Journal of Fractional Calculus and Applications,
Vol. 6(1) Jan. 2015, pp. 101-109.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

ON THE STOCHASTIC FRACTIONAL CALCULUS OPERATORS

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Abstract. The mean square fractional-order integral operator and the mean square fractional-order differential operators, in the Caputo and Riemann-Liouville senses, for the second order stochastic processes have been studied in [1]-[3].
In this work we define the Caputo-via Riemann-Liouville fractional-order operator for the second order stochastic processes and study some equivalent properties for these fractional-order operators and some equivalent Cauchy type problems. Also we define the mild solution of the problems of the non-linear fractional-order stochastic differential equations and give some of its properties and applications.

1. Introduction

Let $I = [a,b]$. Let (Ω, F, P) be a fixed probability space, where Ω is a sample space, F is a σ-algebra and P is a probability measure.
Let $X(t;\omega) = \{ X(t),\ t \in I,\ \omega \in \Omega \}$ be a second order stochastic process, i.e., $E(X^2(t)) < \infty,\ t \in I$.
Let $C = C(I, L_2(\Omega))$ be the space of all second order stochastic processes which is mean square (m.s) continuous on I. This space is a Banach space endowed with the norm

$$||X||_C = \max_t \|X(t)\|_2, \quad \text{where} \quad \|X(t)\|_2 \ = \ (E(X^2(t)))^{\frac{1}{2}}.$$

Definition 1. A function $f : [0,T] \times L_2(\Omega) \to L_2(\Omega)$ is said to satisfy the mean square Lipschitz condition if

$$\|f(t, X(t)) - f(t, Y(t))\|_2 \leq k\|X(t) - Y(t)\|_2, \tag{1}$$

where k is a positive constant.

Let $\Re(I, L_2(\Omega))$ be the class of all second order stochastic processes which is mean square Riemann integrable on I

$$\int_a^b \ E\ X^2(t)\ dt \ < \ \infty.$$

2010 *Mathematics Subject Classification.* 34A12, 34A30, 34D20.
Key words and phrases. Stochastic processes, Stochastic fractional-order derivatives, Stochastic fractional-order integrals.
Submitted Aug. 8, 2014.

The norm of $X \in \Re(I, L_2(\Omega))$ is given by

$$|| X ||_{\Re} = | \int_a^b E X^2(t) \, dt |^{1/2}.$$

2. Fractional-order stochastic integral

2.1. **Mean square continuous processes.** Now we give the definition and some properties of the mean square fractional-order stochastic integral defined on $C = C(I, L_2(\Omega))$ ([2]).

Definition 2. Let $X \in C(I, L_2(\Omega))$ and $\beta \in (0,1)$. The stochastic fractional-order integral $I_a^\beta X(t)$ is defined by

$$I_a^\beta X(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s) ds, \ X \in C(I, L_2(\Omega)). \tag{2}$$

For the existence of the integral (2) we have the following theorem ([2]).

Theorem 1. Let $\alpha, \beta \in (0,1)$. If $X \in C(I, L_2(\Omega))$, then $I_a^\beta X(t)$ exists in m.s. sense as a second order m.s. continuous second order process $I_a^\beta X \in C(I, L_2(\Omega))$ with the following properties

(c1) $I_a^\beta : C(I, L_2(\Omega)) \to C(I, L_2(\Omega))$
(c2) $I_a^\alpha I_a^\beta X(t) = I_a^\beta I_a^\alpha X(t) = I_a^{\alpha+\beta} X(t)$
(c3) $I_a^\beta X(t)|_{t=a} = 0$
(c4) $L.i.m_{\beta \to 1} I_a^\beta X(t) = I_a X(t) = \int_a^t X(s) \, ds$
(c5) $X \in C^1(I, L_2(\Omega)), \Rightarrow$

$$L.i.m_{\beta \to 0} I_a^\beta X(t) = X(t).$$

2.2. **Mean square integrable processes.** Now we give the definition and some properties of the mean square fractional-order stochastic integral defined on $\Re(I, L_2(\Omega))$ ([3]).

