Journal of Fractional Calculus and Applications,
Vol. 6(1) Jan. 2015, pp. 5-20.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

EXISTENCE OF A MILD SOLUTION FOR AN IMPULSIVE NEUTRAL FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION WITH NONLOCAL CONDITIONS

ALKA CHADHA, DWIJENDRA N PANDEY

Abstract. In the present work, we consider an impulsive neutral fractional integro-differential equation with nonlocal condition in arbitrary Banach space X. The existence of mild solution is obtained by using solution operator and Hausdorff measure of noncompactness. To illustrate the theory, we provide an example at the end of the manuscript.

1. Introduction

In recent few decades, fractional calculus has received more and more attention of researchers because of its wide applicability in engineering, physics, quantum mechanics, signal processing, electro-magnetic, fractal theory, economics, electro-chemistry and more fields. The properties of memory and heredity of materials can be described by the fractional derivative which is a major advantage of the fractional derivative compared with integer order derivatives. The fractional differential equation is an important tool for describing the nonlinear oscillation of the earthquake. For a study of fractional calculus, we refer to the books by Kibbas et al.[1], Podlubny [2] and Miller and Ross [3] and references given therein. Neutral fractional differential equations arise in many areas of applied mathematics. The system of rigid heat conduction with finite wave spaces can be modeled in the form of the integro-differential equation of neutral type with delay. For the initial study of the neutral functional differential equations with finite delay, we refer to book by Hale [4] and references given therein.

On the other hand, many real world processes and phenomena which are subjected during their development to short-term external influences can be modeled as impulsive differential equation. Their duration is negligible compared with the total duration of the entire process and phenomena. Such processes are investigated in various areas of sciences such as biology, physics, control theory, population dynamics, medicine and so on. For the general theory of such differential equations,

---

2010 *Mathematics Subject Classification.* 34K37, 34K30, 35R11, 47N20.
Key words and phrases. Caputo derivative, Impulsive differential equation, Fractional calculus, Measure of noncompactness, Neutral fractional differential equation, Resolvent operator.
Submitted May 6, 2014.

we refer to the monographs [5], [6] and papers [7]-[19] and references given therein. The existence of the solution for abstract Cauchy differential equation with nonlocal conditions in a Banach space has been considered first by Byszewski [20]. Since it is shown that nonlocal condition is more realistic than the classical initial condition in dealing with many physical problems. Concerning the developments in the study of nonlocal problems we refer to [20]-[31] and references given therein.

In [8], authors have introduced a new concept of the mild solution for impulsive fractional differential equation and established the existence of solutions of the impulsive Cauchy fractional differential equation in a Banach space with the different assumptions about initial conditions. In [7], authors have extended the results of [8] and studied the existence of the mild solution for impulsive fractional integro-differential equation (1) with infinite delay with the assumption that nonlinear function G satisfies a Lipschitz type condition. The existence of the mild solution for impulsive fractional integro-differential inclusions with nonlocal conditions has been discussed by authors [10] with the help of a fixed-point theorem for discontinuous multi-valued operators due to Dhage and compact semigroup. In [17], authors have studied the controllability of impulsive fractional evolution inclusions and obtained the sufficient conditions for the existence of the mild solution by using a fixed point theorem of multivalued and resolvent operator. The existence of the solution for impulsive fractional differential equation with nonlocal conditions has been investigated by authors [12] with the help of fixed point theorem of Sadoviskii.

The purpose of this work is to establish the existence of mild solution for impulsive fractional differential equation with nonlocal conditions of the form:

$$\begin{aligned} {}^cD_t^q[u(t)+H(t,u_t)] &= A[u(t)+H(t,u_t)]]+J_t^{1-q}G(t,u_t,\mathcal{B}u(t)), \\ & \qquad t\in I=[0,T], \quad t\neq t_k,\ 0<T<\infty, \qquad (1) \\ \Delta u(t_k) &= I_k(u(t_k^-)),\ k=1,\cdots,m, \qquad (2) \\ u(t) &= \phi(t)+g(u),\ \ t\in[-\tau,0], \qquad (3) \end{aligned}$$

where $q\in(0,1)$ and $A:D(A)\subset X\to X$ is a closed and bounded linear operator on Banach space $(X,\|\cdot\|)$ with dense domain $D(A)$. We assume that A is the infinitesimal generator of a solution operator $\{S_q(t)\}_{t\geq 0}$. Here $I_k:X\to X$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=T$, $\Delta u|_{t=t_k}=u(t_k^+)-u(t_k^-)$ and $u(t_k^+)=\lim_{h\to 0+}u(t_k+h)$ and $u(t_k^-)=\lim_{h\to 0-}u(t_k+h)$ denote the right and left limits of $u(t)$ at $t=t_k$, respectively. The function $\mathcal{B}:C([0,T];X)\to C([0,T];X)$ is given by $\mathcal{B}u(t)=\int_0^t B(t,s)u(s)ds$ and $\{B(t,s):0\leq s\leq t\leq T\}$ is a set of bounded linear operator on X with $B(\cdot,s)u\in C([s,T];X)$ and $B(t,\cdot)u\in C([0,t];X)$ for all $t,s\in[0,T]$ and for $u\in X$, the function $u_t:[-\tau,0]\to X$, $u_t(s)=u(t+s)$, $s\in[-\tau,0]$, $H:[0,T]\times C([-\tau,0];X)\to X$, $G:[0,T]\times C([-\tau,0];X)\times X\to X$, $g:C([-\tau,\ 0];X)\to C([-\tau,0];X)$ are appropriate functions and $\phi:[-\tau,\ 0]\to X$ is a given continuous function.

In the present work, we study the solvability of equations (1) and establish the existence result of the equation (1)-(3) by using Hausdorff measure of noncompactness β which is an untreated topics in the literature to the best of our knowledge. We divide this paper into three sections as follows: In section Preliminaries, we recall some basic definition, Lemmas and Theorems. We shall prove the existence of a mild solution for system (1)-(3) in section Existence of mild solution. In the last section, we shall discuss an example to illustrate the application of the abstract results.

2. Preliminaries and Assumptions

In this section, we will provide some basic definition of fractional calculus, resolvent operators, solution operator, theorems and lemmas.
Throughout this work, we assume that $(X, \| \cdot \|)$ is a Banach space and $-A$ is the infinitesimal generator of a solution operator $S_q(t),\ t \geq 0$, on Banach space X. Let $C([0,T];X)$, where $0 < T < \infty$ be the Banach space of all continuous functions from $[0,T]$ into X equipped with the norm $\| u(t)\|_C = \sup_{t\in[0,T]} \| u(t)\|_X$ and $C^m([0,T];X)$, denotes the space of all functions u which are m -times continuous differentiable functions from $[0,T]$ into X, is a Banach space with the norm $\| u\|_{C^m} = \sup_{t\in(a,b)} \sum_{k=0}^{m} \| u^k(t)\|_X$ and $L^p((0,T);X)$ denotes the Banach space of all Bochner-measurable functions from $(0,T)$ into X with the norm $\| u\|_{L^p} = (\int_{(0,T)} \| u(s)\|_X^p ds)^{1/p}$.
Assume that $0 \in \rho(A)$ i.e. A is invertible. Then it can be possible to define the positive fractional power A^α for $0 < \alpha \leq 1$ as a closed linear operator with domain $D(A^\alpha) \subset X$. It is easy to see that $D(A^\alpha)$ which is dense in X is a Banach space endowed with the norm $\| x\| = \| A^\alpha x\|$, for $x \in D(A^\alpha)$. Henceforth, we use X_α as notation of $D(A^\alpha)$. Also, we have $X_\kappa \hookrightarrow X_\alpha$ for $0 < \alpha < \kappa$ and the embedding is continuous. Then for each $\alpha > 0$, we define $X_{-\alpha} = (X_\alpha)^*$, which is the dual space of X_α, is a Banach space with the norm $\| x\|_{-\alpha} = \| A^{-\alpha}x\|$.
Definition 1 The Riemann-Liouville fractional integral for the function F of order $q > 0$ is defined by

