the function f given by (1) be in the class $K_\Sigma(\lambda)$ $(0 \le \lambda < 1, n \in \mathbb{N}_0 = \{0, 1, 2...\})$, if $a_k = 0$ for $2 \le k \le n - 1$, then

$$|a_n| \le \frac{(n+2)\Gamma(n+1-\lambda)}{n\Gamma(2-\lambda)\Gamma(n+1)} \qquad n \ge 4.$$

Proof. First let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be close-to-convex in $\mathbb{U}$. Therefore, there exists a function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*$ so that $\Re\left(\frac{D(D^\lambda f(z))}{g(z)}\right) > 0; \quad for \quad all \quad z \in \mathbb{U}$.

The Faber polynomial expansion for $\frac{D(D^\lambda f(z))}{g(z)}$ and the inverse map $\frac{D(D^\lambda F(w))}{G(w)}$ is given by:

$$\frac{D(D^\lambda f(z))}{g(z)} = 1 + \sum_{n=2}^{\infty} \left[\left(\frac{n\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n - b_n \right) \right.$$
$$\left. + \sum_{s=1}^{n-2} K_s^{-1}(b_2, b_3, ..., b_{s+1}) \left(\frac{(n-s)\Gamma(2-\lambda)\Gamma(n-s+1)}{\Gamma(n-s+1-\lambda)} a_{n-s} - b_{n-s} \right) \right] z^{n-1} \tag{8}$$

and for its inverse map, $F = f^{-1}$ we get

$$\frac{D(D^\lambda F(w))}{G(w)} = 1 + \sum_{n=2}^{\infty} \left[\left(\frac{n\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} A_n - B_n \right) \right.$$

$$+\sum_{s=1}^{n-2} K_s^{-1}(B_2, B_3, ..., B_{s+1}) \left(\frac{(n-s)\Gamma(2-\lambda)\Gamma(n-s+1)}{\Gamma(n-s+1-\lambda)} A_{n-s} - B_{n-s} \right) \Bigg] w^{n-1}. \tag{9}$$

On the other hand, since $\frac{D(D^\lambda f(z))}{g(z)} > 0$ in $\mathbb{U}$, there exists a positive real part function

$\mathtt{p}(z) = 1+\sum_{n=1}^{\infty} c_n z^n \in \mathcal{A}$ *so that,*

$$\frac{D(D^\lambda f(z))}{g(z)} = \mathtt{p}(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{10}$$

Similarly for $\frac{D(D^\lambda F(w))}{G(w)} > 0$ in $\mathbb{U}$, there exists a positive real part function

$\mathtt{q}(w) = 1+\sum_{n=1}^{\infty} d_n w^n \in \mathcal{A}$ *so that,*

$$\frac{D(D^\lambda F(w))}{G(w)} = \mathtt{q}(w) = 1 + \sum_{n=1}^{\infty} d_n w^n. \tag{11}$$

We know that the Carathéodory Lemma [7] gives $|c_n| \leq 2$ and $|d_n| \leq 2$.

Matching the corresponding coefficients of Eqs. (8) and (10) (for any $n \geq 2$) yields,

$$\left(\frac{n\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n - b_n \right)$$
$$+\sum_{s=1}^{n-2} K_s^{-1}(b_2, b_3, ..., b_{s+1}) \left(\frac{(n-s)\Gamma(2-\lambda)\Gamma(n-s+1)}{\Gamma(n-s+1-\lambda)} a_{n-s} - b_{n-s} \right) = c_{n-1}. \tag{12}$$

Similarly, from Eqs. (9) and (11), we can find

$$\left(\frac{n\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} A_n - B_n \right)$$
$$+\sum_{s=1}^{n-2} K_s^{-1}(B_2, B_3, ..., B_{s+1}) \left(\frac{(n-s)\Gamma(2-\lambda)\Gamma(n-s+1)}{\Gamma(n-s+1-\lambda)} A_{n-s} - B_{n-s} \right) = d_{n-1}. \tag{13}$$

For the special case $n = 2$ from Eqs. (12) and (13) respectively yield,

$$\frac{2\Gamma(2-\lambda)\Gamma(3)}{\Gamma(3-\lambda)} a_2 - b_2 = c_1 \qquad \textit{and} \qquad -\frac{2\Gamma(2-\lambda)\Gamma(3)}{\Gamma(3-\lambda)} a_2 - B_2 = d_1$$

solving for a_2 and taking the absolute values we can obtain $|a_2| \leq \frac{2\Gamma(3-\lambda)}{\Gamma(2-\lambda)\Gamma(3)}$.

