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EXISTENCE OF SOLUTIONS FOR FRACTIONAL HAMILTONIAN SYSTEMS WITH NONLINEAR DERIVATIVE DEPENDENCE IN $\mathbb{R}$

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Abstract. In this paper, we investigate the existence of solution for the fractional differential equation with mixed derivatives

$$\begin{aligned} {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) + b(t)u(t) = & \quad f(t, u(t), {}_{-\infty}D^{\alpha}_t u(t)) \\ u \in H^{\alpha}(\mathbb{R}). & \end{aligned} \tag{1}$$

where $\alpha \in (1/2, 1)$ and f is a nonlinearity depending on the fractional derivative of the solution, The existence of a positive solution is stated through an iterative method based on Mountain Pass techniques.

1. Introduction

Fractional differential equations appear naturally in a number of fields such as physics, chemistry, biology, economics, control theory, signal and image processing, blood flow phenomena, etc. During last decades, the theory of fractional differential equations is an area intensively developed, due mainly to the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes (see [1, 2, 4, 8, 17] and the references therein). Therein, the composition of fractional differential operators has got much attention from many scientists, mainly due to its wide applications in modeling physical phenomena exhibiting anomalous diffusion. Specifically, the models involving a fractional differential oscillator equation, which contains a composition of left and right fractional derivatives, are proposed for the description of the processes of emptying the silo [5] and the heat flow through a bulkhead filled with granular material [11], respectively. Their studies show that the proposed models based on fractional calculus are efficient and describe well the processes.

In the aspect of theory, the study of fractional differential equations including both left and right fractional derivatives has attracted much attention by using variational methods [3, 6, 7, 10, 12, 13, 14, 15, 16, 18, 20, 21]. It is not easy to use the critical point theory to study the fractional differential equations including

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both left and right fractional derivatives, since it is often very difficult to establish a suitable space and a variational functional for the fractional problem.

In [13], the author studied the fractional nonlinear Dirichlet problem

$$\begin{aligned} &{}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = f(t,u(t)), \quad t \in [0,T], \\ &u(0) = u(T) = 0, \end{aligned} \tag{2}$$

where $\alpha \in (\frac{1}{2},1)$, and $f \in C([0,T]\times\mathbb{R},\mathbb{R})$ satisfies the Ambrosetti-Rabinowitz condition

(AR) There is a constant $\mu > 2$ such that

$$0 < \mu F(t,u) \leq uf(t,u) \ \text{ for every } \ t \in [0,T] \ \text{ and } \ u \in \mathbb{R}\setminus\{0\}$$

This condition is an effective tool to guarantee the boundedness of the (PS) sequence.

When the nonlinearity f does not depend on ${}_{-\infty}D_t^{\alpha}u$, in [6, 7, 12, 15, 16, 19, 20, 21], the authors use the Mountain Pass Theorem, Fountain Theorems and the genus properties in critical point theory to study the existence and multiplicity results for (1).

In [10], by performing variational methods combined with iterative technique, Sun and Zhang investigated the solvability of the following fractional boundary value problem

$$\begin{gathered} \tfrac{d}{dt}\Big(p\,{}_0D_t^{-\beta}(u'(t)) + q\,{}_tD_1^{-\beta}(u'(t))\Big) + f(t,u(t)) = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{gathered} \tag{3}$$

where $\beta \in (0,1)$, $0 < p = 1-q < 1$, ${}_0D_t^{-\beta}$, and ${}_tD_1^{-\beta}$ denote left and right Riemann-Liouville fractional integrals of order β, respectively, and $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous.

Motivated by the previous works, in [18], by using mountain pass theorem and iterative technique, Xie, Xiao and Luo, studied the existence of solutions for the following nonlinear fractional boundary value problem with dependence on fractional derivative

$$\begin{gathered} {}_tD_T^{\alpha}(p(t)\,{}_0D_t^{\alpha}u(t)) = f(t,u(t),{}_0D_t^{\alpha}u(t)), \quad t \in [0,T], \\ u(0) = u(T) = 0, \end{gathered} \tag{4}$$

where ${}_0D_t^{\alpha}$ and ${}_tD_T^{\alpha}$ are the left and right Riemann-Liouville fractional derivatives of order $1/2 < \alpha \leq 1$, respectively, and $f \in C([0,T]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $p \in C^1([0,T],\mathbb{R})$ with $p(t) > 0$ for $t \in [0,T]$.

In this paper, we investigate the existence of solution for the following fractional differential equation with mixed derivatives

$$\begin{gathered} {}_tD_{\infty}^{\alpha}({}_{-\infty}D_t^{\alpha}u(t)) + b(t)u(t) = \quad f(t,u(t),{}_{-\infty}D_t^{\alpha}u(t)) \\ u \in H^{\alpha}(\mathbb{R}). \end{gathered} \tag{5}$$

where the constant $\alpha \in (1/2,1)$, ${}_{-\infty}D_t^{\alpha}$ and ${}_tD_{\infty}^{\alpha}$ denote left and right Liouville - Weyl fractional derivatives of order α respectively and are defined by

$${}_{-\infty}D_t^{\alpha}u(t) = \frac{d}{dx}{}_{-\infty}I_t^{1-\alpha}u(t), \quad {}_tD_{\infty}^{\alpha}u(t) = -\frac{d}{dt}{}_tI_{\infty}^{1-\alpha}u(t),$$

and $b : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^3 \to \mathbb{R}$ are continuous functions.