Definition 3. Let $X \in \Re(I, L_2(\Omega))$ and $\beta \in (0,1)$. The stochastic fractional-order integral $I_a^\beta X(t)$ is defined by

$$I_a^\beta X(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s) ds, \ X \in \Re(I, L_2(\Omega)). \tag{3}$$

For the existence of this integral we have the following theorem ([3]).

Theorem 2. Let $\alpha, \beta \in (0,1)$. If $X \in \Re(I, L_2(\Omega))$, then $I_a^\beta X(t)$ exists in m.s. sense as a second order Riemann integrable second order process $I_a^\beta X \in \Re(I, L_2(\Omega))$ with the following properties

(r1) $I_a^\beta : \Re(I, L_2(\Omega)) \to \Re(I, L_2(\Omega))$
(r2) $I_a^\alpha I_a^\beta X(t) = I_a^{\alpha+\beta} X(t)$
(r3) $L.i.m_{\beta \to 1} I_a^\beta X(t) = I_a X(t)$.

3. Derivative of the fractional-order integral

Consider firstly the stochastic integral equation

$$X(t) = X_o + \int_a^t F(s) ds, \ t \in I, \tag{4}$$

then we have (see [4]-[6])

(a) If $F \in \Re(I, L_2(\Omega))$, then $X \in C(I, L_2(\Omega))$

(b) If $F \in C(I, L_2(\Omega))$, then $X \in C^1(I, L_2(\Omega))$.

But for the fractional-order (of order $\alpha \in (0,1)$) stochastic integral equation

$$X(t) \ = \ X_o \ + \ \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds, \ t \in I \tag{5}$$

it is proved that (see [1]-[3])

(c) If $F \in \Re(I, L_2(\Omega))$, then $X \in \Re(I, L_2(\Omega))$

(b) If $F \in C(I, L_2(\Omega))$, then $X \in C(I, L_2(\Omega))$.

Now we can prove the following two theorems.

Theorem 3. The necessarily condition for the existence of the derivative $\frac{d}{dt} X(t)$ of the solution of the integral equation (5) is that $F \in C^1(I, L_2(\Omega))$. Moreover

$$\frac{d}{dt} X(t) \ = \ F(a)\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} + I_a^{\alpha} \frac{d}{dt} \ F(t) \ \in \Re(I, L_2(\Omega)). \tag{6}$$

Proof. Write equation (5) in the form

$$X(t) \ = \ X_o \ + \ \int_a^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} F(t-s)ds, \ t \in I. \tag{7}$$

Differentiating (7) and applying the properties of stochastic derivative ([4]-[6]) we obtain the results.

Theorem 4. Let $\alpha \in (0,1)$. If $X \in C^1(I, L_2(\Omega))$, then

$$\frac{d}{dt} \ I_a^{\alpha} \ X(t) \ = \ X(a)\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} + I_a^{\alpha} \ \frac{d}{dt} \ X(t) \tag{8}$$

and

$$\frac{d}{dt} \ I_a^{\alpha} \ \{ \ X(t) \ - \ X(a) \ \} = \ I_a^{\alpha} \ \frac{d}{dt} \ X(t). \tag{9}$$

Proof. Write $I_a^{\alpha} \ X(t)$ in the form

$$I_a^{\alpha} \ X(t) \ = \ \int_a^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} X(t-s)ds, \ t \in I. \tag{10}$$

Differentiating (10) and applying the properties of stochastic derivative ([4]-[6]) we obtain the results.

Corollary 1. Let $\alpha \in (0,1)$ and $X \in C^1(I, L_2(\Omega))$. If $X(a) \ = \ 0$, then

$$\frac{d}{dt} \ I^{\alpha} \ X(t) \ = \ I_a^{\alpha} \frac{d}{dt} \ (t).$$

4. Abel's integral equations I

Let $\beta \in (0,1]$ and $t \in [a,b]$. Consider the stochastic Abel's integral equations of first kind

$$\frac{1}{\Gamma(\beta)} \int_a^t \frac{X(s)}{(t-s)^{1-\beta}} ds = Y(t) \tag{11}$$

which can be written as

$$I_a^{\beta} \ X(t) \ = \ Y(t).$$

Theorem 5. If $Y(t) \in \Re(I, L_2(\Omega))$ and (the mean square derivative) $\frac{d}{dt} I_a^{1-\beta} Y(t) \in \Re(I, L_2(\Omega))$, then the stochastic Abel's integral equation (11) has the (second order process) solution

$$X_\beta(t) = \frac{d}{dt} I_a^{1-\beta} Y(t) \in \Re(I, L_2(\Omega)). \quad (12)$$

