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A NEW SUBCLASS OF BAZILEVIC FUNCTION WITH FIXED ARGUMENT INVOLVING NEW GENERALIZED DIFFERENTIAL OPERATOR AND HYPERGEOMETRIC FUNCTION

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Abstract. In this work, a new subclass of Bazilevic function with fixed argument involving new generalized differential operator and hypergeometric function were considered. Coefficient estimates, distortion theorems, extreme points, radii of convexity and starlikeness for the class $T_n^\alpha(q, s, \mu, \lambda, \delta, a, \beta, A, B)$ and $T_n^\alpha(\theta, q, s, \mu, \lambda, \delta, a, \beta, A, B)$ were defined.

1. Introduction

Let T denote the class of function of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U) \tag{1}$$

which are analytic in the unit disk $E = z : |z| < 1$ and normalized by $f(0) = f' - 1 = 0$.
Recall that $Re\frac{zf'(z)}{f(z)} > 0$ refers to as a starlike function while $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$ refers to as a convex function.
Also, let T^α denote the class of functions of the form

$$f(z)^\alpha = \left(z + \sum_{k=2}^{\infty} a_k z^k\right)^\alpha (z \in E) \tag{2}$$

are analytic in the unit disk $E = z : |z| < 1$ and normalized by $f(0) = f'(0) - 1 = 0$ where $\alpha > 0$ and α is real. Using Binomial expansion in (2) , we obtain

$$f(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} a_k z^{\alpha+k-1} \quad (z \in E) \tag{3}$$

Researchers like [1], [3], [10] , [9] and the likes have used (3) to define several classes of analytic functions and their interesting result are too voluminous to discuss.

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Recently, [4] defined a differential operator as follows :

$$D^0_{\mu,\lambda,\delta}(a,\beta)f(z) = f(z)$$

$$D^1_{\mu,\lambda,\delta}(a,\beta)f(z) = \left(1 - \frac{\beta(\lambda-a)}{\mu+\lambda}\right) f(z) + \frac{\beta(\lambda-a)}{\mu+\lambda} z f'(z) + \frac{\delta}{\mu+\lambda} z^2 f''(z)$$

and for $m = 1, 2, 3, ...$

$$D^m_{\mu,\lambda,\delta}(a,\beta)f(z) = D_{\mu,\lambda,\delta}(a,\beta)\left(D^{m-1}_{\mu,\lambda,\delta}(a,\beta)f(z)\right)$$

If $f(z)$ is given by (1), then from the above operator, we get

$$D^m_{\mu,\lambda,\delta}(a,\beta)f(z) = z + \sum_{k=2}^{\infty}\left[1 + \frac{(k-1)[(\lambda-a)\beta + k\delta]}{\mu+\lambda} a_k z^k\right] \tag{4}$$

for $f \in T, a, \delta \geq 0, \beta, \lambda, \mu > 0$ and $m \in N_0$.
By specializing the parameters surrounded the operator, we obtained numerous operators studied by [2], [6], [13] and [14].
Therefore, applying (4) in (3), we have the following

$$D^0_{\mu,\lambda,\delta}(a,\beta)f(z)^\alpha = f(z)^\alpha$$

$$D^1_{\mu,\lambda,\delta}(a,\beta)f(z)^\alpha = \left(1 - \frac{\beta(\lambda-a)}{\mu+\lambda}\right) f(z)^\alpha + \frac{\beta(\lambda-a)}{\mu+\lambda} z \left(f(z)^\alpha\right)' + \frac{\delta}{\mu+\lambda} z^2 \left(f(z)^\alpha\right)''$$

and for $m = 1, 2, 3, ...$

$$D^m_{\mu,\lambda,\delta}(a,\beta)f(z)^\alpha = D_{\mu,\lambda,\delta}(a,\beta)\left(D^{m-1}_{\mu,\lambda,\delta}(a,\beta)f(z)^\alpha\right)$$