But under the assumption $a_k = 0$, $2 \leq k \leq n-1$ Eqs. (12) and (13) respectively yield,

$$\frac{n\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}a_n - b_n = c_{n-1} \qquad \textit{and} \qquad -\frac{n\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}a_n - B_n = d_{n-1}.$$

Solving either of above equations for a_n and taking the moduli values, also applying the Carathéodory Lemma, we can obtain

$$|a_n| \leq \frac{(n+2)\Gamma(n+1-\lambda)}{n\Gamma(2-\lambda)\Gamma(n+1)}.$$

Noticing that $|b_n| \leq n$ and $|B_n| \leq n$.

When we take $\lambda = 0$ in our class $K_\Sigma(\lambda)$ we obtain, for $\alpha = 0$ the result of Hamidi and Jahangiri [9] as follows,

Corollary 2.2 For $0 \leq \alpha < 1$ let the function $f \in S$ be bi-close-to-convex of order α in $\mathbb{U}$. If $a_k = 0$ for $2 \leq k \leq n-1$, then

$$|a_n| \leq 1 + \frac{2(1-\alpha)}{n}.$$

As a special case to Theorem 1 we derive the resulting estimates for the first coefficients a_2, a_3 and $|a_2^2 - a_3|$ of functions $f \in K_\Sigma(\lambda)$.

Theorem 2.3 Let the function $f \in K_\Sigma(\lambda)$ and $F = f^{-1} \in K_\Sigma(\lambda)$. Then,

$$|a_2| \leq \min\left\{ \sqrt{\frac{2\Gamma(3-\lambda)\Gamma(4-\lambda)}{3\Gamma(3-\lambda)\Gamma(2-\lambda)\Gamma(4) - 2\Gamma(2-\lambda)\Gamma(3)\Gamma(4-\lambda)}},\ \frac{2\Gamma(3-\lambda)}{2\Gamma(2-\lambda)\Gamma(3) - \Gamma(3-\lambda)} \right\},$$

$$|a_3| \leq \frac{[4\Gamma(2-\lambda)\Gamma(3) + 2\Gamma(3-\lambda)]\Gamma(4-\lambda)}{[2\Gamma(2-\lambda)\Gamma(3) - \Gamma(3-\lambda)][3\Gamma(2-\lambda)\Gamma(4) - \Gamma(4-\lambda)]},$$

and

$$\left|a_2^2 - a_3\right| \leq \frac{2\Gamma(4-\lambda)}{3\Gamma(2-\lambda)\Gamma(4) - \Gamma(4-\lambda)}.$$

Proof.

For the function $g(z) = D^\lambda f(z)$ in the proof of Theorem 1, we have $a_n = b_n$.

For $n = 2$ Eqs. (12) and (13) respectively yield,

$$a_2\left[\frac{2\Gamma(2-\lambda)\Gamma(3)}{\Gamma(3-\lambda)} - 1\right] = c_1 \qquad \textit{and} \qquad a_2\left[\frac{-2\Gamma(2-\lambda)\Gamma(3)}{\Gamma(3-\lambda)} + 1\right] = d_1.$$

Taking the absolute values of either of above two equations gives

$$|a_2| \leq \frac{2\Gamma(3-\lambda)}{2\Gamma(2-\lambda)\Gamma(3) - \Gamma(3-\lambda)} \quad .$$