$$J_t^q F(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} F(s) ds, \tag{4}$$

where $F \in L^1((0,T);X)$.
Definition 2 The Riemann-Liouville fractional derivative of the function F with order q is given by

$$D_t^q F(t) = D_t^m J_t^{m-q} F(t), \tag{5}$$

where $D_t^m = \frac{d^m}{dt^m},\ F \in L^1((0,T);X),\ J_t^{m-q} \in W^{m,1}((0,T);X)$.
Definition 3 The Caputo fractional derivative of the function F is given by

$$^C D_t^q F(t) = \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} F^m(t) dt, \quad m-1 < q < m. \tag{6}$$

where $F \in L^1((0,T);X) \cap C^{m-1}((0,T);X)$ and the following holds

$$J_t^q ({}^C D_t^q F(t)) = F(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} F^k(0). \tag{7}$$

Definition 4 [15] An operator A which is closed and linear, is called sectorial operator if there exist constants $\omega \in \mathbb{R}$, $\theta \in [\pi/2, \pi]$, $M > 0$ such that the following two conditions are satisfied:

(1) $\rho(A) \subset \sum_{(\theta,\ \omega)} = \{\lambda \in C : \lambda \neq \omega,\ |\arg(\lambda - \omega)| < \theta\}$,
(2) $\| R(\lambda, A)\|_{L(X)} \leq \frac{M}{|\lambda - \omega|},\ \ \omega \in \sum_{(\theta,\ \omega)}$,

where $\rho(A)$ is the resolvent set of A .
For more details we refer to [40]. Now, we turn to following fractional order Cauchy problem

$$^cD_t^q u(t) = Au(t), \ \ t > 0; \ \ u(0) = x, \ u^k(0) = 0, \ k = 1, \cdots, m-1, \tag{8}$$

where $q > 0$ and $m = \lceil q \rceil$.
Definition 5 [40] A family $\{S_q(t)\}_{t \geq 0}$ is said to be a solution operator (resolvent operator) for equation (8) if $S_q(t)$ satisfies the following conditions:
(1) $S_q(t)$ is strongly continuous for $t \geq 0$ and $S_q(0) = I$;
(2) $S_q(t)D(A) \subset D(A)$ and $AS_q(t)x = S_q(t)Ax \quad \forall \ x \in D(A)$, $t \geq 0$;
(3) $S_q(t)x$ is a solution of following integral equation

$$u(t) = x + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Au(s) ds, \ t \geq 0. \tag{9}$$

Following [40], the problem (8) is well-posed if and only if it admits a solution operator. Also, the solution operator $S_q(t)$ of (8) is defined as (see [40])

$$\lambda^{q-1}(\lambda^q I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_q(t) x dt, \ \ \text{Re } \lambda > \omega, \ x \in X, \tag{10}$$

where $\omega \geq 0$ and $\{\lambda^q : \text{Re}\lambda > \omega\} \subset \rho(A)$.
Definition 6 [40] The solution operator is called exponentially bounded if there exist constants $\delta \geq 0$ and $M \geq 1$ such that $\| S_q(t)\| \leq Me^{\delta t}, \ t \geq 0$.
An operator A is said to belong to $\mathcal{C}^q(X; M, \delta)$, or $\mathcal{C}^q(M, \delta)$ if the problem (8) has a solution operator $S_q(t)$ satisfying $\|S_q(t)\| \leq Me^{\delta t}, \ 0 \leq t$. Denote $\mathcal{C}^q(\delta) = \bigcup\{\mathcal{C}^q(M, \delta); \ M \geq 1\}$, or $\mathcal{C}^q = \bigcup\{\mathcal{C}^q(\delta; \ \delta \geq 0)\}$ (Bazhlekova, [40]).

To define the mild solution for impulsive differential equation (1)-(3), we suggest the following space $\mathcal{PC}([0,T]; X)$ which contains all the continuous functions $u : [0,T] \to X$ such that $u(t)$ is continuous at $t = t_i$ and $u(t_i^-)$, $u(t_i^+)$ exist for all $i = 1, 2, \cdots, m$. We can verify that the space $\mathcal{PC}([0,T]; X)$ is a Banach space endowed with norm $\|u\|_{\mathcal{PC}} = \sup_{t \in [0,T]}\{u(t)\}$. For a function $u \in \mathcal{PC}([0,T]; X)$, define the function $\widetilde{u_i} \in C([t_i, t_{i+1}], X)$ $(i = 1, \cdots, m)$ such that

$$\widetilde{u_i}(t) = \begin{cases} u(t), & \text{for } \ t \in (t_i, t_{i+1}], \\ u(t_i^+), & \text{for } \ t = t_i. \end{cases} \tag{11}$$

For set $F \subset \mathcal{PC}([0,T]; X)$ and $i \in \{0, 1, \cdots, m\}$, we have $\widetilde{F_i} = \{\widetilde{u_i} : u \in F\}$ and we have following Accoli-Arzelà type criteria.
Lemma 1 [26] A set $F \subset \mathcal{PC}([0,T]; X)$ is relatively compact in $\mathcal{PC}([0,T]; X)$ if and only if each set $\widetilde{F_i}$ is relatively compact in $C([t_i, t_{i+1}], X)$.
We now discuss following facts about the measure of noncompactness and condensing map.
Definition 7 [36] The Hausdorff measure of noncompactness β of the set B in Banach space X is the greatest lower bound of those $\epsilon > 0$ for which the set B has in the space X a finite ϵ - net i.e.

$$\beta(B) = \inf\{\epsilon > 0 : \ B \text{ has a finite } \epsilon - \text{net in X}\}, \tag{12}$$

for each bounded subset B in a Banach space X .
Next, we recall the some basic properties about the Hausdorff measure of noncompactness β .