Theorem 6. If $Y(t) \in C^1(I, L_2(\Omega))$ (is m.s. differentiable with m.s. continuous derivative), then the stochastic Abel's integral equation (11) has the (second order process) solution

$$X_\beta(t) = Y(a) \frac{(t-a)^{-\beta}}{\Gamma(1-\beta)} + I_a^{1-\beta} D\, Y(t) \in \Re(I, L_2(\Omega)), \quad (13)$$

from which we obtain

$$L.i.m_{\beta \to 0} X_\beta(t) = Y(t).$$

5. Abel's integral equations II

Consider the stochastic Abel's integral equations of second kind

$$X(t) + \frac{\lambda}{\Gamma(\beta)} \int_a^t \frac{X(s)}{(t-s)^{1-\beta}} ds = Y(t) \quad (14)$$

which can be written as

$$X(t) + \lambda I_a^\beta X(t) = Y(t). \quad (15)$$

Theorem 7. Let $Y(t) \in \Re(I, L_2(\Omega))$. If $|\lambda| < \frac{\Gamma(1+\beta)}{(b-a)^\beta}$, then the stochastic Abel's integral equation (14) has the second order process solution

$$X_\beta(t) = \sum_0^\infty (-\lambda)^n I_a^{n\beta} Y(t) \in \Re(I, L_2(\Omega)). \quad (16)$$

Theorem 8. Let $Y(t) \in C^1(I, L_2(\Omega))$. If $|\lambda| < \frac{\Gamma(1+\beta)}{(b-a)^\beta}$, then the stochastic Abel's integral equation (14) has the second order process solution

$$X_\beta(t) = \sum_0^\infty (-\lambda)^n I_a^{n\beta} Y(t) \in C(I, L_2(\Omega)). \quad (17)$$

From the convergence of the series in equality (17) we can get

$$L.i.m_{\beta \to 0} X_\beta(t) = \frac{1}{1+\lambda} Y(t).$$

which is the solution of (15), as $\beta \to 0$.

6. Fractional-order stochastic derivative

Definition 4. Let $X(t) \in C^1(I, L_2(\Omega))$ (be a second order stochastic process which is m.s. differentiable with m.s. continuous derivative). We define the fractional-order derivative, Caputo sense, of $X(t)$ of order $\alpha \in (0, 1]$ by the second order process ([2]),

$$D_a^\alpha X(t) = I_a^{1-\alpha} D X(t) \in C(I, L_2(\Omega)). \quad (18)$$

For the properties of the fractional order stochastic derivative, we have the following theorems ([2]).

Theorem 9. Let $X(t) \in C^1(I, L_2(\Omega))$, and $\alpha \in (0, 1]$, then

(1) $L.i.m_{\alpha \to 1} D_a^\alpha X(t) = \frac{d}{dt} X(t)$

(2) $L.i.m_{\alpha\to 0}\ D_a^\alpha\ X(t)\ =\ X(t) - X(a)$
(3) $I_a^\alpha D_a^\alpha X(t) = X(t) - X(a)$
(4) $D_a^\alpha I_a^\alpha X(t) = X(t)$.