We note, because the dependence of the nonlinearity on the fractional derivative of the solution, (5) is non-variational, we cannot find some functional such that its critical point is the solution corresponding to the problem (5), so the well-developed critical point theory is of no avail for, at least, a direct attack to the problem (5) above. However, when there is not the presence of the fractional derivative in the nonlinearity term, problem (5) has been studied by establishing corresponding variational structure in some suitable fractional space and applying the critical points theorems, see [15, 16]. Motivated by the works of Xie, Xiao and Luo [18] (see also [10]), for each $w \in H^\alpha(\mathbb{R})$ fixed, we consider the following fractional differential equation with mixed derivatives

$$\begin{aligned} {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u(t)) + b(t)u(t) = \quad & f(t, u(t), {}_{-\infty}D_t^\alpha w(t)) \\ u \in H^\alpha(\mathbb{R}). & \end{aligned} \tag{6}$$

Now problem (6) is variational (see [15, 16]) and we can treat it by variational methods.

Now we state our main assumptions. In order to find solutions of (6), we will assume the following general hypotheses.

(B) There are positive constants $\beta_1, \beta_2 \in \mathbb{R}$ such that

$$0 < \beta_1 < b(t) < \beta_2, \quad \forall t \in \mathbb{R}$$

(f_1) There is $\theta > 2$ such that

$$0 < \theta F(t,\sigma,\xi) \leq \sigma f(t,\sigma,\xi) \quad \forall (t,\sigma,\xi) \in \mathbb{R}\times\mathbb{R}\setminus\{0\}\times\mathbb{R},$$

where $F(t,\sigma,\xi) = \int_0^\sigma f(t,s,\xi)ds$.

(f_2) There exists some positive continuous function $\varrho : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{|t|\to+\infty} \varrho(t) = 0. \tag{7}$$

such that

$$|f(t,\sigma,\xi)| \leq \varrho(t)|\sigma|^{\theta-1} \ \text{ for all } \ (t,\sigma,\xi) \in \mathbb{R}^3,$$

(f_3) There is $\mu > 2$ such that $\lim_{|\sigma|\to+\infty} \frac{f(t,\sigma,\xi)}{|\sigma|^{\mu-1}} = 0$ uniformly with respect to $t, \xi \in \mathbb{R}$.

Remark 1 As a consequence of (f_1), there are constants $\Lambda_1 > 0$ and $\Lambda_2 > 0$ such that

$$F(t,\sigma,\xi) \geq \Lambda_1|\sigma|^\theta, \ \ |\sigma| \geq 1 \tag{8}$$

and

$$F(t,\sigma,\xi) \leq \Lambda_2|\sigma|^\theta, \ \ |\sigma| \leq 1. \tag{9}$$

In fact, by (f_1) we note that: $\theta F(t,s\sigma,\xi) \leq s\sigma f(t,s\sigma,\xi))$. Let $h(s) = F(t,s\sigma,\xi)$, then

$$\frac{d}{ds}\left(h(s)s^{-\theta}\right) \geq 0. \tag{10}$$

Considering $|\sigma| \leq 1$, we integrate (10), from 1 until $\frac{1}{|\sigma|}$ and we get

$$F(t,\sigma,\xi) \leq F(t, \frac{\sigma}{|\sigma|}, \xi)|\sigma|^\theta. \tag{11}$$

By other hand, if $|\sigma| \geq 1$, integrating (10), from $\frac{1}{|\sigma|}$ until 1 we get

$$F(t,\sigma,\xi) \geq |\sigma|^\theta F(t, \frac{\sigma}{|\sigma|}, \xi). \tag{12}$$

Now, since $\frac{\sigma}{|\sigma|} \in B(0,1)$ and $B(0,1)$ is compact, there are $\Lambda_1 > 0$ and $\Lambda_2 > 0$ such that

$$\Lambda_1 \leq F(t,\sigma,\xi) \leq \Lambda_2, \text{ for every } \sigma \in B(0,1).$$

Therefore we get the affirmation.

Before stating our results let us introduce the main ingredients involved in our approach. We let $H^\alpha(\mathbb{R})$ be the usual fractional Sobolev space (see Sect. §2) equipped with the norm

$$\|u\|_\alpha^2 = \int_{\mathbb{R}} u(t)^2 dt + \int_{\mathbb{R}} |w|^{2\alpha} \widehat{u}^2 dw.$$

For $u \in H^\alpha(\mathbb{R})$, b and f satisfying (B), $(f_1) - (f_3)$, as we see in Sect. §3, we may define the functional

$$I_w(u) = \frac{1}{2}\left(\int_{\mathbb{R}} (|_{-\infty}D_t^\alpha u(t)|^2 + b(t)|u(t)|^2]dt\right) - \int_{\mathbb{R}} F(t, u(t), {}_{-\infty}D_t^\alpha w(t))dt \quad (13)$$

which is of class C^1 and we have

$$I_w'(u)v = \int_{\mathbb{R}} [{}_{-\infty}D_t^\alpha u(t) {}_{-\infty}D_t^\alpha v(t) + b(t)u(t)v(t)]dt - \int_{\mathbb{R}} f(t, u(t), {}_{-\infty}D_t^\alpha w(t))v(t)dt, \quad (14)$$

for all $v \in H^\alpha(\mathbb{R})$. We say that $u \in H^\alpha(\mathbb{R})$ is a weak solution of (6) if u is a critical point of I_w.

Now we are in a position to state our existence theorem

Theorem 1 Suppose that (B), $(f_1) - (f_3)$ hold. Then there exist positive constants K_1 and K_2 such that, for each $w \in H^\alpha(\mathbb{R})$, problem (6) has a weak nontrivial solution u_w such that $K_1 \leq \|u_w\|_\alpha \leq K_2$.

We prove the existence of weak solution of (6) by applying the mountain pass theorem to the functional I_w defined on $H^\alpha(\mathbb{R})$. However, since the Palais-Smale sequences lose compactness in $\mathbb{R}$, we need extra arguments. To overcome this difficulty we adapt some ideas from [21] and [18].

