

NON-HOMOGENEOUS FRACTIONAL SCHRÖDINGER EQUATION

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ABSTRACT. In this article we are interested in the non-homogeneous fractional Schrödinger equation

$$(-\Delta)^\alpha u(x) + V(x)u(x) = f(u) + h(x) \text{ in } \mathbb{R}^n. \quad (1)$$

By using mountain pass Theorem and Ekeland's variational principle, we prove the existence of two solutions for (1).

1. INTRODUCTION

Recently, a great attention has been focused on the study of problems involving the fractional Laplacian, from a pure mathematical point of view as well as from concrete applications, since this operator naturally arises in many different contexts, such as, obstacle problems, financial mathematics, phase transitions, anomalous diffusions, crystal dislocations, soft thin films, semipermeable membranes, flame propagations, conservation laws, ultra relativistic limits of quantum mechanics, quasi-geostrophic flows, minimal surfaces, materials science and water waves. The literature is too wide to attempt a reasonable list of references here, so we derive the reader to the work by Di Nezza, Patalluci and Valdinoci [7], where a more extensive bibliography and an introduction to the subject are given.

In the context of fractional quantum mechanics, non-linear fractional Schrödinger equation has been proposed by Laskin [15], [16] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. In the last 10 years, there has been a lot of interest in the study of the fractional Schrödinger equation

$$(-\Delta)^\alpha u + V(x)u = f(x, u) \text{ in } \mathbb{R}^n. \quad (2)$$

where the nonlinearity f satisfies some general conditions. See, for instance, Feng [11], Chang [3], [4], Cheng [5], Dipiero, Palatucci and Valdinoci [8], Dong and Xu [9], Felmer, Quaas and Tan [10], de Oliveira, Costa and Vaz [18] and Secchi [20], [21].

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To the author’s knowledge, most of these works assumed that there exists a trivial solution, namely 0, for (2). There seems to have been very little progress on existence theory for (2) without trivial solutions.

This paper studies the existence of solutions $u \in H^\alpha(\mathbb{R}^n)$ for the fractional equation

$$(-\Delta)^\alpha u(x) + V(x)u(x) = f(u) + h(x) \text{ in } \mathbb{R}^n, \quad n \geq 2, \tag{3}$$

where $0 < \alpha < 1$, $(-\Delta)^\alpha$ stands for the fractional laplacian defined by

$$(-\Delta)^\alpha u(x) = p.v. \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x - z|^{n+2\alpha}} dz.$$

This problem is a model for (2) without trivial solutions and present specific mathematical difficulties.

When $\alpha = 1$ in (3), we have the classical non homogeneous nonlinear Schrödinger equation, which has been studied extensively by many authors in the last few decades, see, for example [2], [22], [23], and references therein, where the existence and multiplicity results have been studied.

Troughout the paper we assume that

- (V₁) $V \in C(\mathbb{R}^n, \mathbb{R})$ and there exists a constant $V_0 > 0$ such that $V(x) \geq V_0, \forall x \in \mathbb{R}^n$,
- (V₂) $\lim_{|x| \rightarrow \infty} V(x) = \infty$,

Regarding f we consider

- (f₁) $f \in C(\mathbb{R}), f(0) = 0$,
- (f₂) $f(t) = o(|t|)$ as $t \rightarrow 0$,
- (f₃) $f(t) = o(|t|^{\frac{n+2\alpha}{n-2\alpha}})$ as $t \rightarrow \infty$,
- (f₄) There is a constant $\mu > 2$ such that

$$0 < \mu F(u) = \mu \int_0^u f(s) ds \leq u f(u), \quad u \neq 0, \tag{4}$$

and for h we consider

- (H) $h \in L^2(\mathbb{R}^n), h \neq 0$ and

$$\|h\|_{L^2(\mathbb{R}^n)} < \frac{\varrho}{S} \left(\frac{1}{2} - \epsilon S^2 - (\epsilon + K_\epsilon) S^{2\alpha^*} \varrho^{2\alpha^* - 2} \right),$$

where $\varrho > 0$ is given by the first geometrical condition of the mountain pass theorem and S is the Sobolev constant.

Our main result is as follows.

Theorem 1.1 Under assumptions (V₁) – (V₂), (f₁) – (f₄) and (H), (3) has at least two solutions.