Lemma 2 [36] Let X be a real Banach space and E, F be bounded subset of X. Then, we have the following results:

(1) $\beta(E) = 0$ iff E is relatively compact ;
(2) $\beta(E) = \beta(convE) = \beta(\overline{E})$, where $conv(E)$ and $\overline{E}$ denotes the convex hull and closure of E respectively;
(3) If $E \subset F$, then $\beta(E) \leq \beta(F)$;
(4) $\beta(E+F) \leq \beta(E) + \beta(F)$, where $E+F = \{x+y:\ x \in E,\ y \in F\}$;
(5) $\beta(E \cup F) \leq \max\{\beta(E),\ \beta(F)\}$;
(6) $\beta(\kappa E) \leq |\kappa|\beta(E)$ for any $\kappa \in R$;
(7) If the map $\mathcal{Q} : D(\mathcal{Q}) \subset X \to Y$ is Lipschitz continuous with a Lipschitz constant μ. Then $\beta_Y(\mathcal{Q}E) \leq \mu\beta(E)$ for every bounded set $E \subset D(\mathcal{Q})$, where Y is a Banach space.

For more study on the measure of noncompactness, we refer to books [33], [36], [35].

Definition 8 [36] A continuous map $\mathcal{Q} : D \subseteq X \to X$ is called a β_X -contraction if there exists a constant $0 < \kappa < 1$ such that $\beta_X(\mathcal{Q}(F)) \leq \kappa\beta_X(F)$, for any bounded closed subset $F \subseteq D$.

Lemma 3 (Darbo-Sadovskii)[36] Let $D \subset X$ be closed, bounded and convex. Assume that the continuous map $\mathcal{Q} : D \to D$ is a β - contraction. Then, there exists at least one fixed point of the map $\mathcal{Q}$ in D.

In this paper, we consider that β denotes the Hausdorff's measure of noncompactness in X, β_C denotes the Hausdorff's measure of noncompactness in $C([0,T];X)$ and $\beta_{\mathcal{PC}}$ denotes the Hausdorff's measure of noncompactness in $\mathcal{PC}([0,T];X)$.

Lemma 4 [36] If $F \subseteq C([0,T];X)$ is bounded, then $\beta(F(t)) \leq \beta(F)$ for all $t \in [0,T]$, where $F(t) = \{x(t); x \in F\} \subseteq X$. Furthermore, if F is equicontinuous on $[0,T]$, then $\beta(F(t))$ is continuous on $[0,T]$ and $\beta_C(F) = \sup\{\beta(F(\tau)), \tau \in [0,T]\}$.

Lemma 5 [36] If $F \subset C([0,T];X)$ is bounded and equicontinuous. Then $\beta(F(t))$ is continuous and

$$\beta(\int_0^t F(\tau)d\tau) \leq \int_0^t \beta(F(\tau))d\tau, \tag{13}$$

for all $t \in [0,T]$, where $\int_0^t F(\tau)d\tau = \{\int_0^t x(\tau)d\tau, x \in F\}$.

Lemma 6 [38] If $F \subseteq \mathcal{PC}([0,T];X)$ is bounded, then $\beta(F(t)) \leq \beta_{\mathcal{PC}}(F)$ for all $t \in [0,T]$. Furthermore, suppose the following conditions are satisfied;

(1) F is equicontinuous on $J_0 = [0,t_1]$ and each $J_i = (t_i, t_{i+1}]$, $i = 1, \cdots, N$,
(2) F is equicontinuous at $t = t_i^+$, $i = 1, \cdots, N$.
Then $\sup\limits_{t \in [0,T]} \beta(F(t)) = \beta_{\mathcal{PC}}(F)$.
(3) If $F \subset \mathcal{PC}([0,T];X)$ is bounded and piecewise equicontinuous, then $\beta(F(t))$ is piecewise continuous for $t \in [0,T]$ and

$$\beta(\int_0^t F(\tau)d\tau) \leq \int_0^t \beta(F(\tau))d\tau, \tag{14}$$

for all $t \in [0,T]$, where $\int_0^t F(\tau)d\tau = \{\int_0^t x(\tau)d\tau, x \in F\}$.

Definition 9 A piece-wise continuous function $u : [-\tau, T] \to X$ is said to be a mild solution for the system (1)-(3) if $u(\cdot)$ satisfies the following fractional integral equation

$$u(t) = \begin{cases} \phi(t) + gu(t), \quad t \in [-\tau, 0], \\ S_q(t)[\phi(0) + gu(0) + H(0, \phi + g(u))] - H(t, u_t) \\ + \int_0^t S_q(t-s)G(s, u_s, \mathcal{B}u(s))ds, \qquad t \in [0, t_1], \\ S_q(t)[\phi(0) + gu(0) + H(0, \phi + g(u))] - H(t, u_t) + S_q(t-t_1)I_1(u(t_1^-)) \\ +S_q(t-t_1)[H(t_1, u_{t_1} + I_1(u_{t_1^-})) - H(t_1, u_{t_1})] \\ + \int_0^t S_q(t-s)G(s, u_s, \mathcal{B}u(s))ds, \qquad t \in (t_1, t_2], \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ S_q(t)[\phi(0) + gu(0) + H(0, \phi + g(u))] - H(t, u_t) + \sum_{i=1}^m S_q(t-t_i)I_i(u(t_i^-)) \\ + \sum_{i=1}^m S_q(t-t_i)[H(t_i, u_{t_i} + I_i(u_{t_i^-})) - H(t_i, u_{t_i})] \\ + \int_0^t S_q(t-s)G(s, u_s, \mathcal{B}u(s))ds, \qquad t \in (t_m, T], \end{cases} \tag{15}$$

To establish the our required result, we made following assumptions :

(A0) The solution operator $\{S_q(t)\}_{t\geq 0}$ is analytic i.e. the map $t \mapsto S_q(t)$ is continuous from $[0,T]$ to $\mathcal{L}(X)$ endowed with the uniform operator norm $\| \cdot \|_{\mathcal{L}(X)}$.
Without loss of generality, we may have that there exist a positive constant M such that $\| S_q(t)\| \leq M$, for $t \geq 0$.

(A1) The function $G : [0,T] \times C([-\tau, 0]; X) \times X \to X$ satisfies the following Carathèodary condition i.e.,
(a) The function $G(\cdot, u, v) : [0,T] \to X$ is strongly measurable for every $u \in C([-\tau,0];X)$ and $v \in X$.
(b) The function $G(t, \cdot, \cdot) : C([-\tau, 0]; X) \times X \to X$ is continuous for each $t \in [0,T]$.
(c) There exist constant functions $m_i(\cdot) \in L^1([0,b], \mathbb{R}_+)$ $(i = 1,2)$ such that

$$\| G(t,x,y)\| \leq m_1(t)\| x\|_{[-\tau,0]} + m_2(t)\| y\|, \tag{16}$$

for almost all $t \in [0,T]$ and $(x,y) \in C([-\tau,0];X) \times X$.

(A2) There exist functions $\eta_i \in L^1([0,T];\mathbb{R}_+)$ $(i = 1,2,)$ such that

$$\beta(G(t, D_1, D_2)) \leq \eta_1(t) \sup_{\theta\in[-\tau,0]} \beta(D_1(\theta)) + \eta_2(t)\beta(D_2), \quad a.e.\ t \in [0,T], \tag{17}$$

for any bounded sets $D_1 \subset C([-\tau,0];X)$ and $D_2 \subset X$.

