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FABER POLYNOMIAL COEFFICIENT ESTIMATES FOR A CLASS OF BI-UNIVALENT FUNCTIONS BASED ON THE SYMMETRIC Q-DERIVATIVE OPERATOR

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ABSTRACT. We introduce a new class of bi-univalent functions defined by using symmetric q-derivative operator. Moreover, using the Faber polynomials, we obtain general coefficient estimates for functions in this class.

1. Introduction, Definitions and Notations

Let A denote the class of functions f which are analytic in the open unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let S be the subclass of A consisting of functions f which are also univalent in U and let P be the class of functions

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} \varphi_n z^n$$

that are analytic in U and satisfy the condition $\Re(\varphi(z)) > 0$ in U. By the Caratheodory's lemma (e.g., see [9]) we have $|\varphi_n| \leq 2$.

It is well known that every function $f \in S$ has an inverse f^{-1}, satisfying $f^{-1}(f(z)) = z,\ (z \in U)$ and $f\left(f^{-1}(w)\right) = w,\ \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right)$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \tag{2}$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions in U given by (2). For a brief history and interesting examples in the class Σ, see the pioneering work on this subject by Srivastava et al. [23], which has apparently revived the study of bi-univalent functions in recent years.

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If the functions f and F analytic in U, then f is said to be subordinate to F, written as

$$f(z) \prec F(z), \quad z \in U$$

if there exists a Schwarz function

$$u(z) = \sum_{n=1}^{\infty} c_n z^n$$

with $|u(z)| < 1$ in U, such that

$$f(z) = F(u(z)).$$

For the Schwarz function $u(z)$ we note that $|c_n| < 1$. (e.g. see Duren [9]).

First formulae in what we now call q-calculus were obtained by Euler in the eighteenth century. In the second half of the twentieth century there was a significant increase of activity in the area of the q-calculus. The fractional calculus operators has gained importance and popularity, mainly due to its vast potential of demonstrated applications in various fields of applied sciences, engineering. The application of q-calculus was initiated by Jackson [14].

In the field of Geometric Function Theory, various subclasses of analytic functions have been studied from different viewpoints. The fractional q-calculus is the important tools that are used to investigate subclasses of analytic functions. Historically speaking, a firm footing of the usage of the the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [22]). In fact, the extension of the theory of univalent functions can be described by using the theory of q-calculus. Moreover, the q-calculus operators, such as fractional q-integral and fractional q-derivative operators, are used to construct several subclasses of analytic functions (see,e.g., [6], [17], [18], [24]). In a recent paper Purohit and Raina [19], investigated applications of fractional q-calculus operators to defined certain new classes of functions which are analytic in the open disk. Later, Mohammed and Darus [16] studied approximation and geometric properties of these q-operators in some subclasses of analytic functions in compact disk.

For the convenience, we provide some basic definitions and concept details of q-calculus which are used in this paper. We suppose throughout the paper that $0 < q < 1$. We shall follow the notation and terminology in [11]. We recall the definitions of fractional q-calculus operators of complex valued function $f(z)$.

Definition 1. *Let $q \in (0,1)$ and define*

$$[n]_q = \frac{1-q^n}{1-q},$$

for $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

Definition 2. *Let $q \in (0,1)$ and define the q-fractional $[n]_q!$ by*

$$[n]_q! = \begin{cases} \prod_{k=1}^{n} [k]_q, & n \in \mathbb{N} \\ 1, & n = 0 \end{cases}.$$

Definition 3. *For $\alpha \in \mathbb{C}$, the q-shifted factorial is defined as a product of $n \in \mathbb{N}_0 = \{0, 1, \ldots\}$ factors by*

$$(\alpha; q)_0 = 1, \quad (\alpha; q)_n = \prod_{i=0}^{n-1}(1 - \alpha q^i), \quad (\alpha; q)_\infty = \prod_{i=0}^{\infty}(1 - \alpha q^i).$$

Definition 4. *(see [14]) The q-derivative of a function f is defined on a subset of $\mathbb{C}$ is given by*

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1 - q)z}, \qquad if \quad z \neq 0, \tag{3}$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

Note that

$$\lim_{q \to 1^-} (D_q f)(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1 - q)z} = \frac{df(z)}{dz}$$

if f is differentiable. From (3), we deduce that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1}. \tag{4}$$

Definition 5. *(see [7]) The symmetric q-derivative $\widetilde{D}_q f$ of a function f given by (1) is defined as follows:*

$$(\widetilde{D}_q f)(z) = \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z}, \qquad if \quad z \neq 0, \tag{5}$$

and $(\widetilde{D}_q f)(0) = f'(0)$ provided $f'(0)$ exists.