Theorem 10. Let $\alpha, \beta \in (0,1],\ \alpha+\beta \in (0,1]$. Let $X(t) \in C^2(I, L_2(\Omega))$, then

$$D_a^\alpha D_a^\beta X(t) = D_a^{\alpha+\beta} X(t), t \in [a,b].$$

Theorem 11. Let $\alpha, \beta \in (0,1],\ \alpha+\beta \in (1,2]$. Let $X(t) \in C^2(I, L_2(\Omega))$. If $DX(t)|_{t=a}\ =\ 0$, then

$$D_a^\alpha D_a^\beta X(t) = D_a^{\alpha+\beta} X(t),\ t \in [a,b].$$

Theorem 12. Let $\alpha, \beta \in (0,1],\ \beta \ge \alpha$. If $X(t) \in C^1(I, L_2(\Omega))$, then

$$D_a^\alpha I_a^\beta X(t) = I_a^{\beta-\alpha} X(t), t \in [a,b] \subset T.$$

Theorem 13. Let $\alpha, \beta \in (0,1]$. Let $X(t) \in C^1(I, L_2(\Omega))$, then

$$\beta \le \alpha \Rightarrow I_a^\beta D_a^\alpha X(t)\ =\ D_a^{\alpha-\beta} X(t).$$

7. Differential equations of fractional order

Consider the following initial value problems.

(I) Firstly, let $f \in C([0,T], L_2(\Omega))$. Then the initial value problem

$$\begin{cases} \frac{d}{dt}X(t)\ =\ f(t),\ t \in (0,T] \\ X(0)\ =\ X_o, \end{cases} \tag{19}$$

has the unique solution

$$X(t)\ =\ X_o\ +\ \int_0^t f(s)\ ds\ \in C^1([0,T], L_2(\Omega)).$$

(II) Let $f \in C([0,T], L_2(\Omega))$ and $\alpha \in (0,1)$. Integrating the initial-value problem

$$\begin{cases} D^\alpha X(t)\ =\ f(t),\ t \in (0,T] \\ X(0)\ =\ X_o, \end{cases} \tag{20}$$

we obtain the corresponding integral equation

$$X(t)\ =\ X(0)\ +\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds\ \in C([0,T], L_2(\Omega)). \tag{21}$$

Now, if $f \in C([0,T], L_2(\Omega))$, then the derivative $X'(t)$ does not exist. Consequently the fractional-order derivative $D^\alpha X(t)$ of the solution of (21) does not exist.
Then there is no equivalence between the initial value problem (20) and its corresponding integral equation (21).
The problem (20) can not be solved.

(III) Let $f \in C^1([0,T], L_2(\Omega))$ and $\alpha \in (0,1]$. Integrating the initial-value problem (20)

$$\frac{d}{dt}X(t)\ =\ f(0)\ \frac{t^{\alpha-1}}{\Gamma(\alpha)}\ +\ \int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)}\ \frac{d}{dt} f(t-s)\ ds\ \in\ L^1([0,T], L_2(\Omega))$$

and

$$D^\alpha X(t)\ =\ f(t),$$

then the integral equation (21) and the initial-value problem (20) are equivalent and (21) is the unique solution of (20).

8. Nonlinear differential equation

Let $f : [0,T] \times L_(\Omega) \to L_2(\Omega)$ be m.s continuous. Let $\alpha \in (0,1)$. Integrating the initial-value problem

$$\begin{cases} D^\alpha X(t) \; = \; f(t, X(t)), \; t \in (0,T] \\ \qquad X(0) \; = \; X_o, \end{cases} \tag{22}$$

we obtain the corresponding integral equation

$$X(t) \; = \; X(0) \; + \; \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) ds \; \in C([0,T], L_2(\Omega)). \tag{23}$$

But the derivative $X'(t)$ does not exist. Consequently the derivative $D^\alpha X(t)$ does not exist and there is no equivalence between the initial-value problem (22) and the integral equation (23). So, the problem (22) can not be solved in this case.

8.1. **Mild Solution.** Now we give the the definition of mild solution of the initial value problem of fractional order differential equations
Definition 5. By a mild solution of the initial value problem (22) (or (18)) we mean an exact solution of the corresponding integral equation (23) (or (19)).

For the existence of the mild solution of the initial value problem (22) we can prove the following theorem.
Theorem 14. Let $f : [0,T] \times L_(\Omega) \to L_2(\Omega)$ be m.s continuous and satisfies the Lipschitz condition (1). If

$$T \leq (\frac{\Gamma(1+\alpha)}{\gamma})^{\frac{1}{\alpha}},$$

then there exists a unique mild solution $X \in C(I, L_2(\Omega))$ of the initial value problem (22). **Proof.** Define the operator

$$F \; X(t) \; = \; X(0) \; + \; \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) ds,$$

then we can prove that $F : \; C(I, L_2(\Omega)) \to \; C(I, L_2(\Omega))$ and is contraction.
Then applying the Banach fixed point theorem we obtain the results.