Our study is motivated by [5], [20], [21]. In [5] Cheng proved the existence of bound state solutions to (2) with $f(t) = t^q$ and unbounded potential by using Lagrange multiplier method and Nehari’s manifold approach. It is worth noticing that under the assumption that potential $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, the embedding $H_V^\alpha(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ is compact, where

$$H_V^\alpha(\mathbb{R}^n) = \left\{ u \in H^\alpha(\mathbb{R}^n) / \int_{\mathbb{R}^n} V(x)u^2(x) dx < \infty \right\}$$

and $2 \leq q < \frac{2n}{n-2\alpha}$. In [20] Secchi has studied the equation (2). Under the same assumption on V , the existence of a ground states is obtained by Mountain pass

Theorem. In [21], Secchi looks for a radially symmetric solution of (3), with f does not depend on x , namely de considered

$$(-\Delta)^\alpha u(x) + V(x)u(x) = f(u)$$

where the nonlinearity f satisfies rather weak assumptions, which are comparable to those in [1]. By using the monoticity trick of Struwe-Jeanjean, Secchi shows the existence of radial solution.

Our theorem extends these result to the case $h \neq 0$. Under this assumption, the problem of existence of solutions is much more delicate, because the extra difficulties arise in studying the properties of the corresponding action functional $I : H_V^\alpha(\mathbb{R}^n) \rightarrow \mathbb{R}$

The problem here is as follows. We are given two sequence of almost critical point in $H_V^\alpha(\mathbb{R}^n)$. The first one, obtained by Ekeland's variational principle, is contained in a small ball centered at 0. Using the mountain pass geometry of the action functional, the existence of the second sequence is established. Both sequence are weakly convergent in $H_V^\alpha(\mathbb{R}^n)$. The question is whether their limits are equal to each other or they define two geometrically distinct solutions of (3). The PS-condition is enough to obtain two solutions. The assumption (V_2) ensure the PS-condition at each level. In fact one needs the PS-condition only at two levels.

This article is organized as follows. In Section §2 we present preliminaries with the main tools and the functional setting of the problem. In Section §3 we prove the Theorem 1.1.

2. PRELIMINARIES

In this section, we collect some information to be used in the paper. Sobolev spaces of fractional order are the convenient setting for our equation. A very complete introduction to fractional Sobolev spaces can be found in [7].

We recall that the fractional Sobolev space $H^\alpha(\mathbb{R}^n)$ is defined for any $\alpha \in (0, 1)$ as

$$H^\alpha(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) / \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(z)|^2}{|x - z|^{n+2\alpha}} < \infty \right\}.$$

This space is endowed with the Gagliardo norm

$$\|u\|_\alpha^2 = \int_{\mathbb{R}^n} u^2(x) dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(z)|^2}{|x - z|^{n+2\alpha}}.$$

Regarding the space $H^\alpha(\mathbb{R}^n)$ we recall the following embedding theorem, whose proof can be found in [7].

Theorem 2.1 Let $\alpha \in (0, 1)$, then there exists a positive constant $C = C(n, \alpha)$ such that

$$\|u\|_{L^{2_\alpha^*}(\mathbb{R}^n)}^2 \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} dy dx \quad (5)$$

and then we have that $H^\alpha(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ is continuous for all $q \in [2, 2_\alpha^*]$.

Moreover, $H^\alpha(\mathbb{R}^n) \hookrightarrow L^q(\Omega)$ is compact for any bounded set $\Omega \subset \mathbb{R}^n$ and for all $q \in [2, 2_\alpha^*)$, where $2_\alpha^* = \frac{2n}{n-2\alpha}$ is the critical exponent.

Now we consider the Hilbert space $H_V^\alpha(\mathbb{R}^n)$ defined by

$$H_V^\alpha = \left\{ u \in H^\alpha(\mathbb{R}^n) / \int_{\mathbb{R}^n} V(x)u^2(x) dx < \infty \right\}$$

endowed with the inner product

$$\langle u, w \rangle_V = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[u(x) - u(z)][w(x) - w(z)]}{|x - z|^{n+2\alpha}} + \int_{\mathbb{R}^n} V(x)u(x)w(x)dx,$$

and norm

$$\|u\|_V^2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(z)|^2}{|x - z|^{n+2\alpha}} + \int_{\mathbb{R}^n} V(x)u^2(x)dx$$

By (V_1) it is standard to prove that $H_V^\alpha(\mathbb{R}^n)$ is continuously embedded in $H^\alpha(\mathbb{R}^n)$ and by Theorem 2.1, we have that $H_V^\alpha(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ is continuous for all $q \in [2, 2_\alpha^*]$. Moreover, we have the following compactness theorem

Theorem 2.2 [5] Suppose that (V_1) and (V_2) holds. Then $H_V^\alpha(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ is compact for all $q \in [2, 2_\alpha^*]$.

