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EXISTENCE OF SOLUTIONS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS OF RIEMANN-LIOUVILLE TYPE

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Abstract. This paper investigates the existence of solutions of the scalar fractional differential equation of Riemann-Liouville type

$$D^q x(t) = f(t, x(t)), \quad \lim_{t\to 0^+} t^{1-q} x(t) = x^0 \tag{1}$$

and of the Volterra integral equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \tag{2}$$

where $q \in (0, 1)$, $f(t, x)$ is continuous for $t > 0$ and $x \in \mathbb{R}$, and x^0 is nonzero. Sufficient conditions for the existence of continuous solutions of (2) are obtained by imposing a growth restriction on f and then applying Schauder's fixed point theorem. It is also shown with an "equivalence theorem" that (1) has the same continuous solutions as (2).

This existence theorem is atypical of the standard existence theorems in the literature in that (i) the growth condition obviates the need for f to be bounded or even to satisfy a Lipschitz condition and (ii) the existence of solutions is dependent on the value of q.

1. Introduction

This paper continues a series of studies [2, 3, 4] of the scalar fractional differential equation

$$D^q x(t) = f(t, x(t)) \quad (t > 0) \tag{1.1a}$$

when it is subject to the initial condition

$$\lim_{t\to 0^+} t^{1-q} x(t) = x^0 \tag{1.1b}$$

where $q \in (0, 1)$, $f\colon (0,\infty)\times\mathbb{R} \to \mathbb{R}$ is continuous, and x^0 is a nonzero real number. D^q denotes the *Riemann-Liouville fractional differential operator of order* q, which for $0 < q < 1$ is defined by

$$D^q x(t) := \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} x(s)\, ds,$$

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where $\Gamma\colon (0,\infty) \to \mathbb{R}$ is *Euler's Gamma function*:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt.$$

Throughout this paper, regardless of whether or not it is explicitly stated in a particular result, the standing assumption is that $q \in (0,1)$ and $x^0 \neq 0$. Also, the term *solution* will always refer to a continuous function satisfying whatever equation is being considered at the time. Finally, the term *initial value problem* will refer to the fractional differential equation (1.1a) together with the initial condition (1.1b):

$$D^q x(t) = f(t, x(t)), \quad \lim_{t\to 0^+} t^{1-q}x(t) = x^0 \quad (t > 0). \tag{1.2}$$

The most salient result of this work is Theorem 3.1 in Section 3, which when combined with Theorem 2.2 in the next section, gives sufficient conditions for the existence of a continuous function satisfying (1.2) on an interval $(0, T]$. The genesis of this theorem came from looking at various existence theorems in the literature and then asking if the conditions imposed by them on $f(t, x)$ could be relaxed so that solutions of (1.2) exist for a larger class of functions. There is a brief history of this given by Kilbas et al. [9, pp. 136–137] in which we find that Pitcher and Sewell [12] in 1938 offered a proof showing the existence of a solution in case f is bounded for certain unbounded x and satisfies a Lipschitz condition. This is a most serious restriction that is disquieting in view of the obvious unboundedness of x in (1.1b). However, the boundedness assumption persists through other papers and we find it still being assumed in recent monographs: see Theorem 5.1 and Lemmas 5.2 and 5.3 in Diethelm [8, pp. 77–80], Theorems 2.4.1 and 2.5.1 in Lakshmikantham et al. [10, pp. 30, 34], and Theorem 3.4 in Podlubny [13, p. 127]. Finally, Kilbas et al. [9, p. 165] prove that if f satisfies a Lipschitz condition then the boundedness of f for x unbounded can be dropped and there is still a unique solution. Yet, as late as 2009, the monograph of Lakshmikantham et al. [10, p. 30] requires that same bound on f for unbounded x in proving a Peano type existence theorem not based on a Lipschitz condition. In our Theorem 3.1 we do obtain existence without the bound and without the Lipschitz condition using Schauder's theorem. At this writing it is our belief that this is entirely new, thereby extending the scope of existence theory to include a much larger class of functions f.

It has long been known that under certain conditions the initial value problem (1.2) is equivalent to the Volterra integral equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, x(s))\,ds. \tag{1.3}$$

However, typical inversion results for transforming (1.2) into (1.3) (e.g., Lemma 5.2 in [8, p. 78]) also require that f be bounded in a certain region. With Theorem 2.2 in the next section we eliminate this requirement.

Recently we obtained an existence and uniqueness theorem [3, Thm. 2.7] for functions f satisfying a Lipschitz-type condition of the form

$$|f(t,x) - f(t,y)| \le K t^{r_1} |x^{r_2} - y^{r_2}|, \tag{1.4}$$

where $K > 0$, $r_1 > -1$, $r_2 = m/n \ge 1$ for positive integers m, n with no common factors and with n odd, and satisfying

$$\mu := 1 + r_1 + (q-1)\, r_2 > 0. \tag{1.5}$$

Notice that allowing r_1 to have negative values allows f to be unbounded for any value of x. In the present work, we take this a step further by replacing the Lipschitz-type condition with a growth condition. This substantially broadens the class of functions f with which to study (1.2) and (1.3).

Instead of starting with the initial value problem (1.2), we first investigate the integral equation (1.3), the outcome of which is Theorem 3.1. It is a general result that ensures the existence of a solution of (1.3) on a short interval $(0,T]$ for functions satisfying the growth condition of which we just spoke, namely

$$|f(t,x)| \leq K_1 + K_2 t^{r_1}|x|^{r_2},$$

where $r_1 > -1$ and $r_2 \geq 0$ with

$$r_1 - r_2 + q(r_2+1) > 0. \tag{1.6}$$

This broadens the spectrum of equations to study by eliminating the stricture that f be bounded. Moreover, because of the growth condition and the absolute integrability of the solution $x(t)$, the function $f(t,x(t))$ is also absolutely integrable. It is this property that enables us (cf. Thm. 2.2 in the next section) to show that a solution of (1.3) is also a solution of (1.2), and vice versa.

What is striking about (1.5) and (1.6) is that they suggest solutions of the fractional and integral equations in this paper may not exist for some values of q, despite being sufficient but not necessary conditions. As a matter of fact, whether or not a solution of the initial value problem

$$D^q x(t) = x^n(t), \quad \lim_{t\to 0^+} t^{1-q}x(t) = x^0 \qquad (x^0 \neq 0)$$

exists on an interval $(0,T]$ for some $T>0$ depends on the value of q. For example, when $n=1$, then it has a unique solution for each $q \in (0,1)$. However, when $n=2$, then it has a unique solution if $q \in (1/2,1)$ but no solution if $q \in (0,1/2]$. And if $n=3$, then the range of values of q for which it has solutions shrinks to $(2/3,1)$. (These assertions are consequences of Theorem 3.11 in Section 3.) Existence theorems such as these which depend on q may seem at first to be at odds with the previously cited theorems from the literature. This apparent discrepancy is resolved when we examine Theorem 3.1 and note that if the function f is bounded or even if it satisfies a Lipschitz condition as stated in our Theorem 2.5 in [3, p. 251] or in Theorem 3.11 in the monograph by Kilbas et al. [9, p. 165], then solutions exist for all values of q in the interval $(0,1)$. Thus, we see that prior to the theorems in this paper and [3], there was no reason to consider the values of q.

Basic properties of solutions emerging from the existence results in [3] reveal that solutions are far more than simply functions satisfying (1.2) and (1.3). Uniqueness of solutions ensured by the theorems in [3] are lost because the Lipschitz-type condition (1.4) is no longer assumed in this paper; however, most of the other properties also hold for the solutions which result from Theorem 3.1.

For functions x that are continuous and absolutely integrable on an interval $(0,T]$, it is proven in [2, Thm. 6.1]) that the initial condition (1.1b) is equivalent to

$$\lim_{t\to 0^+} \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q}x(s)\,ds = x^0\Gamma(q). \tag{1.7}$$

The significance of this is that the results obtained for the initial value problem (1.2) are also valid for the fractional differential equation (1.1a) when it is subject to the initial condition (1.7).