If $f(z)$ is given by (1), then from the above operator, we get

$$D^m_{\mu,\lambda,\delta}(a,\beta)f(z)^\alpha = \left(1 + (\alpha-1)\left(\frac{\beta(\lambda-a)+\alpha\delta}{\mu+\lambda}\right)\right)^m z^\alpha + \sum_{k=2}^{\infty}\left(1 + \frac{(\alpha+k-2)[(\lambda-a)\beta + (\alpha+k-1)\delta]}{\mu+\lambda}\right)^m a_k(\alpha) z^{\alpha+k-1} \tag{5}$$

where the parameters as earlier defined. Varying the parameters involved in (5), it gives birth to many operators which serves as a new generalization in this direction.
Two analytic function f and g written as $f * g$ denote the the convolutions of two functions given by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

For complex parameters,

$$\psi_1, ..., \psi_q \quad \phi_1, ..., \phi_s \left(\phi_j \neq 0, -1, -2, ...; j = 1, ..., s\right),$$

by

$${}_qF_s\left(\psi_1, ..., \psi_q; \phi_1, ..., \phi_s; z\right) = \sum_{k=0}^{\infty} \frac{(\psi_1)_k, ..., (\psi_q)_k}{(\phi_1)_k, ..., (\phi_s)_k} \frac{z^k}{k!}$$

$(q \leq s+1; q, s \in N_0 = 0, 1, 2, ...; z \in E)$, where $(\omega)_k$ is the pochammer symbol defined, in terms of the Gamma function Γ, by

$$(\omega)_k := \frac{\Gamma(\omega+k)}{\Gamma(\omega)} = \begin{cases} 1 & k = 0 \\ \omega(\omega+1)...(\omega+k-1) & (k \in N = N_0/0) \end{cases} \tag{6}$$

Corresponding to a function $h\left(\psi_1, ..., \psi_q; \phi_1, ..., \phi_s; z\right)$ defined by

$$h\left(\psi_1, ..., \psi_q; \phi_1, ..., \phi_s; z\right) = z\,{}_qF_s\left(\psi_1, ..., \psi_q; \phi_1, ..., \phi_s; z\right),$$

we consider a linear operator

$$H\left(\psi_1,...,\psi_q;\phi_1,...,\phi_s\right):A\longrightarrow A,$$

defined by the convolution

$$H\left(\psi_1,...,\psi_q;\phi_1,...,\phi_s\right)f(z):=h\left(\psi_1,...,\psi_q;\phi_1,...,\phi_s\right)f(z).$$

It is observed that, for a function $f(z)^\alpha$ of the form (3), we have

$$H\left(\psi_1,...,\psi_q;\phi_1,...,\phi_s\right)D^m_{\mu,\lambda,\delta}(a,\beta)f(z)^\alpha$$

$$=\left(1+(\alpha-1)\left(\frac{\beta(\lambda-a)+\alpha\delta}{\mu+\lambda}\right)\right)^m z^\alpha+\sum_{k=2}^{\infty}\Gamma_k^{-1}\left(1+\frac{(\alpha+k-2)[(\lambda-a)\beta+(\alpha+k-1)\delta]}{\mu+\lambda}a_kz^k\right)^m a_k(\alpha)z^{\alpha+k-1} \tag{7}$$

where

$$\Gamma_k=\sum_{k=0}^{\infty}\frac{(\phi_1)_{k-1},...,(\phi_s)_{k-1}}{(\psi_1)_{k-1}},...,(\psi_q)_{k-1}.(k-1)!. \tag{8}$$

The linear operator comprises of other operators has been considered by, [5], [6], [8] and [12].

Let $q,s\in N$ and suppose that the parameters $\psi_1,...,\psi_q$ and $\phi_1,...,\phi_s$ are positive real numbers. Also, let

$$0\le\theta\le\pi, 0\le B\le 1 \quad -B\le A<B.$$

2. Lemma and Definitions

For the purpose of this present investigation, the following Lemma and definitions are necessary:

Lemma [11] Let f be a function of the form

$$f(z)=\sum_{k=1}^{\infty}a_kz^k \quad (z\in U) \tag{9}$$

analytic in E. If $f\prec g$ and g belongs to convex then $|a_k|\le 1 \quad (k\in N)$.

