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HARDY'S-SOBOLEV'S-TYPE INEQUALITIES ON TIME SCALE VIA ALPHA CONFORMABLE FRACTIONAL INTEGRAL

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Abstract. In this paper, we obtain some new generalizations of the Hardy's inequality on time scale for function α-fractional integral. Other integral inequalities are established as well, which have as special cases some recent proved Hardy-type inequalities on time scales.

1. Introduction

The classical Hardy inequality states that for $f \geq 0$ and integrable over any finite interval $(0, x)$ and f^p is integrable and convergent over $(0, \infty)$ and $p > 1$, then

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^p \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(t)\, dt, \tag{1}$$

holds and the constant $(p/p-1)^p$ is the best possible. Inequality (1) which is usually referred to in the literature as the classical Hardy inequality, was proved in 1925 by Hardy [1]. In 2005, Řehàk [5] stated that if $a > 0$, $p > 1$, and f be a nonnegative function such that the delta integral $\int_a^\infty f^p(s)\, \Delta s$ exists as a finite number, then

$$\int_a^\infty \left(\frac{1}{\sigma(t) - a} \int_a^{\sigma(t)} f(s)\, \Delta s \right)^p \leq \left(\frac{p}{p-1} \right)^p \int_a^\infty f^p(t)\, \Delta t, \tag{2}$$

unless $f \equiv 0$. If, in addition, $\mu(t)/t \to 0$ as $t \to \infty$, then the constant $(p/p-1)^p$ is the best possible. A family of inequalities that interpolate between Hardy and Sobolev inequalities is given by the Hardy-Sobolev inequality,

$$\int_I \frac{|f(x)|^p}{x^p} dx \leq C_p \int_I \left| f'(x) \right|^p dx, \tag{3}$$

which holds for any function $f \in W_0^{1,p}(I)$, with $I = (0, 1)$, where C_p is a positive constant. This inequality plays an important role in analysis and its applications.

The main aim of this paper is to prove a generalized versions of Hardy-Sobolev inequality on time scales via conformable calculus.

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2. Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, and the backward jump operator $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. If $\sigma(t) > t$ we say that t is right-scattered, while if $\rho(t) < t$ we say that t is left-scattered. Points that are simultaneously right-scattered and left-scattered are said to be isolated. If $\sigma(t) = t$, then t is called right-dense; if $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k := \mathbb{T}$.

The graininess function for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$. For more on the calculus on time scales, we refer the reader to [2, 4]. We review now the conformable fractional derivative and integral [11].

Definition 2.1. [11, Definition 1] *Let $f : \mathbb{T} \to \mathbb{R}$ be a real valued function on a time scale $\mathbb{T}$ and $\alpha \in (0, 1]$, $t \in \mathbb{T}^k$. Then, for $t > 0$, we define $T_\alpha(f)(t)$ to be the number, if one exists, such that for all $\varepsilon > 0$, there is a neighborhood $\mathcal{U}$ of t such that for all $s \in \mathcal{U}$,*

$$\left|[f^\sigma(t) - f(s)]\, t^{1-\alpha} - T_\alpha(f)(t)(\sigma(t) - s)\right| \le \varepsilon\, |\sigma(t) - s|.$$

We call $T_\alpha(f)(t)$ the conformable fractional derivative of f of order α at t, and we define the conformable fractional derivative at 0 as $T_\alpha(f)(0) = \lim_{t \longrightarrow 0} T_\alpha(f)(t)$.

Theorem 2.2. [11, Theorem 15] *Let f, $g : \mathbb{T} \to \mathbb{R}$ are conformable fractional differentiable of order α. Then, the following properties hold:*

(*a*) *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is conformable fractional differentiable with*

$$T_\alpha(f + g) = T_\alpha(f) + T_\alpha(g).$$

(*b*) *For any $\lambda \in \mathbb{R}$, $\lambda f : \mathbb{T} \to \mathbb{R}$ is conformable fractional differentiable with*

$$T_\alpha(\lambda f) = \lambda T_\alpha(f).$$

(*c*) *If f and g are rd-continuous, then the product $fg : \mathbb{T} \to \mathbb{R}$ is conformable fractional differentiable with*

$$\begin{aligned} T_\alpha(fg) &= gT_\alpha(f) + (f \circ \sigma)\, T_\alpha(g) \\ &= T_\alpha(g)\, f + (g \circ \sigma)\, T_\alpha(f). \end{aligned} \tag{4}$$

(*d*) *If f is rd-continuous, then $1/f$ is conformable fractional differentiable with*

$$T_\alpha\left(\frac{1}{f}\right) = \frac{T_\alpha(f)}{f\,(f \circ \sigma)},$$

valid at all points $t \in \mathbb{T}^k$ for which $f(t)\, f(\sigma(t)) \neq 0$,

(*e*) *If f and g are rd-continuous, then f/g is conformable fractional differentiable with*

$$T_\alpha\left(\frac{g}{f}\right) = \frac{T_\alpha(f)\, g - T_\alpha(g)\, f}{g\,(g \circ \sigma)},$$

valid at all points $t \in \mathbb{T}^k$ for which $g(t)\, g(\sigma(t)) \neq 0$.