A final thought before proceeding: One might ask if there are any practical applications of the fractional differential equation (1.1a) with the initial condition (1.7), or alternatively with (1.1b). A partial answer is provided by Abel's equation (cf. [11, pp. 4, 71])

$$\int_0^t \frac{x(s)}{(t-s)^q}\, ds = g(t), \tag{1.8}$$

where $q \in (0,1)$ and g is a given function. Suppose g is continuously differentiable for $t > 0$. Dividing both sides by $\Gamma(1-q)$ and then differentiating with respect to t yields the fractional differential equation

$$D^q x(t) = f(t) \tag{1.9}$$

where $f(t) := g'(t)/\Gamma(1-q)$. Suppose also that $\lim_{t\to 0^+} g(t)$ exists. Then Abel's equation can be written as the fractional differential equation (1.9) subject to the initial condition (1.7) with

$$x^0 := \frac{1}{\Gamma(q)\Gamma(1-q)} \lim_{t\to 0^+} g(t).$$

As a matter of fact, the tautochrone problem of classical physics with $q = 1/2$ can be expressed in this way. The point then is that the initial value problem (1.2) can be viewed as a generalization of Abel's equation, which possibly may have real-world applications.

2. Preparatory results

We begin this section with what is meant in this paper by solutions of the initial value problem (1.2) and of the integral equation (1.3).

Definition 2.1. Let q be some fixed value in the interval $(0,1)$.

- (i) A function ϕ is said to be a *solution* of the initial value problem (1.2) on an interval $(0,T]$ if
  - (a) ϕ is continuous and satisfies the fractional differential equation (1.1a) on this interval and
  - (b) $\lim_{t\to 0^+} t^{1-q}\phi(t) = x^0$.
- (ii) A function ϕ is said to be a *solution* of the integral equation (1.3) on an interval $(0,T]$ if ϕ is continuous and satisfies (1.3) on this interval.

Notice from (i) that because $t^{1-q}\phi(t)$ is continuous on $(0,T]$ and the limit (b) exists, this function can be extended continuously to the closed interval $[0,T]$ by defining it to be x^0 at $t = 0$. Consequently $t^{1-q}\phi(t)$ is uniformly continuous on $(0,T]$ (cf. [1, Thm. 5.4.8]). In Section 3, an example there clearly illustrates this: see the analytical solution (3.16) given in Remark 3.9, which is a solution of both the integral equation and the fractional initial value problem in Example 3.8.

Fortunately, as it turns out, there is an "equivalence theorem" for solutions of the initial value problem (1.2) and of the integral equation (1.3): A solution $x(t)$ on an interval $(0,T]$ of (1.2) is also a solution of (1.3) on $(0,T]$, and conversely, if both $x(t)$ and $f(t,x(t))$ are absolutely integrable. This is precisely stated in the next theorem and proven in [2, Thm. 6.2].

Theorem 2.2. *Let $q \in (0,1)$ and $x^0 \neq 0$. Let $f(t,x)$ be a function that is continuous on the set*

$$\mathfrak{B} = \{(t,x) \in \mathbb{R}^2 \colon 0 < t \leq T, x \in I\}$$

where $I \subseteq \mathbb{R}$ denotes an unbounded interval. Suppose a function $x\colon (0,T] \to I$ is continuous and that both $x(t)$ and $f(t,x(t))$ are absolutely integrable on $(0,T]$. Then $x(t)$ satisfies the initial value problem (1.2) *on $(0,T]$ if and only if it satisfies the Volterra integral equation* (1.3) *on $(0,T]$.*

It follows that if a function $f(t,x)$ is continuous as stated in the theorem and if the hypotheses of some potential existence theorem imply that there is a continuous function $x(t)$ which satisfies the integral equation (1.3) on an interval $(0,T]$ such that both $x(t)$ and $f(t,x(t))$ are absolutely integrable on this interval, then $t^{1-q}x(t)$ must possess the limit given by (b) in Definition 2.1.

The following proposition, though elementary, sets us on the right course to search for solutions of (1.2) and (1.3). It reveals that the limit (1.1b) is the key; for it tells us precisely where to look for potential solutions and that if they exist they will be absolutely integrable.

Proposition 2.3. *Let $x\colon (0,T] \to \mathbb{R}$ be a continuous function with the property*

$$\lim_{t\to 0^+} t^{1-q}x(t) = x^0$$

where $x^0 \neq 0$. Then, for each $\epsilon \in (0,|x^0|)$, there is a $T^* \in (0,T]$ so that

$$(|x^0|-\epsilon)t^{q-1} < |x(t)| < (|x^0|+\epsilon)t^{q-1} < 2|x^0|t^{q-1} \tag{2.1}$$

for $0 < t < T^*$. Also, $\operatorname{sgn}(x(t)) = \operatorname{sgn}(x^0)$; i.e., $x(t)$ has the same sign as x^0 for $t \in (0,T^*)$. Furthermore, x is absolutely integrable on $(0,T]$.

Proof. Let $\epsilon \in (0,|x^0|)$. Then, because of the limit, a $T^* \in (0,T]$ exists such that

$$|t^{1-q}x(t) - x^0| < \epsilon$$

or

$$(x^0-\epsilon)t^{q-1} < x(t) < (x^0+\epsilon)t^{q-1} \tag{2.2}$$

for $0 < t < T^*$. Thus, if $x^0 > 0$, then $\epsilon < x^0$. So

$$x(t) > (x^0-\epsilon)t^{q-1} > 0$$

for $t \in (0,T^*)$. If, on the other hand, $x^0 < 0$, then $\epsilon < -x^0$, which implies

$$x(t) < (x^0+\epsilon)t^{q-1} < 0.$$

The inequalities (2.1) follow from

$$\big||t^{1-q}x(t)| - |x^0|\big| \le |t^{1-q}x(t) - x^0| < \epsilon$$

for $t \in (0,T^*)$. And from (2.1) we see that

$$\begin{aligned}\int_0^T |x(t)|\,dt &< 2|x^0|\int_0^{T^*} t^{q-1}\,dt + \int_{T^*}^T |x(t)|\,dt \\ &= \frac{2|x^0|}{q}(T^*)^q + \int_{T^*}^T |x(t)|\,dt < \infty.\end{aligned}$$

□

It follows from (2.1) that if a continuous function ϕ satisfies the initial condition (1.1b), then $t^{1-q}|\phi(t)| < \infty$ for all $t \in (0,T]$. Consequently, we will look for solutions in the subset of the vector space of all continuous functions ϕ on an interval $(0,T]$ for which

$$\sup\{t^{1-q}|\phi(t)| : 0 < t \le T\} < \infty.$$

This brings us to the following definition.

Definition 2.4. For a fixed $T > 0$ and for $g(t) := t^{q-1}$, let X denote the space of continuous functions $\phi\colon (0,T] \to \mathbb{R}$ for which

$$|\phi|_g := \sup_{0<t\leq T} \frac{|\phi(t)|}{g(t)} \tag{2.3}$$

is finite.

Showing that $|\cdot|_g$ is a norm on X is straightforward. Moreover, $(X, |\cdot|_g)$ is a Banach space (cf. [3, Thm. 2.2]). Clearly the function g itself belongs to this space.

Since Proposition 2.3 tells us to look for potential solutions in the space X, that is exactly the task we take up in the following section.

3. Local Existence of Solutions

We employ the following fixed point theorem of Juliusz Schauder (cf. [14, p. 25], [5, p. 184], or [6, p. 24]) to prove Theorem 3.1, the primary result of this paper. Our most recent paper [4, cf. Appendix] depends on theorems of this type because most of the results therein presuppose the existence of a solution of (1.3) on a short interval.

Schauder's second fixed point theorem. *Let $\mathcal{M}$ be a nonempty convex subset of a normed space. Let $P\colon \mathcal{M} \to \mathcal{K}$ be a continuous mapping, where $\mathcal{K}$ is a compact subset of $\mathcal{M}$. Then P has a fixed point in $\mathcal{K}$.*

The proof of Theorem 3.1 relies on four lemmas that prepare the way for the eventual application of Schauder's theorem. The proofs of the lemmas themselves (Lemma 3.2, Lemmas 3.4–3.6) are given after the conclusion of the proof of this theorem so as not to interrupt the flow of the argument. Notice that f is allowed to be unbounded; compare this with [10, pp. 30, 34], which partially motivated this work.