Definition 1 Let $T_n^\alpha(q,s,\mu,\lambda,\delta,a,\beta,A,B)$ denote the class of functions $f(z)\in T^\alpha$ satisfying the following condition

$$\frac{H\left(\psi_1,...,\psi_q;\phi_1,...,\phi_s\right)D^m_{\mu,\lambda,\delta}(a,\beta)f(z)^\alpha}{\left(1+(\alpha-1)\left(\frac{\beta(\lambda-a)+\alpha\delta}{\mu+\lambda}\right)\right)^m z^\alpha}\prec\frac{1+Az}{1+Bz} \tag{10}$$

Definition 2 Let $T_n^\alpha(\theta,q,s,\mu,\lambda,\delta,a,\beta,A,B)$ denote the subclass of the class $T_n^\alpha(q,s,\mu,\lambda,\delta,a,\beta,A,B)$ of functions of the form (7) such that $arg(a_n)=\theta$ for $a_k\neq 0(k\in N)$.

In particular, for $q=s+1$ and $\psi_{s+1}=1$, we write

$$T_n^\alpha(\theta,q,s,\mu,\lambda,\delta,a,\beta,A,B)=T_n^\alpha(\theta,s,\mu,\lambda,\delta,a,\beta,A,B) \quad (q=s+1,\psi_{s+1}=1).$$

Thus, we can write every function $f(z)^\alpha$ from the the class $T_n^\alpha(\theta,q,s,\mu,\lambda,\delta,a,\beta,A,B)$ in the form

$$f(z)^\alpha=z^\alpha+e^{i\theta}\sum_{k=2}^{\infty}|a_k(\alpha)|z^{\alpha+k-1}. \tag{11}$$

3. Main Results

Here, the coefficient estimates, distortion theorems, extreme points, the radii of convexity and starlikeness for the class $T_n^\alpha(\theta, q, s, \mu, \lambda, \delta, a, \beta, A, B)$. Also, the coefficient estimates for the class $T_n^\alpha(q, s, \mu, \lambda, \delta, a, \beta, A, B)$ were derived.

Theorem 1 If a function $f(z)^\alpha$ of the form (3) belong to the class $T_n^\alpha(\theta, q, s, \mu, \lambda, \delta, a, \beta, A, B)$, then

$$\sum_{k=2}^{\infty} \Gamma_k^{-1} \left(\frac{(\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)}{(\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)} \right)^m |a_k(\alpha)| \leq \frac{B-A}{\sqrt{1-B^2 sin^2\theta} - Bcos\theta} \tag{12}$$

where Γ_k is defined in (8)

Proof. Let a function $f(z)^\alpha$ of the form (3) $T_n^\alpha(\theta, q, s, \mu, \lambda, \delta, a, \beta, A, B)$. Then by (10) and the definition of subordination, we have

$$\left(\left(1+(\alpha-1)\left(\frac{\beta(\lambda-a)+\alpha\delta}{\mu+\lambda} \right) \right)^{-m} z^{-\alpha} \right) \left(H\left(\psi_1, ..., \psi_q; \phi_1, ..., \phi_s\right) D^m_{\mu,\lambda,\delta}(a,\beta) f(z)^\alpha \right) = \frac{1+A\omega(z)}{1+B\omega(z)}$$

where $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in E$. Thus we obtain

$$\left| \frac{\frac{H(\psi_1,...,\psi_q;\phi_1,...,\phi_s) D^m_{\mu,\lambda,\delta}(a,\beta) f(z)^\alpha}{\left(\left(1+(\alpha-1)\left(\frac{\beta(\lambda-a)+\alpha\delta}{\mu+\lambda}\right)\right)^m z^\alpha \right)} - 1}{\frac{BH(\psi_1,...,\psi_q;\phi_1,...,\phi_s) D^m_{\mu,\lambda,\delta}(a,\beta) f(z)^\alpha}{\left(\left(1+(\alpha-1)\left(\frac{\beta(\lambda-a)+\alpha\delta}{\mu+\lambda}\right)\right)^m z^\alpha \right)} - A} \right| < 1.$$