Definition 2.3. [11, Definition 26] *Let $f:\mathbb{T}\to\mathbb{R}$ be a regulated function. Then the α-fractional integral of f is defined by*

$$\int f(t)\,\Delta^{\alpha}t=\int f(t)\,t^{\alpha-1}\Delta t.$$

Theorem 2.4. [11, Theorem 31] *Let $a,b,c\in\mathbb{T}$, $\lambda\in\mathbb{R}$ and let $f;g:\mathbb{T}\to\mathbb{R}$ be two rd-continuous functions. Then, the following properties hold:*

(1) $\int_a^b [f(t)+g(t)]\,\Delta^{\alpha}t=\int_a^b f(t)\,\Delta^{\alpha}t+\int_a^b g(t)\,\Delta^{\alpha}t,$

(2) $\int_a^b \lambda f(t)\,\Delta^{\alpha}t=\lambda\int_a^b f(t)\,\Delta^{\alpha}t,$

(3) $\int_a^b f(t)\,\Delta^{\alpha}t=-\int_b^a f(t)\,\Delta^{\alpha}t,$

(4) $\int_a^b f(t)\,\Delta^{\alpha}t=\int_a^c f(t)\,\Delta^{\alpha}t+\int_a^c f(t)\,\Delta^{\alpha}t,$

(5) $\int_a^a f(t)\,\Delta^{\alpha}t=0.$

Theorem 2.5 (Chain rule)**.** [11, Theorem 21] *Let $\alpha\in(0,1]$. Assume $g:\mathbb{T}\to\mathbb{R}$ is rd-continuous and conformable fractional differentiable of order α at $t\in\mathbb{T}^k$, and $f:\mathbb{R}\to\mathbb{R}$ is differentiable. Then there exists c in the real interval $[t,\sigma(t)]$ with*

$$T_{\alpha}(f\circ g)(t)=f^{'}(g(s))\,T_{\alpha}(g)(t)\,. \tag{5}$$

3. Main Results

Before stating the main results, we begin with the following lemma.

Lemma 3.1. *Let $\mathbb{T}$ be a time scale, $a,b\in\mathbb{T}$ with $a<b$, and let f,g are conformal fractional differentiable of order α. Then the integration by parts formula is given by*

$$\int_a^b T_{\alpha}(f)(t)\,g(t)\,\Delta^{\alpha}t=[f(t)\,g(t)]_a^b-\int_a^b f^{\sigma}(t)\,T(g)(t)\,\Delta^{\alpha}t.$$

Lemma 3.2. *Let $\mathbb{T}$ be a time scale, $a,b\in\mathbb{T}$ with $a<b$, $\alpha\in(0,1]$, $p\in\mathbb{R}$ with $p>1$, and let $\eta_p:(a,b]\cap\mathbb{T}\longrightarrow\mathbb{R}$ is a function defined by:*

$$\eta_p(t):=\frac{1}{(t-a)^p},\qquad \textit{for all } t\in(a,b]_{\mathbb{T}}\,.$$

Then the inequality

$$\eta_{p+1}^{\sigma}(t)\le-\frac{t^{\alpha-1}}{p}T_{\alpha}(\eta_p)(t)\le\eta_{p+1}(t)\,,\quad \textit{for all } t\in(a,b]_{\mathbb{T}}\,. \tag{6}$$

Definition 3.3. *Let $\mathbb{T}$ be a time scale, $a,b\in\mathbb{T}$ with $a<b$, $\alpha\in(0,1]$, $p\in\mathbb{R}$ with $p>1$, and let $f:\mathbb{T}\to\mathbb{R}$, we said that f belongs to $L_{\Delta}^{\alpha,p}([a,b]\cap\mathbb{T})$ provided that*

$$\int_a^b |f(t)|^p\,\Delta^{\alpha}t<\infty.$$

Remark 3.4. *Let $\mathbb{T}$ be a time scale, $a,\, t \in \mathbb{T}$, $\beta \in \mathbb{R}$ with $\beta > 0$, and let $f : \mathbb{T} \to \mathbb{R}$, for simplification, we note*

$$L_f^{\beta}(t) := \lim_{s \to t} \frac{f(s)}{(s-a)^{\beta}}.$$

Now, we are ready to state and prove the main results in this paper. We generalize the Hardy-Sobolev inequality the α-conformable fractional integral on time scales. As particular case we get Δ-inequalities on time scales for $\alpha = 1$. In the sequel we use $[a,b]_{\mathbb{T}}$ to denote $[a,b] \cap \mathbb{T}$.