From (5), we deduce that

$$(\widetilde{D}_q f)(z) = 1 + \sum_{n=2}^{\infty} \widetilde{[n]}_q a_n z^{n-1},$$

where the symbol $\widetilde{[n]}_q$ denotes the number

$$\widetilde{[n]}_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

frequently occurring in the study of q-deformed quantum mechanical simple harmonic oscillator (see [8]).

From (2) and (5), we also deduce that

$$\begin{aligned} (\widetilde{D}_q g)(w) &= \frac{g(qw) - g(q^{-1}w)}{(q - q^{-1})w} \\ &= 1 - \widetilde{[2]}_q a_2 w + \widetilde{[3]}_q \left(2a_2^2 - a_3\right) w^2 - \widetilde{[4]}_q \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^3 + \cdots . \end{aligned} \tag{6}$$

The Faber polynomials introduced by Faber [10] play an important role in various areas of mathematical sciences, especially in geometric function theory. Grunsky [12] succeeded in establishing a set of conditions for a given function which are necessary and in their totality sufficient for the univalency of this function, and in these conditions the coefficients of the Faber polynomials play an important role. Schiffer [20] gave a differential equations for univalent functions solving certain extremum problems with respect to coefficients of such functions; in this differential equation appears again a polynomial which is just the derivative of a Faber polynomial (Schaeffer-Spencer [21]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([5], [15], [13]). The coefficient estimate problem for each of $|a_n|$ $n \in \mathbb{N} \backslash \{1, 2\}$ is still an open problem.

The object of this paper is to introduce a new class of bi-univalent functions defined by using symmetric q-derivative operator. Moreover, we use the Faber polynomial expansions to obtain bounds for the general coefficients $|a_n|$ of bi-univalent functions in $S_{\Sigma}^{p,q}(\varphi)$ as well as we provide estimates for the initial coefficients of these functions.

2. Main results

Definition 6. *A function $f \in \Sigma$ is said to be in the class $\mathrm{S}_{\Sigma}^{p,q}(\varphi)$, for $p \in \mathbb{N}$, if the following subordination holes*

$$\left[(\widetilde{D}_q f)(z)\right]^p \prec \varphi(z)$$

and

$$\left[(\widetilde{D}_q g)(w)\right]^p \prec \varphi(w)$$

where $g(w) = f^{-1}(w)$.

Using the Faber polynomial expansion of functions $f \in A$ of the form (1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as, [3],

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ...) w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} \quad = \quad & \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!} a_2^{n-3} a_3 \\ & + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ & + \frac{(-n)!}{[2(-n+2)]!(n-5)!} a_2^{n-5} \left[a_5 + (-n+2) a_3^2\right] \\ & + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} \left[a_6 + (-2n+5) a_3 a_4\right] \\ & + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned} \tag{7}$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $|a_2|, |a_3|, ..., |a_n|$ [4]. In particular, the first three terms of K_{n-1}^{-n} are

$$\begin{aligned} \frac{1}{2}K_1^{-2} &= -a_2, \\ \frac{1}{3}K_2^{-3} &= 2a_2^2 - a_3, \\ \frac{1}{4}K_3^{-4} &= -\left(5a_2^3 - 5a_2a_3 + a_4\right). \end{aligned} \tag{8}$$

In general, for any $p \in \mathbb{N}$ and $n \geq 2$, an expansion of K_{n-1}^p is as, [3],

$$K_{n-1}^p = pa_n + \frac{p(p-1)}{2}E_{n-1}^2 + \frac{p!}{(p-3)!3!}E_{n-1}^3 + \ldots + \frac{p!}{(p-n+1)!(n-1)!}E_{n-1}^{n-1}, \tag{9}$$

where $E_{n-1}^p = E_{n-1}^p(a_2, a_3, \ldots)$ and by [1],

$$E_{n-1}^m(a_2, \ldots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1} \ldots (a_n)^{\mu_{n-1}}}{\mu_1! \ldots \mu_{n-1}!}, \quad \text{for } m \leq n$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, \ldots, \mu_n$ satisfying

$$\begin{aligned} \mu_1 + \mu_2 + \ \ldots \ + \mu_{n-1} &= m, \\ \mu_1 + 2\mu_2 + \ \ldots \ + (n-1)\mu_{n-1} &= n-1. \end{aligned}$$