Hence, by (7), we have

$$\left| \sum_{k=2}^{\infty} \Gamma_k^{-1} \left(\frac{(\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)}{(\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)} \right)^m |a_k(\alpha)| z^{k-1} \right| \tag{13}$$

$$< \left| B-A+Be^{i\theta} \sum_{k=2}^{\infty} \Gamma_k^{-1} \left(\frac{(\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)}{(\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)} \right)^m |a_k(\alpha)| z^{k-1} \right| \quad z \in E$$

where Γ_k is defined by (8). Putting $z = r (0 \leq r < 1)$, we find that

$$|\omega| < |B-A+B\omega e^\theta| \tag{14}$$

where for convenience

$$\sum_{k=2}^{\infty} \Gamma_k^{-1} \left(\frac{(\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)}{(\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)} \right)^m |a_k(\alpha)| r^{k-1}$$

Since ω is a real number, by (14) we have

$$(1-B^2)\omega^2 - [2B(B-A)cos\theta]\,\omega - (B-A)^2 < 0. \tag{15}$$

Solving this inequality with respect to ω, we obtain

$$\sum_{k=2}^{\infty} \Gamma_k^{-1} \left(\frac{(\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)}{(\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)} \right)^m |a_k(\alpha)| r^{k-1} \leq \frac{B-A}{\sqrt{1-B^2 sin^2\theta} - Bcos\theta} \tag{16}$$

which upon letting $r \longrightarrow 1^-$, readily yields the assertion (12) of Theorem.

Theorem 2 If a function f of the form (3) belong to the class $T_n^\alpha(\pi, q, s, \mu, \lambda, \delta, a, \beta, A, B)$, then

$$\sum_{k=2}^{\infty}\Gamma_k^{-1}\left(\frac{(\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)}{(\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)}\right)^m|a_k(\alpha)|\leq\frac{B-A}{1+B} \tag{17}$$

where Γ_k is defined in (8).

Proof. By the virtue of Theorem 1, we only need to show that the condition (17) is sufficient. Let a function $f(z)^\alpha$ of the form (11) satisfy the condition (17). Then, in view of (13), it is sufficient to prove that

$$\left|\sum_{k=2}^{\infty}\Gamma_k^{-1}\left(\frac{(\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)}{(\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)}\right)^m|a_k(\alpha)|z^{k-1}\right| \tag{18}$$

$$-\left|B-A-B\sum_{k=2}^{\infty}\Gamma_k^{-1}\left(\frac{(\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)}{(\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)}\right)^m|a_k(\alpha)|z^{k-1}\right|<0.\quad z\in E$$

Indeed, letting $|z|=r(0<r<1)$, we have

$$\left|\sum_{k=2}^{\infty}\Gamma_k^{-1}\left(\frac{(\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)}{(\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)}\right)^m|a_k(\alpha)|z^{k-1}\right| \tag{19}$$

$$-\left|B-A-B\sum_{k=2}^{\infty}\Gamma_k^{-1}\left(\frac{(\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)}{(\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)}\right)^m|a_k(\alpha)|z^{k-1}\right|$$

$$\leq\left(\sum_{k=2}^{\infty}\Gamma_k^{-1}\left(\frac{(\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)}{(\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)}\right)^m|a_k(\alpha)|r^{k-1}\right)$$

$$-\left(B-A-B\sum_{k=2}^{\infty}\Gamma_k^{-1}\left(\frac{(\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)}{(\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)}\right)^m|a_k(\alpha)|r^{k-1}\right)$$

$$<(1+B)\sum_{k=2}^{\infty}\Gamma_k^{-1}\left(\frac{(\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)}{(\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)}\right)^m|a_k(\alpha)|-(B-A)\leq 0, \tag{20}$$

which implies that $f\in T_n^\alpha(\pi,q,s,\mu,\lambda,\delta,a,\beta,A,B)$.