Theorem 3.5. *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $0 < a < b$, $\alpha \in (0,1]$, $p \in \mathbb{R}$ with $p > 1$, and let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is conformal fractional differentiable of order α, such that $|f^{\sigma}(t)| \leq |f(t)|$, for $t \in [a,b]_{\mathbb{T}}$,*

$$L_f^{\alpha}(a) < +\infty \qquad \text{and} \qquad L_f^{\alpha}(b) = 0. \tag{7}$$

If $(\sigma(t) - a)^{1-\alpha} T_{\alpha}(f) \in L_{\Delta}^{\alpha,p}([a,b]_{\mathbb{T}})$. Then $(\sigma(t) - a)^{\alpha} f \in L_{\Delta^{\alpha}}^{p}([a,b]_{\mathbb{T}})$, there exist a constant $C_1(p,\alpha,a) > 0$ such that

$$\int_a^b \frac{|f(t)|^p}{(\sigma(t) - a)^{\alpha p}} \Delta^{\alpha} t \leq C_1(p,\alpha,a) \int_a^b \frac{|T_{\alpha}(f)(t)|^p}{(\sigma(t) - a)^{(\alpha-1)p}} \Delta^{\alpha} t, \tag{8}$$

with the constant

$$C_1(p,\alpha,a) = \left(\frac{a^{\alpha-1} p}{\alpha p - 1} \right)^p. \tag{9}$$

Proof. By definition the L_f^{α} in the points a, b and by $L_f^{\alpha}(a) < +\infty$ and $L_f^{\alpha}(b) = 0$, which implies

$$\lim_{t \to a} \eta_{\alpha p - 1}(t) |f(t)|^p = \lim_{t \to b} \eta_{\alpha p - 1}(t) |f(t)|^p = 0. \tag{10}$$

From Lemma 3.1 and (10), we obtain

$$\int_a^b \eta_{\alpha p - 1}(\sigma(t)) T_{\alpha}(|f(t)|^p) \Delta^{\alpha} t = -\int_a^b T_{\alpha}(\eta_{\alpha p - 1})(t) |f(t)|^p \Delta^{\alpha} t.$$

By inequality the (6), we find that

$$\int_a^b \frac{t^{1-\alpha} |f(t)|^p}{(\sigma(t) - a)^{\alpha p}} \Delta^{\alpha} t \leq \frac{1}{\alpha p - 1} \int_a^b \eta_{\alpha p - 1}(\sigma(t)) |T_{\alpha}(|f(t)|^p)| \Delta^{\alpha} t. \tag{11}$$

By using the (5), we obtain

$$T_{\alpha}(|f|^p)(t) = p |f(s)|^{p-2} f(s) T_{\alpha}(f)(t), \quad \text{for all } t \in [a,b]_{\mathbb{T}}, \text{ where } s \in [t, \sigma(t)],$$

which implies that

$$|T_{\alpha}(|f|^p)(t)| \leq p |f(t)|^{p-1} |T_{\alpha}(f)(t)|, \quad \text{for all } t \in [a,b] \tag{12}$$

Substituting (12) into (11), we have

$$\int_a^b \frac{t^{1-\alpha} |f(t)|^p}{(\sigma(t) - a)^{\alpha p}} \Delta^{\alpha} t \leq \frac{p}{\alpha p - 1} \int_a^b \frac{|f(t)|^{p-1} |T_{\alpha}(f)(t)|}{(\sigma(t) - a)^{\alpha p - 1}} \Delta^{\alpha} t. \tag{13}$$