To state our main result concerns the solvability of equation (5), we need a further assumption on f:

(f_4) (i)

$$|f(t,\sigma_1,\xi) - f(t,\sigma_2,\xi)| \leq L_1|\sigma_1 - \sigma_2|,$$

for all $(t,\sigma_1,\xi), (t,\sigma_2,\xi) \in \mathbb{R}^3$ with $\sigma_1, \sigma_2 \in [-\rho_1, \rho_1]$ and $\xi \in \mathbb{R}$

(ii)

$$|f(t,\sigma,\xi_1) - f(t,\sigma,\xi_2)| \leq L_2|\xi_1 - \xi_2|,$$

for all $(t,\sigma,\xi_1), (t,\sigma,\xi_2) \in \mathbb{R}^3$ with $\sigma \in [-\rho_1, \rho_1]$ and $\xi_1, \xi_2 \in \mathbb{R}$, where ρ_1 is a positive constant (see section §4).

(iii) $L_1 + L_2 < \tilde{\gamma}$, where $\tilde{\gamma} = \min\{1, \beta_1\}$.

Theorem 2 Assume conditions $(B), (f_1) - (f_4)$ hold. Then problem (5) has a nontrivial weak solution.

We proof Theorem 2, using iteration methods.

The rest of the paper is organized as follows: In section §2, we describe the Liouville-Weyl fractional calculus and we introduce the fractional space that we use in our work and some proposition are proven which will aid in our analysis. In section §3, we give the proof of Theorem 1. Finally, in section §4, we give the proof of Theorem 2.

2. Preliminary Results

2.1. **Liouville-Weyl Fractional Calculus.** In this section we introduce some basic definitions of fractional calculus which are used further in this paper. For more details we refer the reader to [1].

The Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ are defined as

$$ {}_{-\infty}I_x^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-\xi)^{\alpha-1}u(\xi)d\xi \tag{15}$$

$$ {}_{x}I_{\infty}^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(\xi-x)^{\alpha-1}u(\xi)d\xi \tag{16}$$

The Liouville-Weyl fractional derivative of order $0 < \alpha < 1$ are defined as the left-inverse operators of the corresponding Liouville-Weyl fractional integrals

$$ {}_{-\infty}D_x^{\alpha}u(x) = \frac{d}{dx}\,{}_{-\infty}I_x^{1-\alpha}u(x) \tag{17}$$

$$ {}_{x}D_{\infty}^{\alpha}u(x) = -\frac{d}{dx}\,{}_{x}I_{\infty}^{1-\alpha}u(x) \tag{18}$$

We establish the Fourier transform properties of the fractional integral and fractional differential operators. Recall that the Fourier transform $\widehat{u}(w)$ of $u(x)$ is defined by

$$\widehat{u}(w) = \int_{-\infty}^{\infty} e^{-ix.w}u(x)dx.$$

Let $u(x)$ be defined on $(-\infty, \infty)$. Then the Fourier transform of the Liouville-Weyl integral and differential operator satisfies

$$\widehat{{}_{-\infty}I_x^{\alpha}u(x)}(w) = (iw)^{-\alpha}\widehat{u}(w), \quad \widehat{{}_{x}I_{\infty}^{\alpha}u(x)}(w) = (-iw)^{-\alpha}\widehat{u}(w) \tag{19}$$

$$\widehat{{}_{-\infty}D_x^{\alpha}u(x)}(w) = (iw)^{\alpha}\widehat{u}(w), \quad \widehat{{}_{x}D_{\infty}^{\alpha}u(x)}(w) = (-iw)^{\alpha}\widehat{u}(w) \tag{20}$$

2.2. **Fractional Derivative Spaces.** In this section we introduce some fractional spaces for more detail see [15].

Let $\alpha > 0$. Define the semi-norm

$$|u|_{I_{-\infty}^{\alpha}} = \|{}_{-\infty}D_x^{\alpha}u\|_{L^2}$$

and norm

$$\|u\|_{I_{-\infty}^{\alpha}} = \left(\|u\|_{L^2}^2 + |u|_{I_{-\infty}^{\alpha}}^2\right)^{1/2}, \tag{21}$$

and let

$$I_{-\infty}^{\alpha}(\mathbb{R}) = \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{I_{-\infty}^{\alpha}}}.$$

Now we define the fractional Sobolev space $H^{\alpha}(\mathbb{R})$ in terms of the fourier transform. Let $0 < \alpha < 1$, let the semi-norm

$$|u|_{\alpha} = \||w|^{\alpha}\widehat{u}\|_{L^2} \tag{22}$$

and norm

$$\|u\|_{\alpha} = \left(\|u\|_{L^2}^2 + |u|_{\alpha}^2\right)^{1/2},$$

and let

$$H^{\alpha}(\mathbb{R}) = \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{\alpha}}.$$

We note that a function $u \in L^2(\mathbb{R})$ belong to $I_{-\infty}^{\alpha}(\mathbb{R})$ if and only if

$$|w|^{\alpha}\widehat{u} \in L^2(\mathbb{R}). \tag{23}$$

Moreover

$$|u|_{I^{\alpha}_{-\infty}} = \||w|^{\alpha}\widehat{u}\|_{L^2}. \tag{24}$$

Therefore $I^{\alpha}_{-\infty}(\mathbb{R})$ and $H^{\alpha}(\mathbb{R})$ are equivalent with equivalent semi-norm and norm. Analogous to $I^{\alpha}_{-\infty}(\mathbb{R})$ we introduce $I^{\alpha}_{\infty}(\mathbb{R})$. Let the semi-norm

$$|u|_{I^{\alpha}_{\infty}} = \|{}_xD^{\alpha}_{\infty}u\|_{L^2}$$

and norm

$$\|u\|_{I^{\alpha}_{\infty}} = \left(\|u\|^2_{L^2} + |u|^2_{I^{\alpha}_{\infty}}\right)^{1/2}, \tag{25}$$

and let

$$I^{\alpha}_{\infty}(\mathbb{R}) = \overline{C^{\infty}_0(\mathbb{R})}^{\|.\|_{I^{\alpha}_{\infty}}}.$$

Moreover $I^{\alpha}_{-\infty}(\mathbb{R})$ and $I^{\alpha}_{\infty}(\mathbb{R})$ are equivalent, with equivalent semi-norm and norm [15]. We recall the Sobolev Lemma.