For $n = 3$ Eqs. (12) and (13) respectively yield,

$$\left[\frac{3\Gamma(2-\lambda)\Gamma(4)}{\Gamma(4-\lambda)}a_3 - b_3\right] + \left[\frac{2\Gamma(2-\lambda)\Gamma(3)}{\Gamma(3-\lambda)}a_2 - b_2\right](-b_2) = c_2$$

and

$$\left[\frac{3\Gamma(2-\lambda)\Gamma(4)}{\Gamma(4-\lambda)}A_3 - B_3\right] + \left[\frac{2\Gamma(2-\lambda)\Gamma(3)}{\Gamma(3-\lambda)}A_2 - B_2\right](-B_2) = d_2$$

when we make some simply arrangement we have

$$a_3\left[\frac{3\Gamma(2-\lambda)\Gamma(4)}{\Gamma(4-\lambda)} - 1\right] - a_2^2\left[\frac{2\Gamma(2-\lambda)\Gamma(3)}{\Gamma(3-\lambda)} - 1\right] = c_2 \tag{14}$$

and

$$\left(2a_2^2 - a_3\right)\left[\frac{3\Gamma(2-\lambda)\Gamma(4)}{\Gamma(4-\lambda)} - 1\right] + a_2^2\left[\frac{-2\Gamma(2-\lambda)\Gamma(3)}{\Gamma(3-\lambda)} + 1\right] = d_2. \tag{15}$$

Adding the above two equations and solving for $|a_2|$ by applying the Carathéodory Lemma we obtain

$$|2a_2|^2 = \frac{|c_2 + d_2|\,|\Gamma(3-\lambda)\Gamma(4-\lambda)|}{|3\Gamma(3-\lambda)\Gamma(2-\lambda)\Gamma(4) - 2\Gamma(2-\lambda)\Gamma(3)\Gamma(4-\lambda)|},$$

$$|a_2| \leq \sqrt{\frac{2\Gamma(3-\lambda)\Gamma(4-\lambda)}{3\Gamma(3-\lambda)\Gamma(2-\lambda)\Gamma(4) - 2\Gamma(2-\lambda)\Gamma(3)\Gamma(4-\lambda)}}$$

Substituting $a_2 = c_1\frac{\Gamma(3-\lambda)}{2\Gamma(2-\lambda)\Gamma(3)-\Gamma(3-\lambda)}$ in Eqs. (14) gives

$$a_3\left[\frac{3\Gamma(2-\lambda)\Gamma(4)}{\Gamma(4-\lambda)} - 1\right] - c_1^2\frac{\Gamma(3-\lambda)}{2\Gamma(2-\lambda)\Gamma(3) - \Gamma(3-\lambda)} = c_2$$

$$|a_3| \leq \frac{|c_2|\,|2\Gamma(2-\lambda)\Gamma(3) - \Gamma(3-\lambda)| + |c_1|^2\Gamma(3-\lambda)}{|2\Gamma(2-\lambda)\Gamma(3) - \Gamma(3-\lambda)|}\frac{\Gamma(4-\lambda)}{|3\Gamma(2-\lambda)\Gamma(4) - \Gamma(4-\lambda)|}$$

$$\leq \frac{[4\Gamma(2-\lambda)\Gamma(3) + 2\Gamma(3-\lambda)]\,\Gamma(4-\lambda)}{[2\Gamma(2-\lambda)\Gamma(3) - \Gamma(3-\lambda)]\,[3\Gamma(2-\lambda)\Gamma(4) - \Gamma(4-\lambda)]}.$$

Lastly, Subtracting Eqs. (14) from (15), we have $\left|a_2^2 - a_3\right|$ as follows:

$$\left|a_2^2 - a_3\right| \leq \frac{2\Gamma(4-\lambda)}{3\Gamma(2-\lambda)\Gamma(4) - \Gamma(4-\lambda)}.$$

For $\lambda = 0$ we have for the first initial coefficients of $|a_2|$ and $|a_3|$ (the case $\alpha = 0$) of Hamidi and Jahangiri [9].

Corollory 2.4 For $0 \leq \alpha < 1$ let the function $f \in S^*(\alpha)$ and $F = f^{-1} \in S^*(\alpha)$. Then,

$$|a_2| \leq \begin{cases} \sqrt{2(1-\alpha)}; & 0 \leq \alpha < \frac{1}{2} \\ 2(1-\alpha); & \frac{1}{2} \leq \alpha < 1. \end{cases}$$

and

$$|a_3| \leq \begin{cases} 2(1-\alpha); & 0 \leq \alpha < \frac{1}{2} \\ (1-\alpha)(3-2\alpha); & \frac{1}{2} \leq \alpha < 1. \end{cases}$$

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