Applying Hölder's inequality on time scale, on the term

$$\int_a^b \frac{|f(t)|^{p-1} |T_{\alpha}(f)(t)|}{(\sigma(t) - a)^{\alpha p - 1}} \Delta^{\alpha} t = \int_a^b \frac{|f(t)|^{p-1} t^{\frac{(\alpha-1)(p-1)}{p}}}{(\sigma(t) - a)^{\alpha(p-1)}} \frac{|T_{\alpha}(f)(t)| t^{\frac{\alpha-1}{p}}}{(\sigma(t) - a)^{\alpha - 1}} \Delta t, \tag{14}$$

with indices $p/p-1$ and p, we see that

$$\int_a^b \frac{|f(t)|^{p-1}|T_\alpha(f)(t)|}{(\sigma(t)-a)^{\alpha p-1}}\Delta^\alpha t \le \left(\int_a^b \frac{|f(t)|^p}{(\sigma(t)-a)^{\alpha p}}\Delta^\alpha t\right)^{\frac{p-1}{p}}\left(\int_a^b \frac{|T_\alpha(f)(t)|^p}{(\sigma(t)-a)^{(\alpha-1)p}}\Delta^\alpha t\right)^{\frac{1}{p}}. \tag{15}$$

Substituting (15) into (14), we have

$$\int_a^b \frac{|f(t)|^{p-1}|T_\alpha(f)(t)|}{(\sigma(t)-a)^{\alpha p-1}}\Delta^\alpha t \le \left(\int_a^b \frac{|f(t)|^p}{(\sigma(t)-a)^{\alpha p}}\Delta^\alpha t\right)^{\frac{p-1}{p}}\left(\int_a^b \frac{|T_\alpha(f)(t)|^p}{(\sigma(t)-a)^{(\alpha-1)p}}\Delta^\alpha t\right)^{\frac{1}{p}}. \tag{16}$$

From (13) and (16), we have

$$\left(\int_a^b \frac{t^{1-\alpha}|f(t)|^p}{(\sigma(t)-a)^{\alpha p}}\Delta^\alpha t\right)^p\left(\int_a^b \frac{|f(t)|^p}{(\sigma(t)-a)^{\alpha p}}\Delta^\alpha t\right)^{1-p} \le \left(\int_a^b \frac{|T_\alpha(f)(t)|^p}{(\sigma(t)-a)^{(\alpha-1)p}}\Delta^\alpha t\right).$$

Since $a>0$, we obtain

$$\int_a^b \frac{|f(t)|^p}{(\sigma(t)-a)^{\alpha p}}\Delta^\alpha t \le \left(\frac{a^{\alpha-1}p}{\alpha p-1}\right)^p\int_a^b \frac{|T_\alpha(f)(t)|^p}{(\sigma(t)-a)^{(\alpha-1)p}}\Delta^\alpha t.$$

Which is the desired inequality (8). This proves the Theorem. □

Remark 3.6. *Let $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$, such that $f(b)=f(a)=0$ and $f^\Delta(a)<\infty$, then $L_f^1(a)<\infty$ and $L_f^1(b)=0$.*

The next results provides some useful relationships concerning the space's Sobolev on time scales $W_{0,\Delta}^{1,p}([a,b]_{\mathbb{T}},\mathbb{R})$ and $W_{0,\Delta}^{2,p}([a,b]_{\mathbb{T}},\mathbb{R})$ initiated in [6]. As a special case of Theorem 3.5 when $\alpha=1$, we have the following Hardy-Sobolev inequality on time scales be the generalization the inequality (3).

Remark 3.7. *Assume that α =1 in Theorem 3.5, a, $b\in\mathbb{T}$ such that $0<a<b<\infty$ and let $f\in W_{0,\Delta}^{1,p}([a,b]_{\mathbb{T}})$, such that $|f^\sigma(t)|\le|f(t)|$, for $t\in[a,b]_{\mathbb{T}}$. Then $L_f^1(b)=0$ and $L_f^1(a)=f^\Delta(a)$. It is easy to see that the conditions the Theorem 3.5 are satisfied. Substituting $\alpha=1$ into (9), we have the following Hardy inequality*

$$\int_a^b \frac{|f(t)|^p}{(\sigma(t)-a)^p}\Delta t \le \left(\frac{p}{p-1}\right)^p\int_a^b \left|f^\Delta(t)\right|^p\Delta t.$$

We show some examples of application of Theorem 3.5.

Example 1. *Assume that $\mathbb{T}=\mathbb{R}$ in Theorem 3.5, $\alpha=1$, $a=\varepsilon$ and $b=\infty$, such that $\varepsilon>0$. Let $f\in W_0^{1,p}([\varepsilon,+\infty))$, then $L_f^1(\varepsilon)=f'(\varepsilon)$ and $L_f^1(\infty)=0$. It is easy to see that the conditions the Theorem 3.5 are satisfied. Then the Hardy inequality*

$$\int_\varepsilon^\infty \frac{|f(t)|^p}{(t-\varepsilon)^p}dt \le \left(\frac{p}{p-1}\right)^p\int_\varepsilon^\infty \left|f'(t)\right|^p dt, \qquad \text{for all } \varepsilon>0. \tag{17}$$

Example 2. *By Example 1, we have formula (17) holds for all $\varepsilon>0$ and $f\in W_0^{1,p}([\varepsilon,+\infty))$. If $\varepsilon\to 0$, we have the following Hardy inequality*

$$\int_0^\infty \frac{|f(t)|^p}{t^p}dt \le \left(\frac{p}{p-1}\right)^p\int_0^\infty \left|f'(t)\right|^p dt,$$

hold for all $f\in W_0^{1,p}([0,+\infty))$.