Theorem 1 [12] If $\alpha > \frac{1}{2}$, then $H^{\alpha}(\mathbb{R}) \subset C(\mathbb{R})$ and there is a constant $C = C_{\alpha}$ such that

$$\sup_{x\in\mathbb{R}} |u(x)| \leq C\|u\|_{\alpha} \tag{26}$$

Remark 1 From Theorem 1, we now that if $u \in H^{\alpha}(\mathbb{R})$ with $1/2 < \alpha < 1$, then $u \in L^q(\mathbb{R})$ for all $q \in [2, \infty)$, because

$$\int_{\mathbb{R}} |u(x)|^q dx \leq \|u\|^{q-2}_{\infty}\|u\|^2_{L^2}.$$

Namely, the embedding $H^{\alpha}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ is continuous and there exists positive constant C_q such that

$$\|u\|_{L^q} \leq C_q\|u\|_{\alpha}.$$

In what follows, we introduce the fractional space in which we will construct the variational framework of (6). Let

$$X^{\alpha} = \left\{u \in H^{\alpha}(\mathbb{R}) | \quad \int_{\mathbb{R}} \left[|{}_{-\infty}D^{\alpha}_t u(t)|^2 + b(t)|u(t)|^2\right] dt < \infty\right\},$$

then X^{α} is a reflexive and separable Hilbert space with the inner product

$$\langle u, v\rangle_{X^{\alpha}} = \int_{\mathbb{R}} \left[({}_{-\infty}D^{\alpha}_t u(t),\ {}_{-\infty}D^{\alpha}_t v(t)) + b(t)u(t)v(t)\right] dt$$

and the corresponding norm

$$\|u\|^2_{X^{\alpha}} = \langle u, u\rangle_{X^{\alpha}}.$$

Lemma 1 Suppose b satisfies (B). Then X^{α} and $H^{\alpha}(\mathbb{R})$ are equal with equivalent norms.

Proof. By (B) we have

$$\tilde{\gamma}\|u\|^2_{\alpha} \leq \|u\|^2_{X^{\alpha}} \tag{27}$$

where $\tilde{\gamma} = \min\{1, \beta_1\}$, and

$$\|u\|^2_{X^{\alpha}} \leq \eta\|u\|^2_{\alpha} \tag{28}$$

where $\eta = \max\{1, \beta_2\}$. □

Now we introduce more notations and some necessary definitions. Let $\mathfrak{B}$ be a real Banach space, $I \in C^1(\mathfrak{B}, \mathbb{R})$, which means that I is a continuously Fréchet-differentiable functional defined on $\mathfrak{B}$. Recall that $I \in C^1(\mathfrak{B}, \mathbb{R})$ is said to satisfy the Palais-Smale condition if any sequence $\{u_k\}_{k\in\mathbb{N}} \in \mathfrak{B}$, for which $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$, possesses a convergent subsequence in $\mathfrak{B}$.

Moreover, let B_r be the open ball in $\mathfrak{B}$ with the radius r and centered at 0 and ∂B_r denote its boundary. For the reader convenience we recall the Mountain Pass Theorems, see [9].

Mountain Pass Theorem Let $\mathfrak{B}$ be a real Banach space and $I \in C^1(\mathfrak{B}, \mathbb{R})$ satisfying the (PS) condition. Suppose that $I(0) = 0$ and

(i) there are constants ρ, β such that $I_{\partial B_\rho} \geq \beta$, and

(ii) there is an $e \in \mathfrak{B} \setminus \overline{B}_\rho$ such that $I(e) \leq 0$

Then I possesses a critical value $c \geq \alpha$. Moreover c can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I(\gamma(s)),$$

where

$$\Gamma = \{\gamma \in C([0,1], \mathfrak{B}) : \ \ \gamma(0) = 0, \gamma(1) = e\}.$$

3. Proof of Theorem 1

Let $w \in H^\alpha(\mathbb{R})$. We say that $u \in X^\alpha$ is a weak solution of (6), if

$$\int_{\mathbb{R}} [{}_{-\infty}D_t^\alpha u(t) {}_{-\infty}D_t^\alpha v(t) + b(t)u(t)v(t)]dt = \int_{\mathbb{R}} f(t, u(t), {}_{-\infty}D_t^\alpha w(t))v(t)dt, \quad (29)$$

for all $v \in X^\alpha$.

As usual, a weak solution of a problem as in (6), which is variational, is obtained as a critical point of an associated functional $I_w : X^\alpha \to \mathbb{R}$, defined by

$$I_w(u) = \frac{1}{2}\left(\int_{\mathbb{R}} (|{}_{-\infty}D_t^\alpha u(t)|^2 + b(t)|u(t)|^2]dt\right) - \int_{\mathbb{R}} F(t, u(t), {}_{-\infty}D_t^\alpha w(t))dt.$$

The proof of Theorem 1, is broken into several lemmas. First we prove that the functional I_w has the geometry of the Mountain-Pass Theorem.