Theorem 3.1. *Let $q \in (0,1)$ and $x^0 \in \mathbb{R}$ with $x_0 \neq 0$. Let $r_1 > -1$ and $r_2 \geq 0$ be constants that satisfy the inequality*

$$r_1 - r_2 + q(r_2 + 1) > 0. \tag{3.1}$$

Let $f\colon (0,\infty) \times \mathbb{R} \to \mathbb{R}$ be continuous. Suppose there are nonnegative constants K_1 and K_2 such that

$$|f(t,x)| \leq K_1 + K_2\, t^{r_1} |x|^{r_2} \tag{3.2}$$

for $x \in \mathbb{R}$ and $0 < t < T_0$, where $T_0 \in (0,\infty) \cup \{\infty\}$. Then, for some $T \in (0, T_0)$, there is a continuous function $x\colon (0,T] \to \mathbb{R}$ that satisfies the integral equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \tag{3.3}$$

on $(0,T]$. Furthermore, $|x(t)| \leq 2|x^0| t^{q-1}$ for $t \in (0,T]$.

Proof. Let $g(t) := t^{q-1}$. Recall from Section 2 that for each $T > 0$ the set X of continuous functions $\phi\colon (0,T] \to \mathbb{R}$ for which $|\phi|_g < \infty$ is a Banach space (cf. Def. 2.4 and [3, Thm. 2.2]).

Corresponding to the nonzero x^0 in (3.3), define the set

$$M := \{\phi \in X\colon |\phi|_g \leq 2|x^0|\}. \tag{3.4}$$

Notice that M is a nonempty subset of X as $kt^{q-1} \in M$ for all constants k satisfying $|k| \leq 2|x^0|$. Also, for $\phi, \psi \in M$ and $0 \leq \beta \leq 1$,

$$\big|\beta\phi(t) + (1-\beta)\psi(t)\big| \leq 2|x^0|t^{q-1}$$

for $0 < t \leq T$. Thus, $\beta\phi + (1-\beta)\psi \in M$. So M is a nonempty convex subset of the Banach space $(X, |\cdot|_g)$.

For a fixed $T \in (0, T_0)$ and functions $\phi \in M$, define the mapping P by

$$(P\phi)(t) := x^0 t^{q-1} + (L\phi)(t) \tag{3.5}$$

for $0 < t \leq T$, where

$$(L\phi)(t) := \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \phi(s))\, ds. \tag{3.6}$$

The objective is to prove P has a fixed point in M when $T \in (0, T_0)$ is sufficiently small. This is equivalent to proving the existence of a continuous function satisfying (3.3) on $(0, T]$.

In Lemma 3.2 (immediately following this proof), we show that

(i) $P\colon M \to M$ for any $T \in (0, T_0)$ satisfying inequality (3.10).

Now choose a T so that (i) holds. Then in Lemma 3.4 we show that

(ii) P is continuous on M in the metric provided by the norm $|\cdot|_g$.

Also, for this particular T, define the set LM of functions by

$$LM := \{\psi \mid \exists \phi \in M, \psi(t) = (L\phi)(t) \text{ for } t \in (0, T], \psi(0) = 0\}.$$

In Lemma 3.5 we show that

(iii) LM is *equicontinuous* on $[0, T]$. That is, for every $\mu > 0$ there is a $\delta(\mu) > 0$ such that $t_1, t_2 \in [0, T]$ and $|t_1 - t_2| < \delta$ imply $|\psi(t_1) - \psi(t_2)| < \mu$ for all $\psi \in LM$.

So far we have, corresponding to the fixed T, a nonempty convex subset M of the Banach space $(X, |\cdot|_g)$. We have established that $P\colon M \to M$ and that it is continuous. Consequently, with Schauder's theorem, we can assert that P has a fixed point in M provided we can show that PM is contained in a compact subset of M.

In Lemma 3.6, we prove that LM resides in a compact subset $\mathcal{K}$ of the Banach space $(X, |\cdot|_g)$. Define the set

$$\mathcal{K}_h := \{h + \psi \mid \psi \in \mathcal{K}\},$$

where $h(t) := x^0 t^{q-1}$. For such ψ, we see from Lemma 3.6 that

$$|h + \psi|_g \leq |h|_g + |\psi|_g \leq |x^0| + |x^0| = 2|x^0|.$$

Hence, $\mathcal{K}_h \subseteq M$. If $\Psi \in PM$, then $\Psi = h + L\phi$ for some $\phi \in M$. Since $L\phi \in LM \subseteq \mathcal{K}$, $\Psi \in \mathcal{K}_h$. Therefore,

$$PM \subseteq \mathcal{K}_h \subseteq M.$$

Now let $\{h + \psi_n\}$ be a sequence in $\mathcal{K}_h$. Then as $\{\psi_n\}$ is a sequence in the compact set $\mathcal{K}$, it has a subsequence $\{\psi_{n_k}\}$ that converges in the norm $|\cdot|_g$ to a function $\psi \in \mathcal{K}$. As a result, $\{h + \psi_{n_k}\}$ converges in the norm $|\cdot|_g$ to $h + \psi \in \mathcal{K}_h$. Therefore, $\mathcal{K}_h$ is a compact set in $(X, |\cdot|_g)$.

In conclusion, we have shown that PM is contained in the compact set $\mathcal{K}_h$. It follows from Schauder's theorem that P has a fixed point in $\mathcal{K}_h$. Calling it x, we see from (3.4) that $|x(t)| \leq 2|x^0|t^{q-1}$ for all $t \in (0, T]$. □

Lemma 3.2. *For a given $T \in (0, T_0)$, let M denote the set defined by* (3.4)*. If T is sufficiently small, then $P\colon M \to M$.*

Proof. For $T \in (0, T_0)$ to be determined, let M be defined by (3.4). For any given $\phi \in M$,

$$\begin{aligned}|(P\phi)(t)| &= |x^0 t^{q-1} + (L\phi)(t)| \le \left|x^0 t^{q-1}\right| + |(L\phi)(t)| \\ &\le |x^0| t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, \phi(s))|\, ds.\end{aligned}$$

It then follows from (3.2) and (3.4) that

$$\begin{aligned}|(P\phi)(t)| &\le |x^0| t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \big(K_1 + K_2 s^{r_1} |\phi(s)|^{r_2}\big)\, ds \\ &\le |x^0| t^{q-1} + \frac{K_1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\, ds + \frac{K_2}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^{r_1} |\phi(s)|^{r_2}\, ds \\ &\le |x^0| t^{q-1} + \frac{K_1}{q\Gamma(q)}\, t^q + \frac{K_2}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^{r_1} \big(2|x^0| s^{q-1}\big)^{r_2}\, ds.\end{aligned}$$

Thus,

$$\begin{aligned}|(P\phi)(t)| \le |x^0| t^{q-1} &+ \frac{K_1}{\Gamma(q+1)}\, t^q \\ &+ \frac{K_2}{\Gamma(q)} \big(2|x^0|\big)^{r_2} \int_0^t (t-s)^{q-1} s^{r_1 + (q-1) r_2}\, ds\end{aligned} \tag{3.7}$$

for $t \in (0, T_0)$.

Expressed in terms of $p := r_1 + (q-1) r_2 + 1$, the integral is

$$\int_0^t (t-s)^{q-1} s^{p-1}\, ds.$$

Notice from (3.1) that $p > 0$ since

$$r_1 + (q-1) r_2 + 1 = r_1 - r_2 + q(r_2 + 1) + 1 - q > 1 - q > 0.$$

Since both p and q are positive, the integral is related to the Beta function $B(p, q)$, namely

$$B(p, q) := \int_0^1 v^{p-1} (1-v)^{q-1}\, dv,$$

which can be seen with the change of variable $s = tv$:

$$\begin{aligned}\int_0^t (t-s)^{q-1} s^{p-1}\, ds &= t^{p+q-1} \int_0^1 v^{p-1} (1-v)^{q-1}\, dv \\ &= t^{p+q-1} B(p, q) = t^{p+q-1}\, \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.\end{aligned} \tag{3.8}$$

Thus, the integral in (3.7) is

$$\int_0^t (t-s)^{q-1} s^{r_1 + (q-1) r_2}\, ds = t^{p+q-1} \frac{\Gamma(p)}{\Gamma(p+q)}\, \Gamma(q) = t^{\lambda} \gamma \Gamma(q), \tag{3.9}$$

where

$$\lambda := p + q - 1 = r_1 - r_2 + q(r_2 + 1) \ \text{ and } \ \gamma := \frac{\Gamma(p)}{\Gamma(p+q)}.$$

By (3.1), $\lambda > 0$. Also, $\gamma > 0$ as $p, q > 0$. Consequently,

$$\begin{aligned}|(P\phi)(t)| &\le |x^0|t^{q-1} + \frac{K_1}{\Gamma(q+1)}\, t^q + \frac{K_2}{\Gamma(q)}\big(2|x^0|\big)^{r_2} t^{\lambda}\gamma\Gamma(q) \\ &= |x^0|t^{q-1} + \frac{K_1}{\Gamma(q+1)} t^q + K_2\big(2|x^0|\big)^{r_2}\gamma t^{\lambda} \\ &= \left[|x^0| + \frac{K_1}{\Gamma(q+1)} t + K_2\big(2|x^0|\big)^{r_2}\gamma t^{\lambda+1-q}\right] t^{q-1}\end{aligned}$$

for $t \in (0, T_0)$.