Remark 3.8. *Along the work, we give the main results and for simplification, we note*

$$\gamma(\alpha,p) := \inf_{t\in[a,b]\cap\mathbb{T}} \left\{1-\left(\frac{1-\alpha}{p}\right)\frac{\sigma(t)}{t}\right\}.$$

Theorem 3.9. *Let $\mathbb{T}$ be a time scale, $a,b\in\mathbb{T}$ with $0<a<b$, $\alpha\in(0,1]$, $p\in\mathbb{R}$ with $p>1$, and let $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ is conformal fractional differentiable of order α, such that $|f^{\sigma}(t)|\le|f(t)|$, for $t\in[a,b]_{\mathbb{T}}$,*

$$L_f^{\alpha+1}(a)<+\infty, \qquad L_f^{\alpha+1}(b)=0, \qquad \text{and} \qquad \gamma(\alpha,p)>0.$$

If $(\sigma(t)-a)^{\alpha} T_{\alpha}(f)\in L_{\Delta}^{\alpha,p}([a,b]_{\mathbb{T}})$. Then $(\sigma(t)-a)^{-\alpha-1} f\in L_{\Delta}^{\alpha,p}([a,b]_{\mathbb{T}})$, there exist a constant $C_2(p,\alpha,a)>0$ such that

$$\int_a^b \frac{|f(t)|^p}{(\sigma(t)-a)^{(\alpha+1)p}}\Delta^{\alpha}t \le C_2(p,\alpha,a)\int_a^b \frac{|T_{\alpha}(f)(t)|^p}{(\sigma(t)-a)^{\alpha p}}\Delta^{\alpha}t. \tag{18}$$

with the constant

$$C_2(p,\alpha,a)=\left(\frac{a^{\alpha-1}}{\gamma(\alpha,p)}\right)^p. \tag{19}$$

Proof. By definition the L_f^{α} in the points a,b and by $L_f^{\alpha+1}(a)<+\infty$ and $L_f^{\alpha+1}(b)=0$, which implies

$$\lim_{t\to a}\eta_{(\alpha+1)p-1}(t)|f(t)|^p=\lim_{t\to b}\eta_{(\alpha+1)p-1}(t)|f(t)|^p=0. \tag{20}$$

From Lemma 3.1 and (20), we obtain

$$\begin{aligned}\int_a^b \frac{|f(t)|^p}{(\sigma(t)-a)^{(\alpha+1)p}}\Delta^{\alpha}t &= \int_a^b \eta^{\sigma}_{(\alpha+1)p}(t)|f(t)|^p\Delta^{\alpha}t\\ &\le -\frac{1}{p}\int_a^b T_{\alpha}\left(\eta_{(\alpha+1)p-1}\right)(t)\,t^{\alpha-1}|f(t)|^p\Delta^{\alpha}t\\ &\le \frac{1}{p}\int_a^b \eta^{\sigma}_{(\alpha+1)p-1}(t)\,T_{\alpha}\left(t^{\alpha-1}|f(t)|^p\right)\Delta^{\alpha}t. \end{aligned}\tag{21}$$

Using the product rule the conformable fractional differentiable of order α, we have

$$\begin{aligned}\left|T_{\alpha}\left(t^{\alpha-1}|f(t)|^p\right)\right| &= \left|T_{\alpha}\left(t^{\alpha-1}\right)|f(t)|^p+(\sigma(t))^{\alpha-1}T_{\alpha}\left(|f(t)|^p\right)\right|\\ &\le \left|T_{\alpha}\left(t^{\alpha-1}\right)\right||f(t)|^p+t^{\alpha-1}\left|T_{\alpha}\left(|f(t)|^p\right)\right|. \end{aligned}\tag{22}$$

From (12) and (22), we see that

$$\left|T_{\alpha}\left(t^{\alpha-1}|f(t)|^p\right)\right|\le(1-\alpha)\frac{|f(t)|^p}{t}+pt^{\alpha-1}|f(t)|^{p-1}|T_{\alpha}(f)(t)|, \tag{23}$$