First, we note, due to (7) and from (f_2), it is easy to see that

$$f(t, \sigma, \xi) = o(\sigma) \ \text{ as } \ \sigma \to 0 \quad (30)$$

uniformly with respect to $t, \xi \in \mathbb{R}$ and

$$|F(t, \sigma, \xi)| = \left|\int_0^1 f(t, s\sigma, \xi)\sigma ds\right| \leq \tilde{\varrho}|\sigma|^\mu \int_0^1 s^{\mu-1}ds = \frac{\tilde{\varrho}}{\mu}|\sigma|^\mu,$$

where $\tilde{\varrho} = \max_{t\in\mathbb{R}} \varrho(t)$. Hence, we have

$$|F(t, \sigma, \xi)| = o(\sigma^2)$$

as $\sigma \to 0$ uniformly with respect to $t, \xi \in \mathbb{R}$. That is, for any $\epsilon > 0$, there is $\delta > 0$ such that

$$|F(t, \sigma, \xi)| \leq \epsilon|\sigma|^2, \ \ (t, \sigma, \xi) \in \mathbb{R}^3 \ \text{ and } \ |\sigma| \leq \delta. \quad (31)$$

Lemma 1 Let $w \in H^\alpha(\mathbb{R})$. Under assumptions of Theorem 1, there exists $\rho, \beta > 0$ independent of w such that

$$I_w(u) \geq \beta > 0, \ \text{ if } \ \|u\|_\alpha = \rho.$$

Proof. By (31), for all $\epsilon > 0$, there exists $\delta > 0$ such that $|F(t, u, \xi)| \leq \epsilon|u|^2$ whenever $|u| \leq \delta$. Letting $\rho = \frac{\delta}{C_\alpha}$ and $\|u\|_\alpha = \rho$, we have $\|u\|_\infty \leq \delta$. Hence, we have

$$|F(t, u(t), \xi)| \leq \epsilon|u(t)|^2 \ \text{ for all } \ t, \xi \in \mathbb{R}.$$

Integrating on $\mathbb{R}$, we get

$$\left|\int_{\mathbb{R}} F(t,u,{}_{-\infty}D_t^{\alpha}w)dt\right| \le \epsilon\|u\|_{L^2}^2 \le \epsilon C_2^2\|u\|_{\alpha}^2,$$

where C_2 is defined in Remark 1 - section §2. In consequence, combining this with Lemma 1 - section §2, we obtain that, for $\|u\|_\alpha = \rho$,

$$\begin{aligned} I_w(u) &= \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} F(t,u(t),{}_{-\infty}D_t^{\alpha}w(t))dt \\ &\ge \left(\frac{\tilde{\gamma}}{2} - \epsilon C_2^2\right)\|u\|_{\alpha}^2. \end{aligned} \tag{32}$$

Setting $\epsilon = \frac{\tilde{\gamma}}{4C_2^2}$, the inequality (32) implies that

$$I_w|_{\partial B_\rho} \ge \frac{1}{4}\frac{\tilde{\gamma}\delta^2}{C_\alpha^2} = \beta > 0.$$

□

Lemma 2 Let $w \in H^\alpha(\mathbb{R})$. Under assumptions of Theorem 1, fix $\varphi \in C_0^\infty(\mathbb{R}) \subset H^\alpha(\mathbb{R})$ with $\|\varphi\|_\alpha = 1$, there exists $\mathfrak{s} > 0$ independent of w such that

$$I_w(s\varphi) \le 0, \text{ if } s > \mathfrak{s}.$$

Proof. Let $\varphi \in H^\alpha(\mathbb{R})$ with $|\varphi(t)| = 1$ for all $t \in [0,1]$. For every $s \in [1,\infty)$, by (8) and Lemma 1 - section §2, we get

$$\begin{aligned} I_w(s\varphi) &= \frac{1}{2}\|s\varphi\|_{\alpha}^2 - \int_{\mathbb{R}} F(t,s\varphi(t),{}_{-\infty}D_t^{\alpha}w(t))dt \\ &\le \frac{s^2}{2}\|\varphi\|_\alpha^2 - \Lambda_1 s^\theta\|\varphi\|_{L^\theta}^\theta \end{aligned}$$

Since $\theta > 2$, we conclude taking s big enough. □

Now, our purpose is to show that I_w, satisfies the (PS)-condition. To do this, firsts we prove the following Lemma.

Lemma 3 Let $w \in H^\alpha(\mathbb{R})$. Under the conditions of Theorem 1, ϕ_w' is compact, i.e., $\phi_w'(u_n) \to \phi_w'(u)$ if $u_n \rightharpoonup u$ in $H^\alpha(\mathbb{R})$, where $\phi_w : H^\alpha(\mathbb{R}) \to \mathbb{R}$ is defined by

$$\phi_w(u) = \int_{\mathbb{R}} F(t,u,{}_{-\infty}D_t^{\alpha}w)dt.$$

Proof. Assume that $u_n \rightharpoonup u$ in $H^\alpha(\mathbb{R})$. Then there exists a constant $\mathfrak{K} > 0$ such that

$$\|u_n\|_\alpha \le \mathfrak{K} \text{ and } \|u\|_\alpha \le \mathfrak{K}. \tag{33}$$