Theorem 3 If a function $f(z)^\alpha$ of the form (3) belongs to the class $f\in T_n^\alpha(\theta,q,s,\mu,\lambda,\delta,a,\beta,A,B)$

$$z^\alpha-\frac{(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m}{\Gamma_2\left((\mu+\lambda)+\alpha(\beta(\lambda-a)+(\alpha+1)\delta)\right)^m\sqrt{1-B^2sin^2\theta-Bcos\theta}}z^{\alpha+1}\leq|A^n_{\mu,\lambda,\delta}f(z)^\alpha| \tag{21}$$

$$\leq z^\alpha+\frac{(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m}{\Gamma_2\left((\mu+\lambda)+\alpha(\beta(\lambda-a)+(\alpha+1)\delta)\right)^m\sqrt{1-B^2sin^2\theta-Bcos\theta}}z^{\alpha+1}$$

where Γ_k can be obtained from (8).

Proof. From (3), we have

$$|f(z)^\alpha|\leq z^\alpha+\sum_{k=2}^{\infty}|a_k(\alpha)|z^{\alpha+k-1}\quad 0<|z|<1.$$

Using (12), we get

$$|f(z)^\alpha|\leq z^\alpha+\frac{(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m}{\Gamma_2\left((\mu+\lambda)+\alpha(\beta(\lambda-a)+(\alpha+1)\delta)\right)^m\sqrt{1-B^2sin^2\theta-Bcos\theta}}z^{\alpha+1}\quad 0<|z|<1.$$

Similarly, we get

$$|f(z)^\alpha| \geq z^\alpha - \frac{(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m}{\Gamma_2\left((\mu+\lambda)+\alpha(\beta(\lambda-a)+(\alpha+1)\delta)\right)^m\sqrt{1-B^2sin^2\theta-Bcos\theta}} z^{\alpha+1} \quad 0<|z|<1$$

which completes the proof.

Theorem 4 If a function $f(z)^\alpha$ of the form (3) belongs to the class $f \in T_n^\alpha(\theta, q, s, \mu, \lambda, \delta, a, \beta, A, B)$

$$\alpha z^{\alpha-1} - \frac{(\alpha+1)(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m}{\Gamma_2\left((\mu+\lambda)+\alpha(\beta(\lambda-a)+(\alpha+1)\delta)\right)^m\sqrt{1-B^2sin^2\theta-Bcos\theta}} z^\alpha \leq |\left(A^n_{\mu,\lambda,\delta}f(z)^\alpha\right)'| \tag{22}$$

$$\leq \alpha z^{\alpha-1} + \frac{(\alpha+1)(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m}{\Gamma_2\left((\mu+\lambda)+\alpha(\beta(\lambda-a)+(\alpha+1)\delta)\right)^m\sqrt{1-B^2sin^2\theta-Bcos\theta}} z^\alpha$$

where Γ_2 can be obtained from (8).

Proof. From (3), we have

$$|\left(f(z)^\alpha\right)'| \leq \alpha z^{\alpha-1} + \sum_{k=2}^{\infty}(\alpha+k-1)|a_k(\alpha)|z^{\alpha+k-2} \quad 0<|z|<1 \tag{23}$$

Using (12), we get

$$|\left(f(z)^\alpha\right)'| \leq \alpha z^{\alpha-1} + \frac{(\alpha+1)(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m}{\Gamma_2\left((\mu+\lambda)+\alpha(\beta(\lambda-a)+(\alpha+1)\delta)\right)^m\sqrt{1-B^2sin^2\theta-Bcos\theta}} z^\alpha \quad 0<|z|<1.$$

Similarly, we get

$$|\left(f(z)^\alpha\right)'| \geq \alpha z^{\alpha-1} - \frac{(\alpha+1)(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m}{\Gamma_2\left((\mu+\lambda)+\alpha(\beta(\lambda-a)+(\alpha+1)\delta)\right)^m\sqrt{1-B^2sin^2\theta-Bcos\theta}} z^\alpha \quad 0<|z|<1,$$

which completes the proof.

Theorem 5 Let $f \in T_n^\alpha(\pi, q, s, \mu, \lambda, \delta, a, \beta, A, B)$ and

$$z^{\alpha-1} - \frac{(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m}{(1+B)\Gamma_k\left((\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)\right)^m} z^{\alpha+k-1} \quad k \geq 2$$

if and only if it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty}\varphi_k f_k(z)$$

where $\varphi_k \geq 0$ and $\sum_{k=0}^{\infty} = 1$ and $f_k's$ for $k \geq 0$ are the points for $T_n^\alpha(\pi, q, s, \mu, \lambda, \delta, a, \beta, A, B)$.