9. Differeintegral operators

Let $X(t) \in C([0,T], L_2(\Omega))$. Then we define the differeintegral operator (R-L type fractional derivative) of $X(t)$ of order $\alpha \in (0,1)$ by the second order process,

$${}^R D^\alpha \; X(t) \; = \; \frac{d}{dt} \; I_a^{1-\alpha} X(t). \tag{24}$$

For the existence of the operator (24) we have the following theorem.
Theorem 15. If the second order stochastic process $I^{1-\alpha} \; X(t)$ is mean square differentiable with mean square derivative $\frac{d}{dt} \; I_a^{1-\alpha} X(t) \in \Re(I, L_2(\Omega))$, then the m.s Riemann-Liouville fractional derivative (24) exists.
For the properties of the differeintegral operator we have the following theorem

[1]-[3].

Theorem 16. Let $\alpha, \beta \in (0,1)$. Let $X(t)$ be a second order m.s continuous stochastic process such that ${}^{R}D_a^{\alpha} X(t)$ exists, then

(1) ${}^{R}D_a^{\alpha} I_a^{\alpha} X(t) = X(t) = I_a^{\alpha}\ {}^{R}D_a^{\alpha} X(t)$.
(2) ${}^{R}D_a^{\alpha} I_a^{\beta} X(t) = I_a^{\beta-\alpha} X(t),\ \beta > \alpha$.
(3) ${}^{R}D_a^{\alpha} I_a^{\beta} X(t) = {}^{R}D_a^{\alpha-\beta} X(t),\ \alpha > \beta$.
(4) $I_a^{\beta}\ {}^{R}D_a^{\alpha} X(t) = I_a^{\beta-\alpha} X(t),\ \beta > \alpha$.
(5) $I_a^{\beta}\ {}^{R}D_a^{\alpha} X(t) = {}^{R}D_a^{\alpha-\beta} X(t),\ \alpha > \beta$.
(6) If $\alpha + \beta \in (0,1)$, then

$$ {}^{R}D_a^{\alpha}\ {}^{R}D_a^{\beta} X(t) = {}^{R}D_a^{\alpha+\beta} X(t). $$

Remark 1

From the relation (6) we can deduce that if $X(t) \in C^1(I, L_2(\Omega))$ and $X(0) = 0$, then the fractional-order differeintegral operator ${}^{R}D^{\alpha}\ X(t) = \frac{d}{dt}\ I_a^{1-\alpha} X(t)$ and the fractional-order differential operator (Caputo sense) $D^{\alpha}\ X(t) = I_a^{1-\alpha}\ \frac{d}{dt}\ X(t)$ are equivalent.

9.1. **Differeintegral equations.** Let $f : [0,T] \times L_2(\Omega) \to L_2(\Omega)$ be m.s. continuous. Consider now the following problems.

$$ \begin{cases} {}^{R}D^{\alpha} X(t) = f(t, X(t)),\ t \in (0,T] \\ X(0) = 0, \end{cases} \tag{25} $$

$$ \begin{cases} {}^{R}D^{\alpha} X(t) = f(t, X(t)),\ t \in (0,T] \\ I^{1-\alpha} X(t)|_{t=0} = 0 \end{cases} \tag{26} $$

and

$$ \begin{cases} {}^{R}D^{\alpha} X(t) = f(t, X(t)),\ t \in (0,T] \\ t^{1-\alpha} X(t)|_{t=0} = 0. \end{cases} \tag{27} $$

For these three Cauchy type problems we can prove the following equivalent and existence theorems.