Moreover we consider the following Lemma

Lemma 2.1 [13] Suppose the $\beta > 1$ and the function $f \in C(\mathbb{R})$ satisfies

$$f(t) = o(|t|) \text{ as } |t| \rightarrow 0 \text{ and } f(t) = o(|t|^\beta) \text{ as } |t| \rightarrow \infty$$

If $\{u_k\}_k$ is a bounded sequence in $L^{\beta+1}(\mathbb{R}^n)$ and $u_k \rightarrow u$ in $L^2(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} |f(u_k)(u_k - u)|dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

3. PROOF OF THEOREM 1.1

In this section, our goal is to prove the existence of solutions of equation (3). We start with a precise definition of the notion of solutions for equation (3).

Definition 3.1 We say that $u \in H_V^\alpha(\mathbb{R}^n)$ is a weak solution of (3) if

$$\langle u, w \rangle_V = \int_{\mathbb{R}^n} (f(u(x)) + h(x))w(x)dx, \text{ for all } w \in H_V^\alpha(\mathbb{R}^n).$$

We prove the existence of weak solution of (3) finding a critical point of the functional $I : H_V^\alpha(\mathbb{R}^n) \rightarrow \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\|u\|_V^2 - \int_{\mathbb{R}^n} F(u(x))dx - \int_{\mathbb{R}^n} h(x)u(x)dx. \tag{6}$$

Using the properties of the Nemistky operators and the compact embedding Theorem 2.2, we can prove that the functional $I \in C^1(H_V^\alpha(\mathbb{R}^n), \mathbb{R})$ and we have

$$I'(u)w = \langle u, w \rangle_V - \int_{\mathbb{R}^n} f(u(x))w(x)dx - \int_{\mathbb{R}^n} h(x)w(x)dx, \quad \forall w \in H_V^\alpha(\mathbb{R}^n) \tag{7}$$

In order to prove Theorem 1.1 we use the mountain pass Theorem (see [19] Theorem 2.2) and Ekeland's variational principle (see [17] Theorem 4.1 and Corollary 4.1). The proof will be divided into a sequence of Lemmas.

Lemma 3.1 Suppose that (V_1) , (V_2) , $(f_1) - (f_4)$ and (H) holds. Then the functional $I : H_V^\alpha(\mathbb{R}^n) \rightarrow \mathbb{R}$ satisfies the Palais-Smale condition.

Proof. Let $\{u_k\}$ be a sequence in $H_V^\alpha(\mathbb{R}^n)$ such that

$$|I(u_k)| \leq C, \quad I'(u_k) \rightarrow 0 \text{ in } (H_V^\alpha(\mathbb{R}^n))^* \text{ as } k \rightarrow \infty. \tag{8}$$

There exists k_0 such that for $k \geq k_0$

$$|I'(u_k)u_k| \leq \|u_k\|_V.$$

Then

$$\begin{aligned}
C + \|u_k\|_V &\geq I(u_k) - \frac{1}{\mu} I'(u_k)u_k \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_k\|_V^2 + \int_{\mathbb{R}^n} \left(\frac{1}{\mu} f(u_k)u_k - F(u_k)\right) dx \\
&\quad - \left(1 - \frac{1}{\mu}\right) \int_{\mathbb{R}^n} h(x)u_k(x) dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u\|_V^2 - \left(C_e - \frac{C_e}{\mu}\right) \|h\|_{L^2} \|u_k\|_V,
\end{aligned}$$

so, $\{u_k\}$ is bounded in $H_V^\alpha(\mathbb{R}^n)$. By Theorem 2.2, $H_V^\alpha(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$, compactly and $H_V^\alpha(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$ continuously. Then, there exists a subsequence, still denoted by $\{u_k\}$ such that

$$\begin{aligned}
u_k &\rightharpoonup u \text{ in } H_V^\alpha(\mathbb{R}^n), \\
u_k &\rightarrow u \text{ in } L^2(\mathbb{R}^n).
\end{aligned}$$

By another hand

$$\begin{aligned}
\|u_k\|_V^2 - \langle u_k, u \rangle_V &= I'(u_k)(u_k - u) + \int_{\mathbb{R}^n} f(u_k)(u_k - u) dx \\
&\quad + \int_{\mathbb{R}^n} h(x)(u_k(x) - u(x)) dx.
\end{aligned}$$

Hence, by Lemma 2.1 and (8)

$$\lim_{k \rightarrow \infty} (\|u_k\|_V^2 - \langle u_k, u \rangle_V) = 0.$$

Then $\lim_{k \rightarrow \infty} \|u_k\|_V = \|u\|_V$ and, therefore, the sequence $\{u_k\}$ converges to u strongly in $H_V^\alpha(\mathbb{R}^n)$. \square

Lemma 3.2 Suppose that $(V_1) - (V_2)$, $(f_1) - (f_4)$ and (H) holds. There are $\varrho > 0$ and $\tau > 0$ such that

$$I(u) \geq \tau \text{ for } \|u\|_V = \varrho$$

Proof. By continuous embedding

$$\|u\|_{L^2(\mathbb{R}^n)} \leq C_e \|u\|_V, \quad \|u\|_{L^{2^*}(\mathbb{R}^n)} \leq C_e \|u\|_V. \quad (9)$$