In view of (f_2), for any $\epsilon > 0$, there exists $R > 0$ such that

$$|f(t,u,\xi)| \le \epsilon|u|^{\theta-1} \text{ and } |f(t,u_n,\xi)| \le \epsilon|u_n|^{\theta-1} \tag{34}$$

for $(t,u,\xi)\in\mathbb{R}^3$ with $|t|>R$. Consequently, for n large enough, we have

$$
\begin{aligned}
|\langle \phi_w'(u_n)-\phi_w'(u),v\rangle| &= \left|\int_{\mathbb{R}}[f(t,u_n,{}_{-\infty}D_t^\alpha w)-f(t,u,{}_{-\infty}D_t^\alpha w)]v(t)dt\right|\\
&\le \left|\int_{|t|\le R}(f(t,u_n,{}_{-\infty}D_t^\alpha w)-f(t,u,{}_{-\infty}D_t^\alpha w))vdt\right|\\
&+ \left|\int_{|t|> R}(f(t,u_n,{}_{-\infty}D_t^\alpha w)-f(t,u,{}_{-\infty}D_t^\alpha w))vdt\right|\\
&\le \epsilon\|v\|_\infty+\int_{|t|>R}|f(t,u_n,{}_{-\infty}D_t^\alpha w)||v|dt\\
&\quad +\int_{|t|>R}|f(t,u,{}_{-\infty}D_t^\alpha w)||v|dt\\
&\le \epsilon C_\alpha\|v\|_\alpha+\epsilon\int_{|t|>R}|u_n|^{\theta-1}|v|dt+\epsilon\int_{|t|>R}|u|^{\theta-1}|v|dt\\
&\le \epsilon C_\alpha\|v\|_\alpha+\epsilon\int_{|t|>R}\left(\frac{\theta-1}{\theta}|u_n|^\theta+\frac{1}{\theta}|v|^\theta\right)dt\\
&\quad +\epsilon\int_{|t|>R}\left(\frac{\theta-1}{\theta}|u|^\theta+\frac{1}{\theta}|v|^\theta\right)dt\\
&\le \epsilon C_\alpha\|v\|_\alpha+\frac{\epsilon(\theta-1)}{\theta}\int_{|t|>R}(|u_n|^\theta+|u|^\theta)dt\\
&\quad +\frac{2\epsilon}{\theta}\int_{|t|>R}|v|^\theta dt.
\end{aligned}
$$

Here we apply Young inequality:

$$
ab\le\frac{a^p}{p}+\frac{b^q}{q},\quad a,b>0,\quad p,q>1\ \text{ and }\ \frac{1}{p}+\frac{1}{q}=1.
$$

Consequently, we obtain that

$$
\begin{aligned}
\|\phi_w'(u_n)-\phi_w'(u)\|_{H^{-\alpha}} &= \sup_{\|v\|_\alpha=1}\left|\int_{\mathbb{R}}(f(t,u_n,{}_{-\infty}D_t^\alpha w)-f(t,u,{}_{-\infty}D_t^\alpha w))v(t)dt\right|\\
&\le \epsilon C_\alpha+2\epsilon(C_\theta\mathfrak{K})^\theta\frac{\theta-1}{\theta}+\epsilon C_\theta^\theta\frac{2}{\theta},
\end{aligned}
$$

which implies that ϕ_w' is a compact operator. □

Lemma 4 Under the conditions of Theorem 1, I_w satisfies the (PS) condition.

Proof. Assume that $\{u_n\}_{n\in\mathbb{N}}\subset H^\alpha(\mathbb{R})$ is a sequence such that $\{I_w(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I_w'(u_n)\to 0$ as $n\to+\infty$. Then there exists a constant $\mathfrak{C}>0$ such that

$$
|I_w(u_n)|\le\mathfrak{C}\ \text{ and }\ \|I_w'(u_n)\|_{H^{-\alpha}}\le\mathfrak{C} \tag{35}
$$

for every $n\in\mathbb{R}$, where $H^{-\alpha}(\mathbb{R})$ is the dual space of $H^\alpha(\mathbb{R})$.

Firstly, we show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded. In fact, in view of (f_1) and (35), we obtain that

$$
\frac{1}{2}\|u_n\|_{X^\alpha}^2<\int_{\mathbb{R}}F(t,u_n(t),{}_{-\infty}D_t^\alpha w(t))dt+\mathfrak{C}, \tag{36}
$$

and

$$\int_{\mathbb{R}} f(t,u_n(t),{}_{-\infty}D_t^\alpha w(t))u_n(t)dt < \mathfrak{C}\|u_n\|_{X^\alpha} + \int_{\mathbb{R}}(|{}_{-\infty}D_t^\alpha u_n(t)|^2 + b(t)|u_n(t)|^2)dt. \tag{37}$$

Therefore,

$$\begin{aligned} \frac{1}{2}\|u_n\|_{X^\alpha}^2 &< \frac{1}{\theta}\int_{\mathbb{R}} f(t,u_n(t),{}_{-\infty}D_t^\alpha w(t))u_n(t)dt + \mathfrak{C} \\ &< \frac{1}{\theta}\left[\mathfrak{C}\|u_n\|_{X^\alpha} + \|u_n\|_{X^\alpha}^2\right] + \mathfrak{C}. \end{aligned}$$

So

$$\left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|_{X^\alpha}^2 < \frac{\mathfrak{C}}{\theta}\|u_n\|_{X^\alpha} + \mathfrak{C}, \tag{38}$$

since $\theta > 2$, by (38), the boundness of $\{u_n\}_{n\in\mathbb{N}}$ follows directly.