(A3) There exists a positive constant $0 < \alpha < 1$ such that the nonlinear function $H : [0,T] \times C([-\tau,0];X) \to X$ satisfies the following condition

$$\| A^\alpha H(t,u) - A^\alpha H(t,v)\| \leq L_H\| u - v\|_{[-\tau,0]}, \quad u,v \in C([-\tau,0];X),\ \forall\, t \in [0,T], \tag{18}$$

i.e., H is Lipschitz continuous function and $L_H > 0$ is a constant. Also H satisfies the following conditions

$$\| A^\alpha H(t,u)\| \leq c_1(\| u\|_{[-\tau,0]}) + c_2, \quad u \in C([-\tau,0];X),\ t \in [0,T], \tag{19}$$

where $c_1,\ c_2$ are positive constants.

(A4) The function $g : C([-\tau, 0]; X) \to C([-\tau, 0]; X)$ is Lipschitz continuous in the following sense: there exists a constant $L_g > 0$ such that

$$\| g(u) - g(v)\|_{[-\tau,0]} \leq L_g \| u - v\|_{[0,T]}, \tag{20}$$

for all $u, v \in C([-\tau, 0]; X)$ and g is uniformly bounded i.e., there exists a constant $N > 0$ such that

$$\| g(u)\|_{[-\tau,0]} \leq N, \tag{21}$$

for any $u \in C([-\tau, 0]; X)$.

(A5) The function $I_k : X \to X, (k = 1, \cdots, m)$ are continuous functions and there is a constant $L_I > 0$ such that

$$\| I_k(u) - I_k(v)\| \leq L_I \| u - v\|, \tag{22}$$

and

$$\| I_k(u)\| \leq L, \tag{23}$$

for all $u, v \in X$. Where $L > 0$ is constant.

(A6)

$$\begin{aligned}[ML_g(1 + L_H \| A^{-\alpha}\|) + L_H \| A^{-\alpha}\|) + mML_H(2 + L_I) + ML_I] \\ +M(\| \eta_1\|_{L^1} + B^* \| \eta_2\|_{L^1}) < 1,\end{aligned} \tag{24}$$

where $B^* = \sup_{t\in[0,T]} \int_0^t \| B(t, s)\|ds$.

3. Existence Results

In this section, we discuss the existence of a mild solution for the system (1)-(3).

Theorem 1 Assume that the assumptions $(A0) - (A6)$ are satisfied, then there exists a mild solution for system (1)-(3).

Proof We define the operator $\mathcal{Q} : \mathcal{PC}([-\tau, T]; X) \to \mathcal{PC}([-\tau, T]; X)$ as

$$\mathcal{Q}u(t) = \begin{cases} \phi(t) + gu(t), \quad t \in [-\tau, 0], \\ S_q(t)[\phi(0) + gu(0) + H(0, \phi + g(u))] - H(t, u_t) \\ + \int_0^t S_q(t - s)G(s, u_s, \mathcal{B}u(s))ds, \qquad t \in [0, t_1], \\ S_q(t)[\phi(0) + gu(0) + H(0, \phi + g(u))] - H(t, u_t) + S_q(t - t_1)I_1(u(t_1^-)) \\ +S_q(t - t_1)[H(t_1, u_{t_1} + I_1(u_{t_1^-})) - H(t_1, u_{t_1})] \\ + \int_0^t S_q(t - s)G(s, u_s, \mathcal{B}u(s))ds, \qquad t \in (t_1, t_2], \\ \vdots \qquad \vdots \qquad \vdots \\ S_q(t)[\phi(0) + gu(0) + H(0, \phi + g(u))] - H(t, u_t) + \sum_{i=1}^{m} S_q(t - t_i)I_i(u(t_i^-)) \\ + \sum_{i=1}^{m} S_q(t - t_i)[H(t_i, u_{t_i} + I_i(u_{t_i^-})) - H(t_i, u_{t_i})] \\ + \int_0^t S_q(t - s)G(s, u_s, \mathcal{B}u(s))ds, \qquad t \in (t_m, T], \end{cases} \tag{25}$$

It is easy to verify that $\mathcal{Q}$ is well defined. Firstly we show that $\mathcal{Q}$ is continuous on $\mathcal{PC}([-\tau, T]; X)$. It is obvious that $\mathcal{Q}$ is continuous on $[-\tau, 0]$ by the continuity of ϕ and g . For proving the continuity, let $\{u_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{PC}([-\tau,T];X)$ such that $\lim_{n\to\infty} u_n(t)=u(t)$ in $\mathcal{PC}([-\tau,T];X)$. Since G and H are continuous, therefore we get

$$\begin{aligned} G(t,(u_n)_t,\mathcal{B}u_n(t)) &\to G(t,u_t,\mathcal{B}u(t)), &(26)\\ H(t,(u_n)_t) &\to H(t,u_t), &(27)\end{aligned}$$

as $n\to\infty$. For $t\in[0,t_1]$, we have

$$\begin{aligned}\| \mathcal{Q}u_n(t)-\mathcal{Q}u(t)\| &\leq \| S_q(t)[(gu_n)(0)-(gu)(0)]\| + \| H(t,(u_n)_t)-H(t,u_t)\| \\ &\quad + \int_0^t S_q(t-s)\| G(s,(u_n)_s,\mathcal{B}u_n(s))-G(s,u_s,\mathcal{B}u(s))\| ds,\end{aligned}$$

by Lebesgue's dominate convergence theorem and the usual technique involving the hypothesis $(A1)$, $(A3)$ and $(A4)$, it implies that $\mathcal{Q}$ is continuous. Similarly for $t\in(t_m,T]$,
$\| \mathcal{Q}u_n(t)-\mathcal{Q}u(t)\|$

$$\begin{aligned} &\leq \| S_q(t)[gu_n(0)-gu(0)]\| + \| H(t,(u_n)_t)-H(t,u_t)\| \\ &\quad + \sum_{i=1}^m \| S_q(t-t_i)[I_i(u_n(t_i))-I_i(u(t_i))]\| \\ &\quad + \sum_{i=1}^m \| S_q(t-t_i)[H(t_i,(u_n)_{t_i}+I_i((u_n)_{t_i^-}))-H(t_i,u_{t_i}+I_i(u_{t_i^-}))]\| \\ &\quad + \sum_{i=1}^m \| S_q(t-t_i)[H(t_i,u_n(t_i))-H(t_i,u(t_i))]\| \\ &\quad + \int_0^t \| S_q(t)[G(s,(u_n)_s,\mathcal{B}u_n(s))-G(s,u_s,\mathcal{B}u(s))]\| ds, \qquad (28)\end{aligned}$$

By the assumption $(A1),(A3),(A4)-(A5)$ and Lebesgue's dominate convergence theorem, we have that $\mathcal{Q}$ is continuous on $(t_m,T]$. Hence, $\mathcal{Q}$ is continuous on $[-\tau,T]$.