Substituting (23) into (21), we have

$$\int_a^b \frac{|f(t)|^p}{(\sigma(t)-a)^{(\alpha+1)p}}\Delta^{\alpha}t\le\frac{1-\alpha}{p}\int_a^b\frac{|f(t)|^p}{t(\sigma(t)-a)^{(\alpha+1)p-1}}+\int_a^b\frac{t^{\alpha-1}|f(t)|^{p-1}|T_{\alpha}(f)(t)|}{(\sigma(t)-a)^{(\alpha+1)p-1}}\Delta^{\alpha}t. \tag{24}$$

Applying Hölder's inequality on time scale, on the term

$$\int_a^b\frac{t^{\alpha-1}|f(t)|^{p-1}|T_{\alpha}(f)(t)|}{(\sigma(t)-a)^{(\alpha+1)p-1}}\Delta^{\alpha}t=\int_a^b\frac{|f(t)|^{p-1}t^{\frac{(2\alpha-2)(p-1)}{p}}}{(\sigma(t)-a)^{(\alpha+1)(p-1)}}\frac{|T_{\alpha}(f)(t)|\,t^{\frac{2\alpha-2}{p}}}{(\sigma(t)-a)^{\alpha}}\Delta t.$$

with indices $p/p-1$ and p, we see that

$$\begin{aligned}\int_a^b \frac{t^{\alpha-1}\left|f(t)\right|^{p-1}\left|T_\alpha(f)(t)\right|}{(\sigma(t)-a)^{(\alpha+1)p-1}}\Delta^\alpha t &\le \left(\int_a^b \frac{\left|f(t)\right|^p t^{2\alpha-2}}{(\sigma(t)-a)^{(\alpha+1)p}}\Delta t\right)^{\frac{p-1}{p}}\left(\int_a^b \frac{\left|T_\alpha(f)(t)\right|^p t^{2\alpha-2}}{(\sigma(t)-a)^{\alpha p}}\Delta t\right)^{\frac{1}{p}}\\ &\le a^{\alpha-1}\left(\int_a^b \frac{\left|f(t)\right|^p}{(\sigma(t)-a)^{(\alpha+1)p}}\Delta^\alpha t\right)^{\frac{p-1}{p}}\left(\int_a^b \frac{\left|T_\alpha(f)(t)\right|^p}{(\sigma(t)-a)^{\alpha p}}\Delta^\alpha t\right)^{\frac{1}{p}}.\end{aligned} \tag{25}$$

Therefore, we have

$$\int_a^b \frac{\left|f(t)\right|^p}{t(\sigma(t)-a)^{(\alpha+1)p-1}}\Delta^\alpha t \le \int_a^b \frac{\sigma(t)}{t}\frac{\left|f(t)\right|^p}{(\sigma(t)-a)^{(\alpha+1)p}}\Delta^\alpha t. \tag{26}$$

Substituting (26) into (24), we have

$$\int_a^b \frac{\left|f(t)\right|^p}{(\sigma(t)-a)^{(\alpha+1)p}}\Delta^\alpha t \le \frac{1-\alpha}{p}\int_a^b \frac{\sigma(t)}{t}\frac{\left|f(t)\right|^p}{(\sigma(t)-a)^{(\alpha+1)p}}\Delta^\alpha t + \int_a^b \frac{t^{\alpha-1}\left|f(t)\right|^{p-1}\left|T_\alpha(f)(t)\right|}{(\sigma(t)-a)^{(\alpha+1)p-1}}\Delta^\alpha t.$$

Then

$$\begin{aligned}\gamma(\alpha,p)\int_a^b \frac{\left|f(t)\right|^p}{(\sigma(t)-a)^{(\alpha+1)p}}\Delta^\alpha t &\le \int_a^b \left(1-\frac{(1-\alpha)\sigma(t)}{pt}\right)\frac{\left|f(t)\right|^p}{(\sigma(t)-a)^{(\alpha+1)p}}\Delta^\alpha t\\ &\le \int_a^b \frac{t^{\alpha-1}\left|f(t)\right|^{p-1}\left|T_\alpha(f)(t)\right|}{(\sigma(t)-a)^{(\alpha+1)p-1}}\Delta^\alpha t.\end{aligned} \tag{27}$$

Substituting (27) into (25), we have

$$\int_a^b \frac{\left|f(t)\right|^p}{(\sigma(t)-a)^{(\alpha+1)p}}\Delta^\alpha t \le \left(\frac{a^{\alpha-1}}{\gamma(\alpha,p)}\right)^p \int_a^b \frac{\left|T_\alpha(f)(t)\right|^p}{(\sigma(t)-a)^{\alpha p}}\Delta^\alpha t,$$

which is the desired inequality (18). This proves the Theorem. □

We show some example of application of Theorem 3.9.