By (f_2) and (f_3) , for every ϵ there exists $\rho, \delta > 0$ such that

$$\begin{aligned}
|f(t)| &\leq \epsilon |t|^{\frac{n+2\alpha}{n-2\alpha}} \text{ for } |t| \geq \rho, \text{ and} \\
|f(t)| &\leq \epsilon |t| \text{ for } |t| \leq \delta.
\end{aligned}$$

Therefore we have

$$|f(t)| \leq \epsilon(|t| + |t|^{\frac{n+2\alpha}{n-2\alpha}}) + K_\epsilon |t|^{\frac{n+2\alpha}{n-2\alpha}},$$

where $K_\epsilon = \delta^{-\frac{n+2\alpha}{n-2\alpha}} \max_{\delta \leq |t| \leq \rho} |f(t)|$. Then we have

$$|F(t)| \leq \epsilon(|t|^2 + |t|^{2^*}) + K_\epsilon |t|^{2^*}. \quad (10)$$

Let $0 < \epsilon < \frac{1}{2C_e^2}$. By (9) and (10) we have

$$\begin{aligned}
\int_{\mathbb{R}^n} F(u) dx &\leq \epsilon \left(\|u\|_{L^2(\mathbb{R}^n)}^2 + \|u\|_{L^{2^*}(\mathbb{R}^n)}^{2^*} \right) + K_\epsilon \|u\|_{L^{2^*}(\mathbb{R}^n)}^{2^*} \\
&\leq \epsilon C_e^2 \|u\|_V^2 + (\epsilon + K_\epsilon) C_e^{2^*} \|u\|_V^{2^*}
\end{aligned}$$

and

$$I(u) \geq \left(\frac{1}{2} - \epsilon C_e^2\right) \|u\|_V^2 - (\epsilon + K_\epsilon) C_e^{2^*} \|u\|_V^{2^*} - C_e \|h\|_{L^2(\mathbb{R}^n)} \|u\|_V. \tag{11}$$

Taking $\|u\|_V = \varrho$ then $I(u) \geq \tau > 0$ by (H). \square

Lemma 3.3 Suppose that $(V_1) - (V_2), (f_1) - (f_4)$ and (H) holds. There is $e \in \overline{B(0, \varrho)^c}$ such that $I(e) \leq 0$

Proof. Since $h \neq 0$, we can choose a function $\varphi \in H^\alpha(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} h(x)\varphi(x)dx > 0.$$

By (f_4) it follows that there exists a constant $m > 0$ such that

$$F(u) \geq m|u|^\mu \text{ if } |u| \geq 1, \tag{12}$$

so, for $\lambda \in (0, +\infty)$, we have

$$\begin{aligned} I(\lambda\varphi) &= \frac{\lambda^2}{2} \|\varphi\|_V^2 - \int_{\mathbb{R}^n} F(\lambda\varphi)dx - \lambda \int_{\mathbb{R}^n} h(x)\varphi(x)dx \\ &\leq \frac{\lambda^2}{2} \|\varphi\|_V^2 - m\lambda^\mu \int_{\{|u| \geq 1\}} |\varphi|^\mu dx - \lambda \int_{\mathbb{R}^n} h(x)\varphi(x)dx. \end{aligned}$$

Since $\mu > 2$, $I(\lambda\varphi) \rightarrow -\infty$ as $\lambda \rightarrow +\infty$. Hence, there is $\lambda \in (0, +\infty)$ such that

$$\|\lambda\varphi\|_V > \varrho \text{ and } I(\lambda\varphi) \leq 0.$$

\square

Proof of Theorem 1.1

Since $I(0) = 0$ and I satisfies Lemmas 3.1 - 3.3, it follows by the mountain pass Theorem that I has a critical value c given by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0, 1], H_V^\alpha(\mathbb{R}^n)) / \gamma(0) = 0, I(\gamma(1)) \leq 0\}$. By definition, it follows that $c \geq \varrho > 0$.

From (11), we conclude that I is bounded from below on $\overline{B(0, \varrho)}$. Set

$$\bar{c} = \inf_{\|u\|_V \leq \varrho} I(u). \tag{13}$$

Hence $I(0) = 0$ implies $\bar{c} \leq 0$. Thus $\bar{c} < c$. By Ekeland's variational principle, there is a minimizing sequence $\{w_k\} \subset \overline{B(0, \varrho)}$ such that

$$I(w_k) \rightarrow \bar{c} \text{ and } I'(w_k) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

From Lemma 3.1, \bar{c} is a critical value of I . Consequently, I has at least two critical points. \square

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