Secondly we show that $\mathcal{Q}(B_R)\subset B_R$, where $B_R=B_R(\mathcal{PC}([-\tau,T];X))=\{u\in\mathcal{PC}([-\tau,T];X):\| u\|\leq R\}\subset\mathcal{PC}([-\tau,T];X)$ is a closed and convex ball with center at the origin and radius R and R is a positive integer to be defined later. For $u\in B_R$ and $t\in[-\tau,0]$, we obtain

$$\begin{aligned}\| \mathcal{Q}u(t)\| &\leq \| \phi(t)\|_{[-\tau,0]} + \| gu(t)\|_{[-\tau,0]}, \\ &\leq \| \phi\|_{[-\tau,0]} + N, = R_{-1} \qquad (29)\end{aligned}$$

For $t \in [0, t_1]$, we get
$\| \mathcal{Q}u(t)\|$

$$\begin{aligned}
&\leq \| S_q(t)[\phi(0) + gu(0)]\| + \| S_q(t)(H(0, \phi + g(u)))\| + \| H(t, u_t)\| \\
&\quad + \int_0^t \| S_q(t-s)G(s, u_s, \mathcal{B}u(s))\| ds, \\
&\leq M\| \phi(0) + gu(0)\| + M\| A^{-\alpha}\|[c_1(\|\phi\|_{[-\tau,0]} + N) + c_2] + \| A^{-\alpha}\|(c_1 R + c_2) \\
&\quad + MR \int_0^t (m_1(s) + m_2(s)) ds, \\
&\leq M\| \phi(0) + gu(0)\| + M\| A^{-\alpha}\|[c_1(\|\phi\|_{[-\tau,0]} + N) + c_2] + \| A^{-\alpha}\|(c_1 R + c_2) \\
&\quad + MR[\| m_1\|_{L^1} + \| m_2\|_{L^1}] = R_0, \qquad (30)
\end{aligned}$$

For $t \in (t_1, t_2]$,
$\| \mathcal{Q}u(t)\|$

$$\begin{aligned}
&\leq \| S_q(t)[\phi(0) + gu(0)]\| + \| S_q(t)(H(0, \phi + g(u)))\| + \| H(t, u_t)\| \\
&\quad + \|S_q(t-t_1)I_1(u(t_1^-))\| + \| S_q(t-t_1)[H(t_1, u_{t_1} + I_1(u_{t_1^-})) - H(t_1, u_{t_1})]\| \\
&\quad + \int_0^t \| S_q(t-s)G(s, u_s, \mathcal{B}u(s))\| ds, \\
&\leq M\| \phi(0) + gu(0)\| + M\| A^{-\alpha}\|[c_1(\|\phi\|_{[-\tau,0]} + N) + c_2] + \| A^{-\alpha}\|(c_1 R + c_2) \\
&\quad + ML(1 + L_H\| A^{-\alpha}\|) + MR \int_0^t (m_1(s) + m_2(s)) ds, \\
&\leq M\| \phi(0) + gu(0)\| + M\| A^{-\alpha}\|[c_1(\|\phi\|_{[-\tau,0]} + N) + c_2] + \| A^{-\alpha}\|(c_1 R + c_2) \\
&\quad + ML(1 + L_H\| A^{-\alpha}\|) + MR[\| m_1\|_{L^1} + \| m_2\|_{L^1}] = R_1, \qquad (31)
\end{aligned}$$

For $t \in (t_m, T]$, we get
$\| \mathcal{Q}u(t)\|$

$$\begin{aligned}
&\leq \| S_q(t)[\phi(0) + gu(0)]\| + \| S_q(t)(H(0, \phi + g(u)))\| + \| H(t, u_t)\| \\
&\quad + \|S_q(t-t_1)I_1(u(t_1^-))\| + \| S_q(t-t_1)[H(t_1, u_{t_1} + I_1(u_{t_1^-})) - H(t_1, u_{t_1})]\| \\
&\quad + \int_0^t \| S_q(t-s)G(s, u_s, \mathcal{B}u(s))\| ds, \\
&\leq M\| \phi(0) + gu(0)\| + M\| A^{-\alpha}\|[c_1(\|\phi\|_{[-\tau,0]} + N) + c_2] + \| A^{-\alpha}\|(c_1 R + c_2) \\
&\quad + ML(1 + L_H\| A^{-\alpha}\|) + MR \int_0^t (m_1(s) + m_2(s)) ds, \\
&\leq M\| \phi(0) + gu(0)\| + M\| A^{-\alpha}\|[c_1(\|\phi\|_{[-\tau,0]} + N) + c_2] + \| A^{-\alpha}\|(c_1 R + c_2) \\
&\quad + mML(1 + L_H\| A^{-\alpha}\|) + MR[\| m_1\|_{L^1} + \| m_2\|_{L^1}] = R_m, \qquad (32)
\end{aligned}$$

choose $R = \max\{R_{-1}, R_0, R_1, \cdots, R_m\}$ such that $\mathcal{Q}(B_R) \subset B_R$. Now, we show that $\mathcal{Q}(B_R)$ is equicontinuous on $J_0 = [0, t_1]$, $J_i = (t_i, t_{i+1}]$ and also equicontinuous at $t = t_i^+$, $i = 1, \cdots, m$. To this end, take $u \in B_r$ and $h > 0$ such that $0 \leq t < t + h \leq t_1$ and have that

$$\begin{aligned}\| \mathcal{Q}u(t+h)-\mathcal{Q}u(t)\| \leq & \| [S_q(t+h)-S_q(t)](\phi(0)+gu(0)+H(0,\phi+g(u)))\| \\ & +\| H(t+h,u_{t+h})-H(t,u_t)\| \\ & +\int_t^{t+h}\| S_q(t+h-s)G(s,u_s,\mathcal{B}u(s))\|ds, \\ & +\int_0^t\| [S_q(t+h-s)-S_q(t-s)]G(s,u_s,\mathcal{B}u(s))\|ds,\end{aligned}$$

Since $S_q(t)$ is strongly continuous, the continuity of $t\mapsto \| S_q(t)\|$ allows us to deduce that

$$\lim_{h\to 0}\| S_q(t+h-s)-S_q(t-s)\| =0, \tag{33}$$

Thus, by the assumption $(A3)$ and above inequality, we obtain that

$$\| \mathcal{Q}u(t+h)-\mathcal{Q}u(t)\| \to 0.$$

For $t_m<t<t+h\leq T$,
$\| \mathcal{Q}u(t+h)-\mathcal{Q}u(t)\|$

$$\begin{aligned}\leq & \| [S_q(t+h)-S_q(t)](\phi(0)+gu(0)+H(0,\phi+g(u)))\| \\ & +\| H(t+h,u_{t+h})-H(t,u_t)\| \\ & +\sum_{i=1}^{m}\| [S_q(t+h-t_i)-S_q(t-t_i)]I_i(u(t_i^-))\| \\ & +\sum_{i=1}^{m}\| [S_q(t+h-t_i)-S_q(t-t_i)][H(t_i,u_{t_i}+I_i(u_{t_i^-}))-H(t_i,u_{t_i})]\| \\ & +\int_0^t\| [S_q(t+h-s)-S_q(t-s)]G(s,u_s,\mathcal{B}u(s))\|ds \\ & +\int_t^{t+h}\| S_q(t+h-s)G(s,u_s,\mathcal{B}u(s))\|ds,\end{aligned} \tag{34}$$