On the other hand, according to Lemma 3, ϕ_w' is compact. Therefore, there exists a subsequence, still denotes as $\{u_n\}_{n\in\mathbb{N}}$, such that $u_n \rightharpoonup u$ in $H^\alpha(\mathbb{R})$ and $\phi_w'(u_n) \to \phi_w'(u)$. So

$$\langle I_w'(u_n) - I_w'(u), u_n - u\rangle = \|u_n - u\|_{X^\alpha}^2 - \langle \phi_w'(u_n) - \phi_w'(u), u_n - u\rangle.$$

Therefore, as $I_w'(u_n) \to 0$, we deduce that

$$\|u_n - u\|_{X^\alpha} \to 0 \text{ as } n \to 0$$

and prove that the (PS) condition holds. □

Proof of Theorem 1. It its clear that $I_w(0) = 0$ and by Lemma 1, Lemma 2, $I_w \in C^1(H^\alpha(\mathbb{R}),\mathbb{R})$ satisfies the mountain pass geometry conditions and by Lemma 4, satisfies the (PS) condition. Therefore, by the Mountain Pass Theorem, I_w possesses a critical value $c_w \geq \beta > 0$ given by

$$c_w = \inf_{\gamma\in\Gamma_w}\max_{s\in[0,1]} I_w(\gamma(s)),$$

where

$$\Gamma_w = \{\gamma \in C([0,1], H^\alpha(\mathbb{R})) : \ \gamma(0) = 0, \gamma(1) = e\}.$$

Hence there is $0 \neq u_w \in H^\alpha(\mathbb{R})$ such that

$$I_w(u_w) = c_w \text{ and } I_w'(u_w) = 0.$$

That is, (6) has at least one nontrivial weak solution.

Further, since u_w is weak solution of problem (6), we have

$$\int_{\mathbb{R}}[{}_{-\infty}D_t^\alpha u_w(t){}_{-\infty}D_t^\alpha u_w(t) + b(t)u_w(t)u_w(t)dt = \int_{\mathbb{R}} f(t,u_w(t),{}_{-\infty}D_t^\alpha w(t))u_w(t)dt. \tag{39}$$

By (30) and (f_3), for every $\epsilon > 0$ there is $C_\epsilon > 0$ such that

$$|f(t,\sigma,\xi)| \leq \epsilon|\sigma| + C_\epsilon|\sigma|^{\mu-1}.$$

This implies

$$\|u_w\|_{X^\alpha}^2 \leq \epsilon C_2^2\|u_w\|_\alpha^2 + C_\epsilon C_\mu^\mu\|u_w\|_\alpha^\mu,$$

and by Lemma 1 - section §2, we get

$$\frac{\tilde{\gamma} - \epsilon C_2^2}{C_\epsilon C_\mu^\mu}\|u_w\|_\alpha^2 \leq \|u_w\|_\alpha^\mu.$$

Let $\epsilon > 0$ small enough, such that $\frac{\tilde{\gamma}-\epsilon C_2^2}{C_\epsilon C_\mu^\mu} > 0$. Since $\mu > 2$ we can take

$$K_1 = \left(\frac{\tilde{\gamma} - \epsilon C_2^2}{C_\epsilon C_\mu^\mu}\right)^{\frac{1}{\mu-2}},$$

to get

$$K_1 \leq \|u_w\|_\alpha. \tag{40}$$

On the other hand, by mountain pass characterization of the critical level, we have

$$c_w = I_w(u_w) \leq \max_{s\in[0,\infty)} I_w(s\varphi). \tag{41}$$

Further, by (8) and $\varphi \in H^\alpha(\mathbb{R})$ with $|\varphi(t) = 1|$ for all $t \in [0,1]$, we have

$$I_w(s\varphi) \leq \frac{s^2}{2}\|\varphi\|_{X^\alpha}^2 - \Lambda_1 s^\theta \|\varphi\|_{L^\theta}^\theta.$$

Then

$$c_w \leq \max_{s\geq 0} I_w(s\varphi) \leq \max_{s\geq 0}\left(\frac{s^2}{2}\|\varphi\|_{X^\alpha}^2 - \Lambda_1 s^\theta \|\varphi\|_{L^\theta}^\theta\right) := \tilde{K}.$$

Note that, since $\theta > 2$, $\tilde{K}$ is well defined. Since $I_w'(u_w)u_w = 0$, then

$$\tilde{\gamma}(\frac{1}{2} - \frac{1}{\theta})\|u_w\|_\alpha^2 \leq I_w(u_w) - \frac{1}{\theta}I_w'(u_w)u_w = c_w \leq \tilde{K},$$

taking

$$K_2 = \left(\frac{\tilde{K}}{\tilde{\gamma}\left(\frac{1}{2} - \frac{1}{\theta}\right)}\right)^{1/2},$$

we get

$$\|u_w\|_\alpha \leq K_2. \tag{42}$$

□

4. Proof of Theorem 2

To prove Theorem 2, we construct iterative sequence (u_n) and we show that (u_n) is convergent to a nontrivial solution $u \in H^\alpha(\mathbb{R})$ of problem

$$\begin{aligned} {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u(t)) + b(t)u(t) = \quad & f(t, u(t), {}_{-\infty}D_t^\alpha u(t)) \\ u \in H^\alpha(\mathbb{R}). & \end{aligned} \tag{43}$$