Example 3. *Assume that $\mathbb{T}=\mathbb{N}$ in Theorem 3.9, $\alpha=\frac{1}{2}$, $a=1$, $b=\infty$, $p\in\mathbb{R}$ with $p>1$, and let $f:\mathbb{N}\to\mathbb{R}$, such that $\lim_{n\to\infty} n^{-\frac{1}{2}}f(n)=0$, $\left|f(n+1)\right|\le\left|f(n)\right|$, for $n\in\mathbb{N}$ and $f(1)=0$. Then*

$$\gamma\left(\frac{1}{2},p\right)=\frac{p-1}{p}, \qquad T_{\frac{1}{2}}(f)(n)=\sqrt{n}\Delta f(n), \qquad \text{for all } n\in\mathbb{N}.$$

It is easy to see that the conditions the Theorem 3.9 are satisfied. Furthermore assume that $\sum_{n=1}^{\infty}\frac{\left|\Delta f(n)\right|^p}{\sqrt{n}}$ is convergent. In this case, we have the following discrete Hardy inequality

$$\sum_{n=1}^{\infty}\frac{\left|f(n)\right|^p}{\sqrt{n}^{3p+1}} \le \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty}\frac{\left|\Delta f(n)\right|^p}{\sqrt{n}},$$

where $C_2\left(p,\frac{1}{2},1\right)=(p/p-1)^p$ is defined as in Theorem 3.9.

Remark 3.10. *Let $\mathbb{T}$ be a time scale, $\alpha\in(0,1]$, and let $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$. The following notation*

$$(T_\alpha \circ T_\alpha)(f) = T_\alpha^2(f).$$

Theorem 3.11. *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $0 < a < b$, $\alpha \in (0,1]$, $p \in \mathbb{R}$ with $p > 1$, and let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is conformal fractional differentiable of order 2α, such that $|f^{\sigma}(t)| \leq |f(t)|$, for $t \in [a,b]_{\mathbb{T}}$,*

$$L_f^{\alpha+1}(a) < +\infty, \qquad L_f^{\alpha+1}(b) = 0, \qquad \text{and} \qquad \gamma(\alpha,p) > 0.$$

If $(\sigma(t) - a)^{1-\alpha} T_{2\alpha}(f) \in L_{\Delta}^{\alpha,p}([a,b]_{\mathbb{T}})$. Then $(\sigma(t) - a)^{-(\alpha+1)} f \in L_{\Delta}^{\alpha,p}([a,b]_{\mathbb{T}})$, there exist a constant $C_3(p,\alpha,a) > 0$ such that

$$\int_a^b \frac{|f(t)|^p}{(\sigma(t)-a)^{(\alpha+1)p}} \Delta^{\alpha} t \leq C_3(p,\alpha,a) \int_a^b \frac{\left|T_{\alpha}^2(f)(t)\right|^p}{(\sigma(t)-a)^{(\alpha-1)p}} \Delta^{\alpha} t,$$

with the constant

$$C_3(p,\alpha,a) = \left(\frac{pa^{2\alpha-2}}{\gamma(\alpha,p)(\alpha p - 1)} \right)^p. \tag{28}$$

Proof. This is similar to the proof of the Theorem 3.5 and Theorem 3.9. □

As a special case of Theorem 3.11 when $\alpha = 1$, we have the following Hardy-type inequality.

Remark 3.12. *Let $\mathbb{T}$ be a time scale, assume that $\alpha = 1$ in Theorem 3.11, $a, b \in \mathbb{T}$ such that $a < b < \infty$ and let $f \in W_{0,\Delta}^{2,p}([a,b]_{\mathbb{T}})$, such that $|f^{\sigma}(t)| \leq |f(t)|$, for $t \in [a,b]_{\mathbb{T}}$. Then $\gamma(1,p) = 1$, $L_f^2(b) = 0$ and*

$$L_f^2(a) = \begin{cases} \dfrac{f^{\Delta}(a)}{\mu(a)}, & \text{if } \mu(a) > 0, \\ \dfrac{1}{2} f^{\Delta^2}(a), & \text{if } \mu(a) = 0. \end{cases}$$

It is easy to see that the conditions the Theorem 3.11 are satisfied, therefore, we have

$$C_p := C_3(p,1,a) = \left(\frac{p}{p-1} \right)^p, \tag{29}$$

where C_3 is defined as in Theorem 3.11, we have the Hardy inequality

$$\int_a^b \frac{|f(t)|^p}{(\sigma(t)-a)^{2p}} \Delta t \leq C_p \int_a^b \left| f^{\Delta^2}(t) \right|^p \Delta t.$$

We show some examples of application of Theorem 3.5.