by the strongly continuity and assumption $(A3)$, we obtain that $\| \mathcal{Q}u(t+h)-\mathcal{Q}u(t)\| \to 0$ as $h\to 0$ which implies that $\mathcal{Q}(B_R)$ is equicontinuous on $(t_m,T]$. Therefore $\mathcal{Q}$ is equicontinuous on $[0,T]$. Since g is equicontinuous on $[-\tau,0]$. Hence Q is equicontinuous on $[-\tau,T]$.
Next, we show that $\mathcal{Q}$ is β -contraction. We introduce the decomposition of $\mathcal{Q}=\sum_{i=1}^{2}Q_i$ such that

$$Q_1u(t) = \begin{cases}\phi(t)+gu(t), \quad t\in[-\tau,0], \\ S_q(t)[\phi(0)+gu(0)+H(0,\phi+g(u))]-H(t,u_t)\ t\in[0,t_1], \\ S_q(t)[\phi(0)+gu(0)+H(0,\phi+g(u))] \\ -H(t,u_t)+\sum_{i=1}^{k}S_q(t-t_i)I_i(u(t_i^-)) \\ +\sum_{i=1}^{k}S_q(t-t_i)[H(t_i,u_{t_i}+I_i(u_{t_i^-}))-H(t_i,u_{t_i})],\ t\in(t_k,t_{k+1}],\end{cases} \tag{35}$$

$$Q_2u(t) = \begin{cases}0, \quad t\in[-\tau,0], \\ \int_0^t S_q(t-s)G(s,u_s,\mathcal{B}u(s))ds, \quad t\in[0,T],\ t\neq t_k,\ k=1,\cdots,m.\end{cases} \tag{36}$$

Firstly, we show that Q_1 is Lipschitzian with Lipschitz constant K . For $t \in [-\tau, 0]$ and $u,\ v \in B_R$ and by the assumptions $(A4)$, we obtain

$$\begin{aligned} \| Q_1u(t) - Q_1v(t)\| &\leq \| g(u)(t) - g(v)(t)\|, \\ &\leq \| g(u) - g(v)\|_{[-\tau,0]}, \\ &\leq L_g\| u - v\|_{[-\tau,T]}, \end{aligned} \tag{37}$$

For $t \in [0, t_1]$, we get

$$\begin{aligned} &\| Q_1u(t) - Q_1v(t)\| \\ &\quad\leq M\| [gu(0) - gv(0)]\| + M\| H(0, \phi + g(u)) - H(0, \phi + h(v))\| \\ &\qquad + \| H(t, u_t) - H(t, v_t)\|, \\ &\quad\leq ML_g\| u - v\| + ML_H L_g\| A^{-\alpha}\|\| u - v\|_{[-\tau,T]} + L_H\| A^{-\alpha}\|\| u - v\|_{[-\tau,T]}, \\ &\quad\leq [ML_g(1 + L_H\| A^{-\alpha}\|) + L_H\| A^{-\alpha}\|]\| u - v\|_{[-\tau,T]}, \end{aligned} \tag{38}$$

For $t \in (t_1, t_2]$, we have
$\| Q_1u(t) - Q_1v(t)\|$

$$\begin{aligned} &\leq M\| [gu(0) - gv(0)]\| + M\| H(0, \phi + g(u)) - H(0, \phi + g(v))\| \\ &\quad + M\|I_1(u(t_1^-)) - I_1(v(t_1^-))\| + M[\| H(t_1, u_{t_1} + I_1(u_{t_1^-})) - H(t_1, v_{t_1} + I_1(v_{t_1^-}))\| \\ &\quad + \|H(t_1, u_{t_1}) - H(t_1, v_{t_1})\|] + \| H(t, u_t) - H(t, v_t)\|, \\ &\leq ML_g(1 + L_H\| A^{-\alpha}\|) + L_H\| A^{-\alpha}\|\| u - v\|_{[-\tau,T]} + ML_I\| u - v\|_{[-\tau,T]} \\ &\quad + ML_H(2 + L_I)\| u - v\|_{[-\tau,T]}, \\ &\leq [ML_g(1 + L_H\| A^{-\alpha}\|) + L_H\| A^{-\alpha}\| + ML_I + ML_H(2 + L_I)]\| u - v\|_{[-\tau,T]}, \end{aligned} \tag{39}$$

and for $t \in (t_m, T]$, we get
$\| Q_1u(t) - Q_1v(t)\|$

$$\begin{aligned} &\leq [ML_g(1 + L_H\| A^{-\alpha}\|) + L_H\| A^{-\alpha}\|) + mML_H(2 + L_I) + mML_I] \\ &\quad \times\| u - v\|_{[-\tau,T]}, \end{aligned} \tag{40}$$

Thus for all $t \in [-\tau, T]$, we conclude that
$\| Q_1u(t) - Q_1v(t)\|$

$$\begin{aligned} &\leq [ML_g(1 + L_H\| A^{-\alpha}\|) + L_H\| A^{-\alpha}\|) + mML_H(2 + L_I) + mML_I] \\ &\quad \times\| u - v\|_{[-\tau,T]}, \end{aligned} \tag{41}$$

It follows that

$$\| Q_1u(t) - Q_1v(t)\| \leq K\| u - v\|_{[-\tau,T]}, \tag{42}$$

where $K = [ML_g(1 + L_H\| A^{-\alpha}\|) + L_H\| A^{-\alpha}\|) + mML_H(2 + L_I) + mML_I]$.
On the other hand, since $S_q(t)$, for $t \geq 0$ is an equicontinuous solution operator which is generated by $-A$. By Lemma 2, 4, 6 and $(A1)(c)$, we obtain that for

any bounded set $W \subset \mathcal{PC}([-\tau,T];X)$,

$$\begin{aligned}
\beta_{\mathcal{PC}}(Q_2W) &= \sup_{t\in[0,T]} \beta(Q_2W(t)),\\
&\leq \sup_{t\in[0,T]} \beta(\int_0^t S_q(t-s)G(s,W_s,\mathcal{B}W(s))ds),\\
&\leq \sup_{t\in[0,T]} \int_0^t \beta(S_q(t-s)G(s,W_s,\mathcal{B}W(s)))ds,\\
&\leq M \sup_{t\in[0,T]} \int_0^t [\eta_1(s)(\sup_{\theta\in[-\tau,T]} \beta(W(s+\theta))) + \eta_2(s)B^*\beta(W(s))]ds,\\
&\leq M\beta_{\mathcal{PC}}(W)\int_0^t [\eta_1(s)+B^*\eta_2(s)]ds,\\
&\leq M(\| \eta_1\|_{L^1} + B^*\| \eta_2\|_{L^1})\beta_{\mathcal{PC}}(W) \qquad (43)
\end{aligned}$$

where $W(t)=\{u(t):u\in W\}\subset \mathcal{PC}$ and $W_t=\{u_t:u\in W\}\subset \mathcal{PC}([-\tau,0]:X)$ and for $t\in[-\tau,0]$, we have $\beta_{\mathcal{PC}}(Q_2W)=0$.
Thus, from Lemma 2 we get that for any bounded set $W\subset \mathcal{PC}([-\tau,T];X)$.

$$\begin{aligned}
\beta_{\mathcal{PC}}(QW) &\leq \beta_{\mathcal{PC}}(Q_1W)+\beta_{\mathcal{PC}}(Q_2W),\\
&\leq [K+M(\| \eta_1\|_{L^1}+B^*\| \eta_2\|_{L^1})]\beta_{\mathcal{PC}}(W). \qquad (44)
\end{aligned}$$