Theorem 17. Let $f : [0,T] \times L_2(\Omega) \to L_2(\Omega)$ be m.s. continuous. Then the three Cauchy type problems (25)-(27) are equivalent and they equivalent to the stochastic fractional-order integral equation

$$ X(t) = I_a^{\alpha}\ f(t, X(t)). \tag{28} $$

Theorem 18. Let $f : [0,T] \times L_2(\Omega) \to L_2(\Omega)$ satisfies the Lipschitz condition (1). If

$$ T \le (\frac{\Gamma(1+\alpha)}{\gamma})^{\frac{1}{\alpha}}. $$

Then three Cauchy type problems (25)-(27) has a unique solution

$$ X \in C([0,T], L_2(\Omega)). $$

This solution is the solution of the integral equation

$$ X(t) = I^{\alpha}\ f(t, X(t)) $$

10. Caputo derivative via Riemann-Liouville

Now we give the definition of the Caputo fractional-order derivative, for the second order stochastic process $X(t)$, via the Riemann-Liouville one.

Definition 6. The Caputo fractional-order derivative, of the second order process $X(t)$, via the Riemann-Liouville one is defined by

$$^{C-(R-L)}D_a^{\alpha} X(t) = I_a^{1-\alpha} \frac{d}{dt} (X(t) - X(a)). \tag{29}$$

For the existence of the operator (29) we have the following theorem

Theorem 19. The necessarily condition for the existence of the Caputo fractional-order derivative, for the second order stochastic process $X(t)$, via the Riemann-Liouville one is that second order stochastic process $I_a^{1-\alpha} X(t)$ is mean square differentiable with mean square derivative $\frac{d}{dt} I_a^{1-\alpha} X(t) \in \Re(I, L_2(\Omega))$.

Consider now the Cauchy problem

$$\begin{cases} ^{C-(R-L)}D^{\alpha} X(t) = f(t, X(t)),\ t \in (0, T] \\ X(0) = X_o. \end{cases} \tag{30}$$

Theorem 20. Let $f : [0, T] \times L_(\Omega) \to L_2(\Omega)$ be m.s continuous. Then the problem (30) is equivalent to the integral equation

$$X(t) = X_o + I^{\alpha} f(t, X(t)). \tag{31}$$

Proof. Integrating the equation

$$\frac{d}{dt} I^{1-\alpha} (X(t) - X(0)) = f(t, X(t))$$

we obtain

$$I^{1-\alpha} (X(t) - X(0)) = I^{1-\alpha} (X(t) - X(0))|_{t=0} + I f(t, X(t)) = I f(t, X(t)),$$

operating with I^{α} we obtain

$$I (X(t) - X(0)) = I^{1+\alpha} f(t, X(t)),$$

differentiating we get the result.

Theorem 21. Let $f : [0, T] \times L_(\Omega) \to L_2(\Omega)$ be m.s continuous and satisfies the Lipschitz condition (1). If

$$T \leq (\frac{\Gamma(1+\alpha)}{\gamma})^{\frac{1}{\alpha}},$$

then there exists a unique solution $X \in C(I, L_2(\Omega))$ of the initial value problem (30). **Proof.** Define the operator

$$F X(t) = X(0) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) ds,$$

then we can prove that $F : C(I, L_2(\Omega)) \to C(I, L_2(\Omega))$ is contraction.
Then applying the Banach fixed point theorem we obtain the results.

Now we can deduce the following theorem.

Theorem 22. Any solution of the problem (30) is a mild solution of the problem (22).

Remark 2

It must be noticed that in all the fractional calculus differential operators Caputo, Riemann-Liouville and Caputo via Riemann-Liouville the fpllowing two relations not holds.
1- The derivative of two multiplied functions

$$\frac{d^\alpha}{dt^\alpha} f(t)g(t) \neq g(t) \frac{d^\alpha}{dt^\alpha} f(t) + f(t) \frac{d^\alpha}{dt^\alpha} g(t)$$

2- The chain rule

$$\frac{d^\alpha}{dt^\alpha} f(g(t)) \neq \frac{d^\alpha}{dg^\alpha} f(g) \cdot \frac{d^\alpha}{dt^\alpha} g(t)$$

or

$$\frac{d^\alpha}{dt^\alpha} f(g(t)) \neq \frac{d^\alpha}{dg^\alpha} f(g) \cdot \frac{d}{dt} g(t)$$

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