Evidently, $E_{n-1}^{n-1}(a_2, \ldots, a_n) = a_2^{n-1}$, [2];
or equivalently,

$$E_n^m(a_1, a_2, \ldots, a_n) = \sum_{n=1}^{\infty} \frac{m!(a_1)^{\mu_1} \ldots (a_n)^{\mu_n}}{\mu_1! \ldots \mu_n!}, \quad \text{for } m \leq n$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, \ldots, \mu_n$ satisfying

$$\begin{aligned} \mu_1 + \mu_2 + \ \ldots \ + \mu_n &= m, \\ \mu_1 + 2\mu_2 + \ \ldots \ + n\mu_n &= n. \end{aligned}$$

It is clear that $E_n^n(a_1, a_2, \ldots, a_n) = a_1^n$. The first and the last polynomials are:

$$E_n^1 = a_n \qquad E_n^n = a_1^n.$$

Theorem 7. *For* $p \in \mathbb{N}$, *let* $f \in \mathrm{S}_\Sigma^{p,q}(\varphi)$. *If* $a_m = 0$; $2 \leq m \leq n-1$, *then*

$$|a_n| \leq \frac{2}{\widetilde{[n]}_q p}; \qquad n \geq 4 \tag{10}$$

Proof. Let f be given by (1). We have

$$\left[(\widetilde{D}_q f)(z)\right]^p = 1 + \sum_{n=2}^{\infty} K_n^p(\widetilde{[2]}_q a_2, \widetilde{[3]}_q a_3, \ldots, \widetilde{[n+1]}_q a_{n+1}) z^n, \tag{11}$$

and for $\left(\widetilde{D}_q g(w)\right)^p$, from (6), we have

$$\left[(\widetilde{D}_q g)(w)\right]^p = 1 + \sum_{n=2}^{\infty} K_n^p(b_1, b_2, \ldots, b_n) w^n. \tag{12}$$

On the other hand, for $f \in S_{\Sigma}^{p,q}(\varphi)$ and $\varphi \in P$ there are two Schwarz functions

$$u(z) = \sum_{n=1}^{\infty} c_n z^n$$

and

$$v(w) = \sum_{n=1}^{\infty} d_n w^n$$

such that

$$\left(\widetilde{D}_q f(z)\right)^p = \varphi(u(z)) \tag{13}$$

and

$$\left(\widetilde{D}_q g(w)\right)^p = \varphi(v(w)) \tag{14}$$

where

$$\varphi(u(z)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \varphi_k E_n^k(c_1, c_2, ..., c_n) z^n, \tag{15}$$

and

$$\varphi(v(w)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \varphi_k E_n^k(d_1, d_2, ..., d_n) w^n. \tag{16}$$

Comparing the corresponding coefficients of (11) and (15) yields

$$\widetilde{[n]}_q p a_n = \sum_{k=1}^{n-1} \varphi_k E_{n-1}^k(c_1, c_2, ..., c_{n-1}), \ n \geq 2$$

or

$$K_{n-1}^p(\widetilde{[2]}_q a_2, \widetilde{[3]}_q a_3, \ldots, \widetilde{[n]}_q a_n) = \varphi_1 c_{n-1} \tag{17}$$

and similarly, from (12) and (16) we obtain

$$\widetilde{[n]}_q p b_n = \sum_{k=1}^{n-1} \varphi_k E_{n-1}^k(d_1, d_2, ..., d_{n-1}), \quad n \geq 2.$$

or

$$K_{n-1}^p(b_1, b_2, ..., b_n) = \varphi_1 d_{n-1} \tag{18}$$

Note that for $a_m = 0$; $2 \leq m \leq n-1$ we have $b_n = -a_n$ and so

$$\begin{aligned} \widetilde{[n]}_q p a_n &= \varphi_1 c_{n-1} \\ -\widetilde{[n]}_q p a_n &= \varphi_1 d_{n-1} \end{aligned} \tag{19}$$