From the assumption $(A6)$, we have that $[ML_g(1+L_H\| A^{-\alpha}\|)+L_H\| A^{-\alpha}\|)+ mML_H(2+L_I)+ML_I+M(\| \eta_1\|_{L^1}+B^*\| \eta_2\|_{L^1})]<1$. Hence, it implies that $\mathcal{Q}$ is a contraction i.e., there exists a fixed point $u\in X$ by Darbo-Sadovskii's fixed point theorem. The fixed point of the map $\mathcal{Q}$ is a mild solution for the system (1)-(3). This complete the proof of the theorem.
If we replace the conditions $(A1)(c)$ and $(A2)$ of Theorem 3.1 by

(A1)(c') There is an integrable function $m_G:[0,T]\to\mathbb{R}_+$ and a continuous nondecreasing function $\Omega:[0,\infty)\to[0,\infty)$ such that

$$\| G(t,u,v)\| \leq m_G(t)\Omega(\| u\|_{[-\tau,0]}+\| v\|), \qquad (45)$$

for all $t\in[0,T]$ and $(u,v)\in C([-\tau,0];X)\times X$.

(A2') There exist integrable functions $\eta_1,\ \eta_2:[0,T]\to[0,\infty)$ such that for any bounded subset $D_1\subset C([-\tau,0];X)$, $D_2\subset X$

$$\beta(S_q(t)G(t,D_1,D_2))\leq \eta_1(t)(\sup_{\theta\in[-\tau,0]}\beta(D_1(\theta)))+\eta_2(t)\beta(D_2)). \qquad (46)$$

Then, we can have the following result:
Theorem 2 Suppose that the assumptions $(A0),(A1),(A2'),(A3)-(A5)$ are satisfied and

$$\begin{aligned}
&[ML_g(1+L_H\| A^{-\alpha}\|)+L_H\| A^{-\alpha}\|)+mML_H(2+L_I)\\
&\qquad +ML_I+\| \eta_1\|_{L^1}+B^*\| \eta_2\|_{L^1})]<1. \qquad (47)
\end{aligned}$$

Then, nonlocal impulsive fractional integro-differential equation has at least one mild solution.
Theorem 3 If assumptions $(A0)-(A1)[(a),(b),(c')],(A2'),(A3)-(A5)$ holds and

$$\begin{aligned}
&[ML_g(1+L_H\| A^{-\alpha}\|)+L_H\| A^{-\alpha}\|)+mML_H(2+L_I)+ML_I\\
&\qquad +\|\eta_1\|_{L^1}+B^*\|\eta_2\|_{L^1}]<1, \qquad (48)
\end{aligned}$$

and

$$\| A^{-\alpha}\|c_1 + M \lim \inf_{k\to\infty} \frac{\Omega((1+B^*)r)}{r}\int_0^T m_G(s)ds < 1. \tag{49}$$

Then, there exists a mild solution for system (1)-(3).

4. Application

In this section, we consider the following fractional integro-differential equation to illustrate the application of the theory

$$\begin{aligned} D_t^q[u(t,x) &+& e^{-t}\int_{-r}^0 \frac{a_1(\theta)}{1+|u(t+\theta,x)|}d\theta] \\ &=& \frac{\partial^2}{\partial x^2}[u(t,x) + e^{-t}\int_{-r}^0 \frac{a_1(\theta)}{1+|u(t+\theta,x)|}d\theta] \\ && +J_t^{1-q}[\int_{-r}^0 a_2(\theta)t^{2/3}\sin(\frac{|u(t+\theta,x)|}{t}) \\ && +\int_0^t B(t,s)s^l \sin|u(s,x)|ds],\ x\in[0,1],\ t\in[0,1],\ t\neq t_n, \quad (50) \\ u(t,0) &=& e^{-t}\int_{-r}^0 \frac{a_1(\theta)}{1+|u(t+\theta,0)|}d\theta, \quad (51) \\ u(t,1) &=& e^{-t}\int_{-r}^0 \frac{a_1(\theta)}{1+|u(t+\theta,1)|}d\theta, \quad (52) \\ u(\theta,x) &=& u_0(\theta,x) + \frac{e^{\mu\theta}}{l^2}\times\frac{|u(\theta,x)|}{1+|u(\theta,x)|},\ -r\le\theta\le 0, \quad (53) \\ \Delta u(t_i,x) &=& \int_0^1 p_i(x,y)dy\cos^2 u(t_i,x)ds,\ \ x\in[0,1],\ 1\le i\le n, \quad (54) \end{aligned}$$

where ${}^cD_t^q$ denotes the Caputo's fractional derivative of order q, $0<q<1$, $l\in\mathbb{N}$, $r>0$, $0<t_1<t_2<\cdots,<t_n<T$ are prefixed numbers and $\phi\in C([-r,0];X)$, $u_0:[-r,0]\to[0,1]$ is continuous functions and $a_1,\ a_2:[-r,0]\to\mathbb{R}$, $p_i(x,y)\in L^2([0,1]\times[0,1];\mathbb{R})$ satisfy the following conditions

(i) a_1 is a continuous function such that

$$\int_{-r}^0 |a_1(\theta)|d\theta < 1, \tag{55}$$

(ii) a_2 is a continuous function such that

$$\int_{-r}^0 |a_2(\theta)|d\theta < \infty. \tag{56}$$

(iii) For $i=1,\cdots,n$, the function $p_i(x,y),\ y\in[0,1]$ is measurable function such that

$$(\int_0^1(\int_0^t p_i(x,y)dy)^2dx)^{1/2}\le N_p. \tag{57}$$

Consider $X=L^2([0,1];\mathbb{R})$. We define an operator $A:D(A)\subset X\to X$ by $Av=v''$ with the domain $D(A)=H^2([0,1])\cap H_0^1([0,1])$. Then, $-A$ generates an analytic semigroup of bounded linear operators $\{T(t)\}_{t\ge0}$ on X. By the subordination principle of solution operator [Thm, 3.1 in [40]], we get that $-A$

generates a solution operator $\{S_q(t)\}_{t\geq 0}$. Since $S_q(t)$ is strongly continuous on $[0,\infty)$, therefore from the uniformly bounded theorem, we have that there exists a constant $M>0$ such that $\| S_q(t)\| \leq M$ for $t\in[0,T]$.
Then we can reformulate the equation (50) as the equation (1). If we set, for $x\in[0,1]$ and $\varphi\in C([-r,0];X)$

$$\begin{aligned} w(t)(x) &= u(t,x),\\ \phi(\theta)(x) &= u_0(\theta,x),\ \ \theta\in[-r,0],\\ g(t,\varphi)(x) &= e^{-t}\int_{-r}^{0}\frac{a_1(\theta)}{1+|\varphi(\theta)x|}d\theta,\\ h(\varphi(\theta))(x) &= \frac{e^{\mu\theta}}{l^2}\cdot\frac{|\varphi(\theta)(x)|}{1+\varphi(\theta)(x)},\\ B &= B(t-s),\\ f(t,\varphi,\mathcal{B}w(t))(x) &= \int_{-r}^{0}a_2(\theta)t^{2/3}\cdot\sin(\frac{|\varphi(\theta)(x)|}{t})d\theta+\int_0^t B(t,s)s^l\sin|w(s)x|ds. \end{aligned} \tag{58}$$