Since $\lambda + 1 - q > 0$, a $T \in (0, T_0)$ exists so that

$$\frac{K_1}{\Gamma(q+1)}\, T + K_2\big(2|x^0|\big)^{r_2}\gamma T^{\lambda+1-q} \le |x^0|. \tag{3.10}$$

It then follows that

$$|(P\phi)(t)| \le 2|x^0|t^{q-1}$$

for $0 < t \le T$. This completes the proof since for a fixed $T \in (0, T_0)$ satisfying (3.10) we have shown that $P\colon M \to M$. □

Remark 3.3. From an inspection of the proof, we see that

$$\begin{aligned}|(L\phi)(t)| &\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, \phi(s))|\, ds \qquad (3.11) \\ &\le \frac{K_1}{\Gamma(q+1)}\, t^q + K_2\big(2|x^0|\big)^{r_2}\gamma t^{\lambda}\end{aligned}$$

for each $\phi \in M$ and for all $t \in (0, T]$. This along with (3.10) implies $|L\phi|_g \le |x^0|$. Thus, for T satisfying (3.10), $L\colon M \to M$.

Lemma 3.4. *If $T \in (0, T_0)$ satisfies (3.10), then P is continuous on M in the metric provided by the norm $|\cdot|_g$.*

Proof. Let $\phi, \psi \in M$. Let $\mu > 0$. We show a $\delta(\mu) > 0$ exists such that $|P\phi - P\psi|_g < \mu$ if $|\phi - \psi|_g < \delta$.

It follows from (3.5), (3.6), and (3.11) that

$$\begin{aligned}&|(P\phi)(t) - (P\psi)(t)| = |(L\phi)(t) - (L\psi)(t)| \le |(L\phi)(t)| + |(L\psi)(t)| \\ &\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, \phi(s))|\, ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, \psi(s))|\, ds \\ &\le \frac{2K_1}{\Gamma(q+1)}\, t^q + 2K_2\big(2|x^0|\big)^{r_2}\gamma t^{\lambda}\end{aligned}$$

for all $t \in (0, T]$. Let $\epsilon \in (0, T)$. Then for $0 < t \le \epsilon$, we have

$$\frac{1}{t^{q-1}}|(P\phi)(t) - (P\psi)(t)| \le \left[\frac{2K_1}{\Gamma(q+1)}\, \epsilon^q + 2K_2\big(2|x^0|\big)^{r_2}\gamma\epsilon^{\lambda}\right] T^{1-q}.$$

Since the bracketed quantity tends to 0 as $\epsilon \to 0^+$, we can choose ϵ small enough so that

$$\frac{1}{t^{q-1}}|(P\phi)(t) - (P\psi)(t)| < \frac{\mu}{2} \tag{3.12}$$

for all $t \in (0, \epsilon]$.

Now suppose $t \in [\epsilon, T]$. Then

$$\begin{aligned}
\frac{1}{t^{q-1}}|(P\phi)(t)-(P\psi)(t)| &= \frac{1}{t^{q-1}}|(L\phi)(t)-(L\psi)(t)| \\
&= \frac{1}{t^{q-1}\Gamma(q)}\left|\int_0^t (t-s)^{q-1}\big[f(s,\phi(s))-f(s,\psi(s))\big]\,ds\right| \\
&\le \frac{1}{t^{q-1}\Gamma(q)}\int_0^{\epsilon} (t-s)^{q-1}\big|f(s,\phi(s))-f(s,\psi(s))\big|\,ds \\
&\quad + \frac{1}{t^{q-1}\Gamma(q)}\int_{\epsilon}^{t} (t-s)^{q-1}\big|f(s,\phi(s))-f(s,\psi(s))\big|\,ds.
\end{aligned}$$

Consider the integral over the interval $[0, \epsilon]$. Since $t \ge \epsilon$, $t-s \ge \epsilon-s$; so $(t-s)^{q-1} \le (\epsilon-s)^{q-1}$ as $q-1<0$. Thus,

$$\begin{aligned}
&\frac{1}{t^{q-1}\Gamma(q)}\int_0^{\epsilon} (t-s)^{q-1}\big|f(s,\phi(s))-f(s,\psi(s))\big|\,ds \\
&\le \frac{1}{t^{q-1}\Gamma(q)}\int_0^{\epsilon} (\epsilon-s)^{q-1}\big|f(s,\phi(s))-f(s,\psi(s))\big|\,ds \\
&\le \frac{1}{t^{q-1}\Gamma(q)}\int_0^{\epsilon} (\epsilon-s)^{q-1}\big|f(s,\phi(s))\big|\,ds \\
&\quad + \frac{1}{t^{q-1}\Gamma(q)}\int_0^{\epsilon} (\epsilon-s)^{q-1}\big|f(s,\psi(s))\big|\,ds \\
&\le 2t^{1-q}\left[\frac{K_1}{\Gamma(q+1)}\,\epsilon^q + K_2\big(2|x^0|\big)^{r_2}\gamma\epsilon^{\lambda}\right],
\end{aligned}$$

where the last inequality follows from (3.11). It follows from the inequality before (3.12) that

$$\begin{aligned}
&\frac{1}{t^{q-1}\Gamma(q)}\int_0^{\epsilon} (t-s)^{q-1}\big|f(s,\phi(s))-f(s,\psi(s))\big|\,ds \\
&\le \left[\frac{2K_1}{\Gamma(q+1)}\,\epsilon^q + 2K_2\big(2|x^0|\big)^{r_2}\gamma\epsilon^{\lambda}\right]T^{1-q} < \frac{\mu}{2}
\end{aligned}$$

for $\epsilon \le t \le T$.

Now let us find a bound for the integral over $[\epsilon, T]$. Since $f(s,x)$ is uniformly continuous on the compact set

$$\big[\epsilon, T\big] \times \big[-2|x^0|\epsilon^{q-1}, 2|x^0|\epsilon^{q-1}\big], \tag{3.13}$$

there exists a $\delta > 0$ such that

$$|f(s,x_1)-f(s,x_2)| \le \frac{\mu\Gamma(q+1)}{3T}$$

for all (s,x_1) and (s,x_2) in the set with $|x_1-x_2| < \delta\epsilon^{q-1}$. Hence, for $\phi,\psi \in M$ with $|\phi-\psi|_g < \delta$ and $s \in [\epsilon, T]$, we have

$$\begin{aligned}
|\phi(s)-\psi(s)| &= \frac{|\phi(s)-\psi(s)|}{s^{q-1}}\,s^{q-1} \le s^{q-1}\sup_{\epsilon\le s\le T}\frac{|\phi(s)-\psi(s)|}{s^{q-1}} \\
&\le s^{q-1}|\phi-\psi|_g < s^{q-1}\delta \le \delta\epsilon^{q-1}.
\end{aligned}$$

Consequently,

$$|f(s,\phi(s))-f(s,\psi(s))| \le \frac{\mu\Gamma(q+1)}{3T}.$$

And so

$$\begin{aligned}
&\frac{1}{t^{q-1}\Gamma(q)}\int_\epsilon^t (t-s)^{q-1}\big|f(s,\phi(s))-f(s,\psi(s))\big|\,ds\\
&\qquad\le \frac{1}{t^{q-1}\Gamma(q)}\int_\epsilon^t (t-s)^{q-1}\frac{\mu\Gamma(q+1)}{3T}\,ds\\
&\qquad\le \frac{\mu\Gamma(q+1)}{t^{q-1}\Gamma(q)\,3T}\int_\epsilon^t (t-s)^{q-1}\,ds \le \frac{\mu q\Gamma(q)}{t^{q-1}\Gamma(q)\,3T}\cdot\frac{t^q}{q}\le\frac{\mu t}{3T}\le\frac{\mu}{3}.
\end{aligned}$$

Thus, for $t \in [\epsilon, T]$, we have shown that

$$\frac{1}{t^{q-1}}|(P\phi)(t)-(P\psi)(t)| < \frac{\mu}{2}+\frac{\mu}{3}=\frac{5\mu}{6} \tag{3.14}$$

for $\phi, \psi \in M$ and $|\phi-\psi|_g < \delta$.