Now taking the absolute values of either of the above two equations in (19) and using the facts that $|\varphi_1| \leq 2$, $|c_{n-1}| \leq 1$and $|d_{n-1}| \leq 1$, we obtain

$$|a_n| = \frac{|\varphi_1 c_{n-1}|}{\widetilde{[n]}_q p} = \frac{|\varphi_1 d_{n-1}|}{\widetilde{[n]}_q p} \leq \frac{2}{\widetilde{[n]}_q p}.$$

□

Theorem 8. *Let* $f \in \mathrm{S}_{\Sigma}^{p,q}(\varphi)$. *Then*

(*i*) $$|a_2| \leq \min\left\{\frac{2q}{(1+q^2)p}, \frac{2q}{\sqrt{(1+q+q^2)(1-q+q^2)p}}\right\} = \frac{2q}{(1+q^2)p}$$

(*ii*) $$|a_3| \leq \begin{cases} \frac{4q^2}{(1+q^2)^2p^2} + \frac{2q^2}{(1+q+q^2)(1-q+q^2)p} & p \geq 2 \\ \frac{4q^2}{(1+q+q^2)(1-q+q^2)p} & p = 1 \end{cases}$$

(*iii*) $$\left|a_3 - a_2^2\right| \leq \frac{4q^2}{(1+q+q^2)(1-q+q^2)p}.$$

Proof. Replacing n by 2 and 3 in (17) and (18), respectively, we find that

$$\widetilde{[2]}_q p a_2 = \varphi_1 c_1, \tag{20}$$

$$\widetilde{[3]}_q p a_3 = \varphi_1 c_2 + \varphi_2 c_1^2, \tag{21}$$

$$-\widetilde{[2]}_q p a_2 = \varphi_1 d_1, \tag{22}$$

$$\widetilde{[3]}_q p(2a_2^2 - a_3) = \varphi_1 d_2 + \varphi_2 d_1^2 \tag{23}$$

From (20) or (22) we obtain

$$|a_2| = \frac{|\varphi_1 c_1|}{\widetilde{[2]}_q p} = \frac{|\varphi_1 d_1|}{\widetilde{[2]}_q p} \leq \frac{2}{\widetilde{[2]}_q p} = \frac{2q}{(1+q^2)p}. \tag{24}$$

Adding (21) to (23) implies

$$2\widetilde{[3]}_q p a_2^2 = \varphi_1(c_2 + d_2) + \varphi_2\left(c_1^2 + d_1^2\right)$$

or, equivalently,

$$|a_2| \leq \frac{2q}{\sqrt{(1+q+q^2)(1-q+q^2)p}}. \tag{25}$$

From (21),

$$|a_3| = \frac{\left|\varphi_1 c_2 + \varphi_2 c_1^2\right|}{\widetilde{[3]}_q p} \leq \frac{4q^2}{(1+q+q^2)(1-q+q^2)p}. \tag{26}$$

Next, in order to find the bound on the coefficient $|a_3|$, we substract (23) from (21). We thus get

$$2\widetilde{[3]}_q p\left(a_3 - a_2^2\right) = \varphi_1(c_2 - d_2) + \varphi_2\left(c_1^2 - d_1^2\right) \tag{27}$$

or

$$|a_3| \leq |a_2|^2 + \frac{|\varphi_1(c_2 - d_2)|}{2\widetilde{[3]}_q p} \leq |a_2|^2 + \frac{2}{\widetilde{[3]}_q p}. \tag{28}$$

Upon substituting the value of a_2^2 from (24) and (25) into (28), it follows that

$$|a_3| \leq \frac{4q^2}{(1+q^2)^2p^2} + \frac{2q^2}{(1+q+q^2)(1-q+q^2)p}$$

and

$$|a_3| \leq \frac{6q^2}{(1+q+q^2)(1-q+q^2)p}.$$

Finally, solving the equation (27) for $\left(a_3 - a_2^2\right)$, we obtain

$$\left|a_3 - a_2^2\right| = \frac{\left|\varphi_1\left(c_2 - d_2\right) + \varphi_2\left(c_1^2 - d_1^2\right)\right|}{2\widetilde{[3]}_q p} \leq \frac{4q^2}{(1+q+q^2)(1-q+q^2)p}.$$

□

References

[1] H. Airault, Symmetric sums associated to the factorization of Grunsky coefficients, in Conference, Groups and Symmetries, Montreal, Canada, April, 2007.