We consider the solution (u_n) of the following problem

$$(P_n)\begin{cases} {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u_n(t)) + b(t)u_n(t) = f(t, u_n(t), {}_{-\infty}D_t^\alpha u_{n-1}(t)) \\ u_n \in H^\alpha(\mathbb{R}), \end{cases}$$

starting with an arbitrary $u_0 \in H^\alpha(\mathbb{R})$. By iterative technique, we can get a sequence $\{u_n\}$, the nontrivial point obtained by Theorem 1. Moreover, by Theorem 1, we know that: $0 < K_1 \leq \|u_n\|_\alpha \leq K_2$ and by Theorem 1 - section §2, there exists positive constant ρ_1, such that

$$\|u_n\|_\infty \leq \rho_1. \tag{44}$$

By (14) and $I'_{u_n}(u_{n+1})(u_{n+1}-u_n)=0$ and $I'_{u_{n-1}}(u_n)(u_{n+1}-u_n)=0$, we obtain

$$\int_{\mathbb{R}}[{}_{-\infty}D_t^{\alpha}u_{n+1}\cdot{}_{-\infty}D_t^{\alpha}(u_{n+1}-u_n)+b(t)u_{n+1}(u_{n+1}-u_n)]dt$$
$$=\int_{\mathbb{R}}f(t,u_{n+1},{}_{-\infty}D_t^{\alpha}u_n)(u_{n+1}-u_n)dt,$$

and

$$\int_{\mathbb{R}}[{}_{-\infty}D_t^{\alpha}u_n\cdot{}_{-\infty}D_t^{\alpha}(u_{n+1}-u_n)+b(t)u_n(u_{n+1}-u_n)]dt$$
$$=\int_{\mathbb{R}}f(t,u_n,{}_{-\infty}D_t^{\alpha}u_{n-1})(u_{n+1}-u_n)dt,$$

hence

$$\|u_{n+1}-u_n\|^2_{X^\alpha}=\int_{\mathbb{R}}(f(t,u_{n+1},{}_{-\infty}D_t^{\alpha}u_n)-f(t,u_n,{}_{-\infty}D_t^{\alpha}u_{n-1}))(u_{n+1}-u_n)dt.$$

So, we have

$$\begin{aligned}\|u_{n+1}-u_n\|^2_{X^\alpha} &= \int_{\mathbb{R}}(f(t,u_{n+1},{}_{-\infty}D_t^{\alpha}u_n)-f(t,u_n,{}_{-\infty}D_t^{\alpha}u_n))(u_{n+1}-u_n)dt\\ &\quad+\int_{\mathbb{R}}(f(t,u_n,{}_{-\infty}D_t^{\alpha}u_n)-f(t,u_n,{}_{-\infty}D_t^{\alpha}u_{n-1}))(u_{n+1}-u_n)dt\\ &\le L_1\int_{\mathbb{R}}|u_{n+1}-u_n|^2dt+L_2\int_{\mathbb{R}}|{}_{-\infty}D_t^{\alpha}(u_n-u_{n-1})|\cdot|u_{n+1}-u_n|dt\\ &\le L_1\|u_{n+1}-u_n\|^2_{L^2}+L_2\|u_{n+1}-u_n\|_{L^2}\|{}_{-\infty}D_t^{\alpha}(u_n-u_{n-1})\|_{L^2}\\ &\le L_1\|u_{n+1}-u_n\|^2_{\alpha}+L_2\|u_{n+1}-u_n\|_{\alpha}\|u_n-u_{n-1}\|_{\alpha}.\end{aligned}$$

By Lemma 1 - section §2, we obtain

$$\|u_{n+1}-u_n\|^2_{X^\alpha}\le\frac{L_1}{\tilde{\gamma}}\|u_{n+1}-u_n\|^2_{X^\alpha}+\frac{L_2}{\tilde{\gamma}}\|u_{n+1}-u_n\|_{X^\alpha}\|u_n-u_{n-1}\|_{X^\alpha}.$$

Since $L_1+L_2<\tilde{\gamma}$, then

$$\|u_{n+1}-u_n\|_{X^\alpha}\le\frac{L_2}{\tilde{\gamma}-L_1}\|u_n-u_{n-1}\|_{X^\alpha},\tag{45}$$

and then (u_n) be a Cauchy sequence in $H^\alpha(\mathbb{R})$, so there exists a $u\in H^\alpha(\mathbb{R})$ such that (u_n) converges strongly to u in $H^\alpha(\mathbb{R})$ and by (40), we know that $u\neq 0$.

In order to show that u is a weak solution of problem (43), we need to prove that

$$\int_{\mathbb{R}}[{}_{-\infty}D_t^{\alpha}u\,{}_{-\infty}D_t^{\alpha}v+b(t)uv]dt=\int_{\mathbb{R}}f(t,u,{}_{-\infty}D_t^{\alpha}u)vdt\ \ \forall v\in H^\alpha(\mathbb{R}).$$

It suffices to show that

$$\int_{\mathbb{R}}f(t,u_n,{}_{-\infty}D_t^{\alpha}u_{n-1})vdt\to\int_{\mathbb{R}}f(t,u,{}_{-\infty}D_t^{\alpha}u)vdt\ \text{ as }\ n\to\infty.$$

Indeed, it follows from the assumption (f_4) that

$$\begin{aligned}
&\int_{\mathbb{R}}[f(t,u_n,{}_{-\infty}D_t^{\alpha}u_{n-1}) - f(t,u,{}_{-\infty}D_t^{\alpha}u)]vdt\\
&= \int_{\mathbb{R}}[f(t,u_n,{}_{-\infty}D_t^{\alpha}u_{n-1}) - f(t,u_n,{}_{-\infty}D_t^{\alpha}u)]vdt\\
&+ \int_{\mathbb{R}}[f(t,u_n,{}_{-\infty}D_t^{\alpha}u) - f(t,u,{}_{-\infty}D_t^{\alpha}u)]v(t)dt\\
&\leq L_1\int_{\mathbb{R}}|u_n-u||v|dt + L_2\int_{\mathbb{R}}|{}_{-\infty}D_t^{\alpha}(u_n-u_{n-1})||v|dt\\
&\leq [L_1\|u_n-u\|_{\alpha} + L_2\|u_{n-1}-u\|_{\alpha}]\,\|v\|_{\alpha}\\
&\to 0, \quad n\to\infty.
\end{aligned}$$

Therefore, we obtain a nontrivial solution of problem (43). □

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