It follows from (3.12) and (3.14) that for each $\mu > 0$ there is a $\delta > 0$ such that

$$\sup_{0<t\le T}\frac{|(P\phi)(t)-(P\psi)(t)|}{t^{q-1}} < \mu$$

if $\phi, \psi \in M$ and $|\phi-\psi|_g < \delta$. □

Lemma 3.5. *Let $T \in (0, T_0)$ satisfy* (3.10). *The set of functions*

$$LM = \{\psi \mid \exists\phi\in M, \psi(t)=(L\phi)(t) \text{ for } t\in(0,T], \psi(0)=0\}$$

is equicontinuous on the interval $[0,T]$.

Proof. Let $\mu > 0$. We show a $\delta > 0$ exists such that $t_1, t_2 \in [0,T]$ and $|t_2-t_1| < \delta$ imply

$$|\psi(t_2)-\psi(t_1)| < \mu$$

for all $\psi \in LM$.

To this end, take any $\psi \in LM$. Then there is a $\phi \in M$ such that $\psi(t) = L\phi(t)$ for $t \in (0,T]$. Moreover, we see from (3.11) that

$$|\psi(t)| \le \frac{K_1}{\Gamma(q+1)}\,t^q + K_2\big(2|x^0|\big)^{r_2}\gamma t^\lambda$$

for $0 < t \le T$. Since $q, \lambda > 0$, the bounding function approaches 0 as $t \to 0^+$; hence

$$\lim_{t\to0^+}\psi(t)=0.$$

Thus, as $\psi(0) = 0$, the function ψ is continuous on $[0,T]$. Furthermore, we can choose $\epsilon \in (0,T)$ so that $|\psi(t)| < \mu/4$ for $0 \le t \le \epsilon$. Hence, if $t_1, t_2 \in [0,\epsilon]$, then

$$|\psi(t_2)-\psi(t_1)| < \frac{\mu}{2}.$$

Now consider $\psi = L\phi$ on $[\epsilon, T]$. Since $\phi \in M$ is continuous and

$$|\phi(t)| \le 2|x^0|t^{q-1} \le 2|x^0|\epsilon^{q-1}$$

for $\epsilon \le t \le T$ and as $f(t,x)$ is continuous on the compact set (3.13), there is a constant $k > 0$ such that

$$\big|f(t,\phi(t))\big| \le k$$

for $t \in [\epsilon, T]$. It then follows from [7, Thm. 5.1] (with minor alterations in its proof) that a constant $H > 0$ exists such

$$|\psi(t_2)-\psi(t_1)| \le H|t_2-t_1|^q$$

for all $t_1, t_2 \in [\epsilon, T]$. For the given μ, let $\delta := (\mu/2H)^{1/q}$. Then $t_1, t_2 \in [\epsilon, T]$ and $|t_2 - t_1| < \delta$ imply that

$$|\psi(t_2) - \psi(t_1)| \le H|t_2 - t_1|^q < H\delta^q = \frac{\mu}{2}.$$

Finally, there is the case with a $t_i \in [0, \epsilon]$ while the other is in $[\epsilon, T]$. For definiteness, suppose $t_1 \in [0, \epsilon]$ and $t_2 \in [\epsilon, T]$. Then, if $|t_2 - t_1| < \delta$,

$$|\psi(t_2) - \psi(t_1)| \le |\psi(t_2) - \psi(\epsilon)| + |\psi(\epsilon) - \psi(t_1)| < \frac{\mu}{2} + \frac{\mu}{2} < \mu.$$

□

Lemma 3.6. *Let $\mathcal{C}$ denote the Banach space of continuous functions on $[0, T]$ with the supremum norm $\|\cdot\|$. Let $\mathcal{K}$ denote the closure of LM in $\mathcal{C}$. Then $\mathcal{K}$ is a compact subset of the Banach space X with the norm $|\cdot|_g$. Furthermore,*

$$\mathcal{K} \subseteq \{\phi \in X : |\phi|_g \le |x^0|\} \subseteq M \subseteq X.$$

Proof. First notice that each function $\varphi \in \mathcal{C}$ also belongs to the Banach space X as $|\varphi|_g < \infty$. By Lemma 3.5, $LM \subseteq \mathcal{C}$. Hence, $LM \subseteq X$.

For a given $\psi \in LM$, a function $\phi \in M$ exists with $\psi(t) = (L\phi)(t)$. The bound given by (3.11) and $\psi(0) = 0$ imply that

$$|\psi(t)| \le \frac{K_1}{\Gamma(q+1)} T^q + K_2\big(2|x^0|\big)^{r_2}\gamma T^\lambda$$

for $0 \le t \le T$. Thus the set of functions LM is uniformly bounded on $[0, T]$. By Lemma 3.5, LM is equicontinuous on $[0, T]$.

The set $\mathcal{K}$, namely the closure of LM, is also equicontinuous on $[0, T]$. This can be seen by choosing a limit point of LM, say a function ψ_L. And so there is a sequence $\{\psi_n\}$ of functions in LM converging to ψ_L, i.e., $\|\psi_n - \psi_L\| \to 0$ as $n \to \infty$. This implies $\{\psi_n\}$ converges uniformly on $[0, T]$ to ψ_L. Consequently, ψ_L is also continuous on $[0, T]$. In point of fact, each limit point of LM satisfies the same equicontinuity condition as do all the functions constituting LM. This can be shown with a classical $\epsilon/3$ argument applied to the sequence $\{\psi_n\}$, which we defer to the reader. In sum, $\mathcal{K}$ is equicontinuous on $[0, T]$. Clearly the uniform boundedness of LM implies that the functions constituting $\mathcal{K}$ are also uniformly bounded on $[0, T]$.

Since the set $\mathcal{K}$ of functions is uniformly bounded and equicontinuous on $[0, T]$ and of course closed, it follows from the Arzelà-Ascoli theorem that $\mathcal{K}$ is a compact subset of $\mathcal{C}$.

Now consider $\mathcal{K}$ as a subset of X. Let $\{\Psi_n\}$ be a sequence in $\mathcal{K}$. Since $\mathcal{K}$ is compact in $\mathcal{C}$, it has a subsequence $\{\Psi_{n_k}\}$ that converges in the supremum metric to a function $\Psi \in \mathcal{K}$. In other words, for every $\epsilon > 0$, there is an N such that $k > N$ implies

$$|\Psi_{n_k}(t) - \Psi(t)| < \frac{\epsilon}{2T^{1-q}}$$

for $t \in [0, T]$. Consequently,

$$|\Psi_{n_k} - \Psi|_g = \sup_{0<t\le T} \frac{|\Psi_{n_k}(t) - \Psi(t)|}{t^{q-1}} \le \frac{\epsilon}{2} < \epsilon.$$

So each sequence $\{\Psi_n\}$ in $\mathcal{K}$ has a subsequence that converges in the metric provided by the norm $|\cdot|_g$ to a function in $\mathcal{K}$. As a result, we conclude that $\mathcal{K}$ is also a compact subset of the Banach space X.

In fact, $\mathcal{K}$ lies in the closed ball

$$\{\phi \in X \colon |\phi|_g \leq |x^0|\}.$$

To see this, let $\Psi \in \mathcal{K}$. If $\Psi \in LM$, then $|\Psi|_g \leq |x^0|$ (cf. Remark 3.3). If $\Psi \notin LM$, then for each $\delta > 0$ a function $\Psi^* \in LM$ exists such that $|\Psi - \Psi^*|_g < \delta$. And so

$$|\Psi|_g \leq |\Psi - \Psi^*|_g + |\Psi^*|_g < \delta + |x^0|.$$

Since this holds for all $\delta > 0$, $|\Psi|_g \leq |x^0|$. Therefore, $|\Psi|_g \leq |x^0|$ for all $\Psi \in \mathcal{K}$. □

In Theorems 2.2 and 3.1, q denotes a fixed value in $(0, 1)$ while x^0 denotes any fixed nonzero real number. As mentioned earlier, these are the standing assumptions in this paper and will not always be stated. Let us now show that the solution ensured by Theorem 3.1 is also a solution of the initial value problem (1.2).