Example 4. *Assume that $\mathbb{T} = \mathbb{R}$ in Theorem 3.11, $\alpha = 1$, $a = \varepsilon$ and $b = \infty$, such that $\varepsilon > 0$. Let $f \in W_0^{2,p}([\varepsilon, +\infty))$, then*

$$\gamma(1,p) = 1 > 0, \qquad L_f^2(\varepsilon) = f''(\varepsilon) \qquad \text{and} \qquad L_f^2(\infty) = 0.$$

It is easy to see that the conditions the Theorem 3.11 are satisfied. Then the Hardy inequality

$$\int_{\varepsilon}^{\infty} \frac{|f(t)|^p}{(t-\varepsilon)^{2p}} dt \leq C_P \int_{\varepsilon}^{\infty} \left| f''(t) \right|^p dt, \qquad \text{for all } \varepsilon > 0, \tag{30}$$

where C_p is defined as in (29).

Example 5. *By Example 4, we have formula* (30) *holds for all* $\varepsilon > 0$ *and* $f \in W_0^{2,p}([\varepsilon, +\infty))$. *If* $\varepsilon \to 0$, *we have the following Hardy inequality*

$$\int_0^\infty \frac{|f(t)|^p}{t^{2p}} dt \leq C_P \int_0^\infty \left| f''(t) \right|^p dt, \qquad \text{hold for all } f \in W_0^{2,p}([0, +\infty)),$$

where C_p *is defined as in* (29).

Corollary 3.13. *Let* $\mathbb{T}$ *be a time scale,* $a, b \in \mathbb{T}$ *with* $0 < a < b$, $\alpha \in (0, 1]$, $p \in \mathbb{R}$ *with* $p > 1$, *and let* $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ *is conformal fractional differentiable of order* 2α, *such that* $|f^\sigma(t)| \leq |f(t)|$, *for* $t \in [a, b]_{\mathbb{T}}$,

$$L_f^{\alpha+1}(a) < +\infty, \qquad L_f^{\alpha+1}(b) = 0, \qquad \text{and} \qquad \gamma(\alpha, p) > 0.$$

If $(\sigma(t) - a)^{1-\alpha} T_{2\alpha}(f) \in L_\Delta^{\alpha,p}([a, b]_{\mathbb{T}})$. *Then* $(\sigma(t) - a)^{-(\alpha+1)} f \in L_\Delta^{\alpha,p}([a, b]_{\mathbb{T}})$ *and* $(\sigma(t) - a)^{-\alpha} T_\alpha(f) \in L_\Delta^{\alpha,p}([a, b]_{\mathbb{T}})$, *there exist a constant* $C_4(p, \alpha, a) > 0$ *such that*

$$\int_a^b \frac{|f(t)|^p}{(\sigma(t) - a)^{(\alpha+1)p}} \Delta^\alpha t + \int_a^b \frac{|T_\alpha(f)(t)|^p}{(\sigma(t) - a)^{\alpha p}} \Delta^\alpha t \leq C_4(p, \alpha, a) \int_a^b \frac{\left| T_\alpha^2(f)(t) \right|^p}{(\sigma(t) - a)^{(\alpha-1)p}} \Delta^\alpha t.$$

Remark 3.14. *Let* $\mathbb{T}$ *be a time scale, assume that* $\alpha = 1$ *in Corollary 3.13,* $a, b \in \mathbb{T}$ *such that* $0 < a < b < \infty$ *and let* $f \in W_{0,\Delta}^{2,p}([a, b]_{\mathbb{T}})$, *such that* $|f^\sigma(t)| \leq |f(t)|$, *for* $t \in [a, b]_{\mathbb{T}}$. *Then, we have the Hardy inequality*

$$\int_a^b \frac{|f(t)|^p}{(\sigma(t) - a)^{2p}} \Delta t + \int_a^b \frac{\left| f^\Delta(t) \right|^p}{(\sigma(t) - a)^p} \Delta t \leq C_p \int_a^b \left| f^{\Delta^2}(t) \right|^p \Delta t,$$

where C_p *is constant.*

4. Conclusion

The study of integral inequalities on time scales via the α-fractional integral. In this paper we generalize integral inequalities on time scales to α-fractional integral. As special cases, one obtains previous Hardy's-sobolev's inequalities.

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