[2] H. Airault, Remarks on Faber polynomials, Int. Math. Forum, 3, 449-456, 2008.

[3] H. Airault, and H. Bouali, Differential calculus on the Faber polynomials, Bulletin des Sciences Mathematiques, 130, 179-222, 2006.

[4] H. Airault, and J. Ren, An algebra of differential operators and generating functions on the set of univalent functions, Bulletin des Sciences Mathematiques, 126, 343-367. 2002.

[5] Ş. Altınkaya, and S. Yalçın, Coefficient bounds for a subclass of bi-univalent functions, TWMS Journal of Pure and Applied Mathematics, 6, 180-185, 2015.

[6] M. Aydoğan, Y. Kahramaner, and Y. Polatoğlu, Close-to-Convex Functions Defined by Fractional Operator, Appl. Math. Sci. 7, 2769-2775, 2013.

[7] K.L. Brahim, and Y. Sidomou, On some symmetric q-special functions, Le Matematiche, LXVIII, 107-122, 2013.

[8] L.C. Biedenharn, The Quantum Group $SU_q(2)$ and a q-Analogue of the Boson Operators, J. Phys., A. 22, L873-L878, 1984.

[9] P.L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Springer, New York, USA, vol. 259, 1983.

[10] G. Faber, Über polynomische entwickelungen, Math. Ann., 57, 1569-1573, 1903.

[11] G. Gasper, and M. Rahman, Basic Hypergeometric Series (with a Foreword by Richard Askey), Encyclopedia of Mathematics and Its Applications, Vol. 35, Cambridge University Press, Cambridge, New York, Port Chester, Melbourne and Sydney, 1990; Second edition, Encyclopedia of Mathematics and Its Applications, Vol. 96, Cambridge University Press, Cambridge, London and New York, 2004.

[12] H. Grunsky, Koeffizientenbedingungen für schlicht abbildende meromorphe funktionen, Math. Zeit., 45, 29-61, 1939.

[13] S.G. Hamidi, and J.M. Jahangiri, Faber polynomial coefficients of bi-subordinate functions, C. R. Acad. Sci. Paris, Ser. I 354, 365-370, 2016.

[14] F.H. Jackson, On q-functions and a certain difference operator, Transactions of the Royal Society of Edinburgh, 46, 253–281, 1908.

[15] J.M. Jahangiri, and S.G. Hamidi, Coefficient estimates for certain classes of bi-univalent functions, International Journal of Mathematics and Mathematical Sciences, Volume 2013, Article ID 190560, 4 pages.

[16] A. Mohammed, and M. Darus, A generalized operator involving the q-hypergeometric function, Mat. Vesnik, 65, 454-465, 2013.

[17] G. Murugusundaramoorthy, and T. Janani, Meromorphic parabolic starlike functions associated with q-hypergeometric series, ISRN Mathematical Analysis, Volume 2014, Article ID 923607, 9 pages.

[18] Y. Polatoğlu, Growth and distortion theorems for generalized q-starlike functions, Advances in Mathematics: Scientific Journal, 5, 7-12, 2016.

[19] S.D. Purohit, and R.K. Raina, Fractional q-calculus and certain subclass of univalent analytic functions, Mathematica, 55, 62–74, 2013.

[20] M. Schiffer, A method of variation within the family of simple functions, Proc. London Math. Soc. 44, 432-449, 1938.

[21] A.C. Schaeffer, and D.C. Spencer, The coefficients of schlicht functions, Duke Math. J. 10, 611-635, 1943.

[22] H.M. Srivastava, Univalent functions, fractional calculus, and associated generalized hypergeometric functions, in Univalent Functions; Fractional Calculus; and Their Applications (H. M. Srivastava and S. Owa, Editors), Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[23] H.M. Srivastava, A.K. Mishra, and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. 23, 1188-1192, 2010.
[24] H.E. Özkan Uçar, Coefficient inequalities for q-starlike functions, Appl. Math. Comp. 276, 122-126, 2016.

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