Corollary 3.7. *If a function $f(t, x)$ satisfies the conditions of Theorem 3.1, then for some $T > 0$ there is a continuous function that is a solution of both the integral equation* (3.3) *and the initial value problem*

$$D^q x(t) = f(t, x(t)), \quad \lim_{t\to 0^+} t^{1-q}x(t) = x^0 \tag{3.15}$$

for $0 < t \leq T$.

Proof. We have already established the existence of an interval $(0, T]$ and a function x that satisfies (3.3) on $(0, T]$. Now let us show that x is also a solution of (3.15) on $(0, T]$.

First, from Theorem 3.1 note that $|x(t)| \leq 2|x^0|t^{q-1}$ for $0 < t \leq T$; thus $x(t)$ is absolutely integrable on $(0, T]$. Second, $f(t, x(t))$ is also absolutely integrable on $(0, T]$ because $r_1 + r_2(q - 1) > -1$ implies

$$\begin{aligned}
\int_0^T |f(t, x(t))|\, dt &\leq \int_0^T (K_1 + K_2\, t^{r_1} |x(t)|^{r_2})\, dt \\
&\leq K_1 T + K_2 \int_0^T t^{r_1} \left(2|x^0|t^{q-1}\right)^{r_2} dt \\
&\leq K_1 T + K_2 \left(2|x^0|\right)^{r_2} \int_0^T t^{r_1 + r_2(q-1)}\, dt < \infty.
\end{aligned}$$

Thus both $x(t)$ and $f(t, x(t))$ are absolutely integrable on $(0, T]$. It then follows from Theorem 2.2 that x is also a solution of (3.15) on $(0, T]$. □

Example 3.8. For $t \geq 0$, define the function

$$f(t, x) := \begin{cases} 0, & x < 0 \\ -\dfrac{\sqrt{\pi}}{2}(\sqrt{t}x)^{3/2}, & x \geq 0. \end{cases}$$

There is a continuous function that is a solution of both

$$x(t) = \frac{1}{\sqrt{t}} + \frac{1}{\sqrt{\pi}} \int_0^t \frac{f(s, x(s))}{\sqrt{t - s}}\, ds$$

and

$$D^{1/2}x(t) = f(t, x(t)), \quad \lim_{t\to 0^+} \sqrt{t}x(t) = 1$$

on the interval $(0, \sqrt{2}/4)$.

Proof. From (3.3) and (3.15), we see that $q = 1/2$ and $x^0 = 1$. Since

$$|f(t,x)| \le \frac{\sqrt{\pi}}{2} t^{3/4} |x|^{3/2}$$

for $x \in \mathbb{R}$ and $0 \le t < \infty$, condition (3.2) is satisfied with $K_1 = 0$, $K_2 = \sqrt{\pi}/2$, $r_1 = 3/4$, and $r_2 = 3/2$. Condition (3.1) is also satisfied because

$$r_1 - r_2 + q(r_2 + 1) = \tfrac{3}{4} - \tfrac{3}{2} + \tfrac{1}{2}\left(\tfrac{3}{2} + 1\right) = \tfrac{1}{2}.$$

Since all of the conditions of Theorem 3.1 are met, the existence of a continuous function $x(t)$ satisfying both the integral equation and the initial value problem on a common interval $(0, T]$ follows from Corollary 3.7.

Perusing the proof of Theorem 3.1 and supporting lemmas, we see that a value for T can be obtained from (3.10), where the values of the constants p, λ, and γ are:

$$p = r_1 + (q-1)r_2 + 1 = 1, \quad \lambda = p + q - 1 = \frac{1}{2}$$

$$\gamma = \frac{\Gamma(p)}{\Gamma(p+q)} = \frac{\Gamma(1)}{\Gamma(1+\frac{1}{2})} = \frac{1}{\frac{1}{2}\Gamma(\frac{1}{2})} = \frac{2}{\sqrt{\pi}}.$$

So T satisfies (3.10), namely

$$\frac{K_1}{\Gamma(q+1)} T + K_2 \big(2|x^0|\big)^{r_2} \gamma T^{\lambda+1-q} \le |x^0|,$$

if

$$\frac{\sqrt{\pi}}{2} \cdot 2^{3/2} \cdot \frac{2}{\sqrt{\pi}} T \le 1.$$

We conclude a common solution exists on $(0, \sqrt{2}/4)$. □

Remark 3.9. It is shown with Examples 4.12, 5.2, and 6.4 in [2] that the function

$$x(t) = \frac{1}{\sqrt{t}(1+t)} \tag{3.16}$$

is a solution of both

$$D^{1/2}x(t) = -\frac{\sqrt{\pi}}{2}(\sqrt{t}x(t))^{3/2}, \quad \lim_{t\to 0^+} \sqrt{t}x(t) = 1$$

and

$$x(t) = \frac{1}{\sqrt{t}} - \frac{1}{2}\int_0^t \frac{(\sqrt{s}x(s))^{3/2}}{\sqrt{t-s}}\, ds$$

on the interval $(0, \infty)$. Since (3.16) is positive for all $t > 0$, we conclude that it is a solution of the initial value problem and the integral equation in Example 3.8, not only on $(0, \sqrt{2}/4)$ but on the entire interval $(0, \infty)$.

If in Theorem 3.1 $r_2 \le r_1$, then q may assume any value in the interval $(0, 1)$ and the existence of a solution of (3.3) is guaranteed. However, this is not the case if $r_2 > r_1$ or if $r_1 \in (-1, 0)$; for then condition (3.1) restricts q to

$$q > \frac{r_2 - r_1}{r_2 + 1} > 0.$$

The function $f(t,x) := x^n$ serves as an example of $r_2 > r_1$ since $r_2 = n$ and $r_1 = 0$. As a consequence, we have the following sufficient condition for the existence of a solution of the integral equation (3.17).

Example 3.10. A solution of the integral equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} x^n(s)\, ds \qquad (t > 0) \tag{3.17}$$

exists if

$$q > \frac{n}{n+1}.$$

However, by appealing to Theorem 2.7 in [3, p. 255], we can do much better than this: the next theorem and its corollary give necessary and sufficient conditions for the existence of a unique solution of (3.17) and the complementary initial value problem (3.18).

Theorem 3.11. *Let $n \in \mathbb{N}$, $q \in (0,1)$, and $x^0 \neq 0$. The initial value problem*

$$D^q x(t) = x^n(t), \qquad \lim_{t\to 0^+} t^{1-q} x(t) = x^0 \qquad (t > 0) \tag{3.18}$$

has a solution if and only if

$$q > \frac{n-1}{n}. \tag{3.19}$$

Moreover, the solution is unique.

Proof. Let us begin with the "only if" part of the proof. Accordingly, suppose there is a function x that satisfies the fractional differential equation on an interval $(0,T]$ and the initial condition in (3.18). Then we see from (2.2) in the proof of Proposition 2.3 that there exists a $T^* \in (0,T]$ corresponding to $\epsilon = \frac{1}{2}|x^0|$ such that

$$(x^0 - \tfrac{1}{2}|x^0|)t^{q-1} < x(t) < (x^0 + \tfrac{1}{2}|x^0|)t^{q-1} \tag{3.20}$$

for $t \in (0, T^*)$. Moreover, for this particular ϵ, it follows from (2.1) that

$$\tfrac{1}{2}|x^0| t^{q-1} < |x(t)| < \tfrac{3}{2}|x^0| t^{q-1} \tag{3.21}$$

for $t \in (0, T^*)$. The convergence of the improper integral on the right-hand side implies the convergence of $\int_0^{T^*} |x(t)|\, dt$. It follows that $x(t)$ is absolutely integrable on $(0,T]$.

Likewise, $x^n(t)$ is also absolutely integrable on $(0,T]$. We will see in the following argument that this is a consequence of x satisfying the fractional differential equation

$$\frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} x(s)\, ds = x^n(t)$$

on $(0,T]$. Since the integration

$$\frac{1}{\Gamma(1-q)} \int_\xi^t \frac{d}{du} \int_0^u (u-s)^{-q} x(s)\, ds\, du = \int_\xi^t x^n(u)\, du$$

yields

$$\frac{1}{\Gamma(1-q)} \left[\int_0^t (t-s)^{-q} x(s)\, ds - \int_0^\xi (\xi - s)^{-q} x(s)\, ds \right] = \int_\xi^t x^n(u)\, du,$$

it follows from (1.7), which is equivalent to the initial condition (3.18), that the left-hand side converges as $\xi \to 0^+$. Thus the integral on the right-hand side converges to

$$\int_0^t x^n(u)\, du = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} x(s)\, ds - x^0 \Gamma(q) \tag{3.22}$$

for $0 < t \le T$. Consequently, $x^n(t)$ is absolutely integrable on $(0, T]$ when n is even.