Proof. Let

$$f(z) = \sum_{k=0}^{\infty}\varphi_k f_k(z)$$

$$= \left(1-\sum_{k=2}^{\infty}\varphi_k\right) z^\alpha + \sum_{k=2}^{\infty}\varphi_k\left(z^{\alpha-1} - \frac{(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m}{(1+B)\Gamma_k\left((\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)\right)^m} z^{\alpha+k-1}\right)$$

$$= z^\alpha - \sum_{k=2}^{\infty}\varphi_k \frac{(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m}{(1+B)\Gamma_k\left((\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)\right)^m} z^{\alpha+k-1}$$

which proves that $f \in T_n^\alpha(\pi, q, s, \mu, \lambda, \delta, a, \beta, A, B)$. Since

$$= \sum_{k=2}^{\infty}\frac{(1+B)\Gamma_k\left((\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)\right)^m}{(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m}$$

$$\left[\varphi_k \frac{(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m}{(1+B)\Gamma_k\left((\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)\right)^m}\right]$$

$$\sum_{k=2}^{\infty}\varphi_k < 1-\varphi_1 \leq 1.$$

Conversely, suppose that $f \in T_n^{\alpha}(\pi, q, s, \mu, \lambda, \delta, a, \beta, A, B)$ then using we set

$$\varphi_k = \frac{(1+B)\Gamma_k\left((\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)\right)^m}{(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m} a_k(\alpha)$$

for $k \geq 2$ and $\varphi_1 = 1-\sum_{k=2}^{\infty}\varphi_k$. Thus

$$f(z)^{\alpha} = z^{\alpha} - \sum_{k=2}^{\infty}|a_k(\alpha)|z^{\alpha+k-1} = f_0(z) - \sum_{k=2}^{\infty}\frac{(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m}{(1+B)\Gamma_k\left((\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)\right)^m} z^{\alpha+k-1}$$

$$= f_0(z) - \sum_{k=2}^{\infty}\left[z^{\alpha}f_k(z)\right]\varphi_k = \sum_{k=2}^{\infty}\varphi_k f_k(z)$$

this completes the proof.

Theorem 6 If a function f belongs to the class $T_n^{\alpha}(\theta, q, s, \mu, \lambda, \delta, a, \beta, A, B)$, then f is convex of $\xi(0 \leq \xi < 1)$ in the disk $U(r^c)$ where

$$r^c := \inf_{k\geq 2}\left\{\frac{(1-\xi)(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m}{k(k-\xi)(1+B)\Gamma_k\left((\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)\right)^m}\right\}^{\frac{1}{k-1}} \tag{24}$$

and Γ_k is defined by (8). For $\theta = \pi$, the result is sharp that is

$$R^c\left(T_n^{\alpha}(\pi, q, s, \mu, \lambda, \delta, a, \beta, A, B)\right) = r^c$$

where r^c is defined by (24).

Theorem 7 If a function f belongs to the class $T_n^{\alpha}(\theta, q, s, \mu, \lambda, \delta, a, \beta, A, B)$, then f is starlike of $\rho(0 \leq \rho < 1)$ in the disk $U(r^*)$ where

$$r^* := \inf_{k\geq 2}\left\{\frac{(1-\rho)(B-A)\left((\mu+\lambda)+(\alpha-1)(\beta(\lambda-a)+\alpha\delta)\right)^m}{(k-\rho)(1+B)\Gamma_k\left((\mu+\lambda)+(\alpha+k-2)(\beta(\lambda-a)+(\alpha+k-1)\delta)\right)^m}\right\}^{\frac{1}{k-1}} \tag{25}$$

and Γ_k is defined by (8). For $\theta = \pi$, the result is sharp that is

$$R^*\left(T_n^{\alpha}(\pi, q, s, \mu, \lambda, \delta, a, \beta, A, B)\right) = r^*$$

where r^* is defined by (25).

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