Now suppose n is odd. Choose any t in $(0, T^*)$. If $x^0 > 0$, then from (3.20) we see that $x(u) > \frac{1}{2}x^0 u^{q-1} > 0$ for $0 < u \le t$. And so on this interval $|x^n(u)| = x^n(u)$. But if $x^0 < 0$, then $x(u) < \frac{1}{2}x^0 u^{q-1} < 0$; so $|x^n(u)| = -x^n(u)$ for $0 < u \le t$. In both of these cases, it follows from (3.22) that $x^n(u)$ is absolutely integrable on $(0, t]$. As a result, we can conclude that $x^n(t)$ is also absolutely integrable on $(0, T]$ when n is odd.

Let $f(t, x) := x^n$. It follows from Theorem 2.2 that the continuity and absolute integrability that has been established for $x(t)$ and $f(t, x(t))$ on $(0, T]$ imply that $x(t)$ is also a solution of the complementary integral equation (3.17) on $(0, T]$.

We see from (3.20) and (3.21) that

$$x^n(s) > (\tfrac{1}{2}|x^0|)^n s^{(q-1)n} \quad (0 < s < T^*) \tag{3.23}$$

when n is even and also when n is odd and $x^0 > 0$. (The case $x^0 < 0$ will be considered later.) Thus, for a fixed $t \in (0, T^*]$,

$$(t-s)^{q-1} x^n(s) > (\tfrac{1}{2}|x^0|)^n (t-s)^{q-1} s^{(q-1)n}$$

for $0 < s < t$. Since (3.17) implies that $\int_0^t (t-s)^{q-1} x^n(s)\, ds$ converges, it follows that $\int_0^t (t-s)^{q-1} s^{(q-1)n}\, ds$ also converges. With the change of variable $s = tv$, we can express this latter integral in terms of the Beta function, namely

$$B(p, q) := \int_0^1 v^{p-1}(1-v)^{q-1}\, dv,$$

as follows:

$$\begin{aligned}\int_0^t (t-s)^{q-1} s^{(q-1)n}\, ds &= \int_0^1 (t-tv)^{q-1}(tv)^{(q-1)n} t\, dv \\ &= t^{q+(q-1)n} \int_0^1 v^{(q-1)n}(1-v)^{q-1}\, dv.\end{aligned}$$

Thus,

$$\int_0^t (t-s)^{q-1} s^{(q-1)n}\, ds = t^{q+p-1} \int_0^1 v^{p-1}(1-v)^{q-1}\, dv = t^{q+p-1} B(p, q)$$

where $p := (q-1)n + 1$. Since the integral on the left-hand side converges, so must $B(p, q)$. However, it is well-known that the Beta function converges if and only if both of its arguments are positive. Therefore $p > 0$, which is (3.19).

Now let us dispose of the remaining case: n odd and $x^0 < 0$. Since $x(t)$ is a solution of (3.18), the function $-x(t)$ is a solution of

$$D^q y(t) = y^n(t), \quad \lim_{t \to 0^+} t^{1-q} y(t) = y^0 \quad (t > 0) \tag{3.24}$$

if $y^0 = -x^0$. But this is the previous case: n odd and $y^0 > 0$. So (3.19) obtains in this case too.

Conversely, if (3.19) holds, then (1.4) and (1.5) along with the other conditions of Theorem 2.7 in [3, p. 255] are fulfilled. Consequently, for some $T > 0$, a unique solution x of the integral equation (3.17) exists on $(0, T]$. Furthermore, this theorem also states that both $x(t)$ and $x^n(t)$ are absolutely integrable on $(0, T]$. Finally, it follows from the equivalence theorem (Thm. 2.2) that x is also the unique solution of the complementary initial value problem (3.18). □

Corollary 3.12. *The integral equation* (3.17) *has a unique solution if and only if* q *satisfies the inequality* (3.19).

Proof. In the course of proving Theorem 3.11, we showed that the solution $x(t)$ of the initial value problem (3.18) is unique and absolutely integrable on some interval $(0, T]$. We also established that $x^n(t)$ is absolutely integrable on this same interval. Therefore, by Theorem 2.2, $x(t)$ must also be the unique continuous solution of (3.17). □

4. A generalization

We can easily generalize Theorem 3.1 by replacing the second term in the upper bound for f in (3.2) with the sum in (4.2) below.

Theorem 4.1. *Let* $r_{1,i} > -1$ *and* $r_{2,i} \geq 0$ *be constants that satisfy the inequalities*

$$\lambda_i := r_{1,i} - r_{2,i} + q\,(r_{2,i} + 1) > 0 \tag{4.1}$$

for $i = 1, \ldots, n$. *Let* $f\colon (0,\infty) \times \mathbb{R} \to \mathbb{R}$ *be continuous. Suppose there are constants* $K_1 \geq 0$ *and* $K_{2,i} \geq 0$ $(i = 1, \ldots, n)$ *such that*

$$|f(t,x)| \leq K_1 + \sum_{i=1}^{n} K_{2,i}\, t^{r_{1,i}} |x|^{r_{2,i}} \tag{4.2}$$

for $x \in \mathbb{R}$ *and* $0 < t < T_0$, *where* $T_0 \in (0,\infty) \cup \{\infty\}$. *Then there is a* $T \in (0, T_0)$ *and a continuous function* $x\colon (0,T] \to \mathbb{R}$ *that satisfies the integral equation* (3.3) *on* $(0,T]$. *Furthermore,* $|x(t)| \leq 2|x^0| t^{q-1}$ *for* $t \in (0,T]$.

Proof. Consider the mappings P and L defined by (3.5) and (3.6), respectively. Following the proof of Lemma 3.2 and making appropriate modifications, we see that the effect of (4.2) is to change (3.7) to

$$\begin{aligned} |(P\phi)(t)| \leq |x^0| t^{q-1} &+ \frac{K_1}{\Gamma(q+1)}\, t^q \\ &+ \sum_{i=1}^{n} \frac{K_{2,i}}{\Gamma(q)} \big(2|x^0|\big)^{r_{2,i}} \int_0^t (t-s)^{q-1} s^{r_{1,i} + (q-1) r_{2,i}}\, ds \end{aligned} \tag{4.3}$$

for each $\phi \in M$ and $t \in (0, T_0)$.

Let $p_i := r_{1,i} + (q-1) r_{2,i} + 1$ for $i = 1, \ldots, n$. Referring to (3.9), we see that as each $p_i > 0$ each integral in (4.3) is

$$\int_0^t (t-s)^{q-1} s^{r_{1,i} + (q-1) r_{2,i}}\, ds = \int_0^t (t-s)^{q-1} s^{p_i - 1}\, ds = t^{\lambda_i} \gamma_i \Gamma(q),$$

where

$$\lambda_i := p_i + q - 1 > 0 \quad \text{and} \quad \gamma_i := \frac{\Gamma(p_i)}{\Gamma(p_i + q)}. \tag{4.4}$$

Consequently,

$$\begin{aligned}|(P\phi)(t)| &\le |x^0|t^{q-1} + \frac{K_1}{\Gamma(q+1)}\, t^q + \sum_{i=1}^{n} \frac{K_{2,i}}{\Gamma(q)} \big(2|x^0|\big)^{r_{2,i}} t^{\lambda_i} \gamma_i \Gamma(q) \\ &= |x^0|t^{q-1} + \frac{K_1}{\Gamma(q+1)} t^q + \sum_{i=1}^{n} K_{2,i} \big(2|x^0|\big)^{r_{2,i}} \gamma_i t^{\lambda_i} \\ &= \left[|x^0| + \frac{K_1}{\Gamma(q+1)} t + \sum_{i=1}^{n} K_{2,i} \big(2|x^0|\big)^{r_{2,i}} \gamma_i t^{\lambda_i+1-q} \right] t^{q-1}\end{aligned}$$

for $t \in (0, T_0)$. Since each $\lambda_i + 1 - q > 0$, a $T \in (0, T_0)$ exists such that

$$\frac{K_1}{\Gamma(q+1)} T + \sum_{i=1}^{n} K_{2,i} \big(2|x^0|\big)^{r_{2,i}} \gamma_i T^{\lambda_i+1-q} \le |x^0|. \tag{4.5}$$

Therefore, for such a fixed T and each $\phi \in M$, we have

$$|(P\phi)(t)| \le 2|x^0|t^{q-1}, \tag{4.6}$$

which proves that $P\colon M \to M$.

It can be readily seen from the previous work that the counterpart of (3.11) is

$$|(L\phi)(t)| \le \frac{K_1}{\Gamma(q+1)} t^q + \sum_{i=1}^{n} K_{2,i} \big(2|x^0|\big)^{r_{2,i}} \gamma_i t^{\lambda_i} \tag{4.7}$$

for $0 < t \le T$. Observe from the proofs of Lemmas 3.4–3.6 that they are still valid in the present situation; the only alterations needed in their proofs is to replace every use of (3.11) with (4.7).

In conclusion, the lemmas that were used to prove Theorem 3.1 are still true here. As a result, the proof of Theorem 3.1 also serves as a proof of Theorem 4.1. □

Theorem 4.1 gives sufficient conditions for the existence of a solution $x(t)$ of the integral equation (1.3) on an interval $(0, T]$. Now we argue that $x(t)$ must also be a solution of the initial value problem (1.2) on this interval, similar to the way we showed Corollary 3.7 followed from Theorem 3.1. First we note that $x(t)$ is absolutely integrable on $(0, T]$ because $|x(t)| \le 2|x^0|t^{q-1}$. Then it follows from (4.2) that

$$\begin{aligned}|f(t, x(t))| &\le K_1 + \sum_{i=1}^{n} K_{2,i}\, t^{r_{1,i}} |x(t)|^{r_{2,i}} \le K_1 + \sum_{i=1}^{n} K_{2,i}\, t^{r_{1,i}} |2x^0 t^{q-1}|^{r_{2,i}} \\ &\le K_1 + \sum_{i=1}^{n} K_{2,i} |2x^0|^{r_{2,i}} t^{r_{1,i}+(q-1)r_{2,i}}\end{aligned}$$

for $0 < t \le T$. Therefore, as $r_{1,i}+(q-1)\, r_{2,i} > -q > -1$, the integral $\int_0^T |f(t, x(t))|\, dt$ converges. So we have proven the following:

Proposition 4.2. *Let $f\colon (0,\infty) \times \mathbb{R} \to \mathbb{R}$ be continuous. Suppose for constants satisfying the inequalities* (4.1) *that f is bounded as in* (4.2) *for $x \in \mathbb{R}$ and $0 < t \le T$. If $x\colon (0, T] \to \mathbb{R}$ is a continuous function such that $|x(t)| \le 2|x^0|t^{q-1}$, then $f(t, x(t))$ is absolutely integrable on $(0, T]$.*

In sum, we have proved that if the conditions of Theorem 4.1 hold, then there exists a function $x(t)$ that satisfies the integral equation (3.3) (or (1.3)) and both it and $f(t, x(t))$ are absolutely integrable. As a result, because of Theorem 2.2, we have the following result.

Corollary 4.3. *Assume that* $f\colon (0,\infty)\times\mathbb{R}\to\mathbb{R}$ *is continuous and satisfies conditions* (4.1)–(4.2) *in Theorem 4.1. Then there is a* $T>0$ *and a continuous function* $x\colon (0,T]\to\mathbb{R}$ *that is a solution of both the initial value problem* (1.2) *and the integral equation* (1.3) *on the interval* $(0,T]$.

The final example in this paper applies Corollary 4.3 and illustrates the variety of functions that can satisfy condition (4.2).

Example 4.4. Let g be a continuous, bounded function on $(0,\infty)$ and h a polynomial of degree n. Then a continuous function exists that is a solution of both the initial value problem

$$D^q x(t) = x\sin t + g(t) + h(tx), \qquad \lim_{t\to 0^+} t^{1-q}x(t) = x^0$$

and the integral equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[x(s)\sin s + g(s) + h(sx(s))\big]\,ds$$

on some interval $(0,T]$.

Proof. Let

$$f(t,x) := x\sin t + g(t) + h(tx)$$

for $t>0$ and $x\in\mathbb{R}$. Since h is a polynomial of degree n,

$$h(tx) = a_0 + \sum_{i=1}^{n} a_i (tx)^i,$$

for $a_i\in\mathbb{R}$ and $a_n\neq 0$. It suffices to show that f satisfies conditions (4.1)–(4.2).

Let k be a bound for g. Then, as $|x\sin t|\le t|x|$ for $t\ge 0$,

$$\begin{aligned}|f(t,x)| &\le t|x| + k + |a_0| + \sum_{i=1}^{n}|a_i|t^i|x|^i\\ &= (k+|a_0|) + (1+|a_1|)t|x| + \sum_{i=2}^{n}|a_i|t^i|x|^i = K_1 + \sum_{i=1}^{n} K_{2,i}\, t^{r_{1,i}}|x|^{r_{2,i}}\end{aligned}$$

where $r_{1,i}=r_{2,i}=i$ for $i=1,\ldots,n$, $K_1 = k+|a_0|$, $K_{2,1} = 1+|a_1|$, and $K_{2,i}=|a_i|$ for $i=2,\ldots,n$. Thus, (4.2) is satisfied. Also, (4.1) is satisfied since

$$r_{1,i} - r_{2,i} + q\,(r_{2,i}+1) = q(i+1) > 0$$

for $i=1,\ldots,n$. The result follows from Corollary 4.3. □

5. Epilogue

The results in this paper are for establishing the existence of solutions of (1.2) and (1.3). Generally speaking, the intervals on which they guarantee solutions are short. Take Example 3.8 for instance. There a solution was shown to exist on the interval $(0,\sqrt{2}/4)$, which is about 0.35 units in length. However, as we pointed out earlier in Remark 3.9, the solution actually exists on the entire interval $(0,\infty)$. This evokes the question: Can solutions be continued beyond such short intervals? This matter is in fact addressed in the recent papers [3] and [4]; see especially Sections 3 and 4 of [3].

References

[1] R. G. Bartle and D. R. Sherbert, Introduction to Real Analysis, third ed., John-Wiley & Sons, New York, 2000.

[2] L. C. Becker, T. A. Burton, and I. K. Purnaras, Complementary equations: A fractional differential equation and a Volterra integral equation, Electron. J. Qual. Theory Differ. Equ., No. 12, 1–24, 2015.

[3] L. C. Becker, T. A. Burton, and I. K. Purnaras, An inversion of a fractional differential equation and fixed points, Nonlinear Dyn. Syst. Theory, Vol. 15, No. 3, 242–271, 2015.

[4] L. C. Becker, T. A. Burton, and I. K. Purnaras, Fractional differential equations, transformations, and fixed points, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. Vol. 22, 333–361, 2015.

[5] T. A. Burton, Stability and Periodic Solutions of Ordinary and Functional Differential Equations, Academic Press, Orlando, FL, 1985 (unabridged, slightly corrected version reprinted by Dover, Mineola, NY, 2005).

[6] T. A. Burton, Stability by Fixed Point Theory for Functional Differential Equations, Dover, Mineola, NY, 2006.

[7] T. A. Burton and Bo Zhang, Fixed points and fractional differential equations: examples, Fixed Point Theory, Vol. 14, No. 2, 313–326, 2013.

[8] Kai Diethelm, The Analysis of Fractional Differential Equations, Springer, Heidelberg, 2010.

[9] A. Kilbas, H. Srivastava, and J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematical Studies, Vol. 204, Elsevier, 2006.

[10] V. Lakshmikantham, S. Leela, and J. Vasundhara Devi, Theory of Fractional Dynamic Systems, Cambridge Scientific Publishers, Cottenham, Cambridge, 2009.

[11] Peter Linz, Analytical and Numerical Methods for Volterra Equations, SIAM Studies in Applied Mathematics, Philadelphia, 1985.

[12] E. Pitcher, and W. E. Sewell, Existence theorems for solutions of differential equations of non-integral order, Bull. Amer. Math. Soc., Vol. 44, No. 2, 100–107, 1938.

[13] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, CA, 1999.

[14] D. R. Smart, Fixed Point Theorems, Cambridge Univ. Press, 1980.

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