Further, for $t\in(0,1]$, we have that

$$\begin{aligned} \| f(t,\varphi,\mathcal{B}w(t))\| &\leq t^{-3/2}\|\varphi\|_{[-r,0]}\int_{-r}^{0}|a_2(\theta)|d\theta+B^*t^l\| w(t)\|,\\ &\leq m_1(t)\|\varphi\|_{[-r,0]}+m_2(t)\| w(t)\|, \end{aligned} \tag{59}$$

where $m_1(t)=t^{-3/2}\int_{-r}^{0}|a_2(\theta)|d\theta$ and $m_2(t)=B^*\,t^l,\ B^*=\sup_{t\in[0,T]}\int_0^t\|B(t,s)\|ds$.
Next, for $w_1,w_2\in X$ and $\varphi_1,\varphi_2\in C([-r,0];X)$, we obtain

$$\begin{aligned} \| f(t,\varphi_1,\mathcal{B}w_1(t))(x) &- f(t,\varphi_2,\mathcal{B}w_2(t))(x)\|\\ &\leq t^{-3/2}\int_{-r}^{0}|a_2(\theta)|\|\varphi_1(\theta)(x)-\varphi(\theta)(x)\|d\theta\\ &\quad +B^*t^l\| w_1(t)-w_2(t)\|, \end{aligned} \tag{60}$$

Thus, for any bounded sets $D_1\subset C([-r,0];X)$, $D_2\subset X$, we get

$$\begin{aligned} \beta(f(t,D_1,D_2)) &\leq t^{-3/2}\int_{-r}^{0}|a_2(\theta)|\beta(D_1(\theta))d\theta+B^*t^l\beta(D_1),\\ &\leq t^{-3/2}\sup_{\theta\in[-r,0]}\beta(D_1(\theta))\int_{-r}^{0}|a_2(\theta)|d\theta+B^*t^l\beta(D_1),\\ &\leq \eta_1(t)(\sup_{\theta\in[-r,0]}\beta(D_1(\theta)))+\eta_2(t)\beta(D_2), \end{aligned} \tag{61}$$

where $\eta_1,\ \eta_2$ are defined as $\eta_1(t)=t^{-3/2}\int_{-r}^{0}|a_2(\theta)|d\theta,\ \eta_2(t)=B^*t^l$.
Now, we can see that for $\varphi_1,\varphi_2\in C([-r,0];X),\ \theta\in[-r,0]$,

$$\| h(\varphi_1)(x)-\varphi_2(x)\|\leq\frac{e^{\mu\theta}}{l^2}\cdot\|\varphi_1-\varphi_2\|\leq\frac{\|\varphi_1-\varphi_2\|}{l^2}, \tag{62}$$

we take $L_h=1/l^2$ and $\| h(\varphi)(x)\|\leq 1/l^2=N$, for $\varphi\in C([-r,0];X)$. Similarly we can see that g satisfies the assumption $(A3)$. Applying Theorem 3 , we get that system (50) has a mild solution.

Acknowledgment

The authors would like to thank the referee for valuable comments and suggestions. The work of the first author is supported by the University Grants Commission (UGC), Government of India, New Delhi.

References

[1] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of fractional differential equations, *Elsevier Science B. V.*, Amsterdam, The Netherlands, 2006.

[2] Igor Podlubny, Frcational Differential Equations, *Academic Press*, San Diego, Calif, USA, 1999.

[3] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, Inc., New York, 1993.

[4] J. K. Hale, L. Verduyn and M. Sjoerd, Introduction to Functional Differential Equations, *Applied Mathematical Sciences*, **99**, Springer-Verlag, New York, 1993.

[5] V. Lakshmikantham, Dimitŭr Baǐnov and Pavel S. Simeonov, Theory of impulsive differential equations, Series in Modern Applied Mathematics, *World Scientific Publishing Co., Inc.*, Teaneck, NJ, 1989.

[6] M. Benchohra, J. Henderson and S. K. Ntouyas, Impulsive differential equations and inclusions, *Contemporary Mathematics and Its Applications, Vol.2*, Hindawi Publishing Corporation, New York, 2006.

[7] Jaydev Dabas and Archana Chauhan, Existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equation with infinite delay, *Mathematical and Computer Modelling*, **57** (2013), 754-763.

[8] JinRong Wang, Michal Fečkan and Yong Zhou, On the new concept of solutions and existence results for impulsive fractional evolution equations, *Dynamics of PDE*, **8** (2011), 345-361.

[9] JinRong Wang, Yong Zhou and Michal Fečkan, Abstract Cauchy problem for fractional differential equations, *Nonlinear Dyn*, **71** (2013), 685-700.

[10] Mohamed I. Abbas, Existence for fractional order impulsive integrodifferential inclusions with nonlocal initial conditions, *International Journal of Mathematical Analysis*, **(6)**, 1813-1828 (2012).

[11] Lakshman Mahto, Syed Abbas and Angelo Favini, Analysis of Caputo impulsive fractional order differential equations with applications, *International Journal of Differential Equations*, 2013 (2013), pp-11.

[12] Hamdy M. Ahmed, A. A. M. Hassan and A. S. Ghanem, Existence of Mild Solution for Impulsive Fractional Differential Equations with Non-local Conditions in Banach Space, *British Journal of Mathematics and Computer Science*, **4** (6) (2014), 73-83.

[13] Michal Fečkan, Yong Zhou and JinRong Wang, On the concept and existence of solution for impulsive fractional differential equations, *Communications in Nonlinear Science and Numerical Simulation*, **17** (2012), 3050-3060.

[14] K. Balachandran and F. C. Samuel, Existence of mild solutions for integrodifferential equations with impulsive conditions, *Electronic Journal of Differential Equations*, **84** (2009), 1-9.

[15] Xiao-Bao Shu, Y. Lai and Y. Chen, The existence of mild solutions for impulsive fractional partial differential equations, *Nonlinear Analysis: TMA*, **74** (2011), 2003-2011.

[16] X. Zhang, X. Huang and Z. Liu, The Existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions and infinite delay, *Noninear Analysis: Hybrid Systems*, **4** (2010), 775-781.

[17] Zhenhai Liu and Xiuwen Li, On the controllability of impulsive fractional evolution inclusions in Banach Spaces, *Journal of Optimization Theory and Applications*, **156** (2013), 167182.

[18] S. Farahi and T. Guendouzi, Approximate controllability of fractional neutral stochastic evolution equations with nonlocal conditions, *Results in Mathematics*, **2014**, pp-21 (2014).

[19] M. Li and M. Han, Existence for neutral impulsive functional differential equations with nonlocal conditions, *Indagationes Mathematicae*, **20** (2009), 435-451.

[20] L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *Journal of Mathematical Analysis and Applications*, **162** (1991), 497-505.

[21] L. Byszewski and V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Applied Analysis*, **40** (1990), 11-19.
[22] Fang Li and Gaston M. N' Guérékata, An existence result for neutral delay integrodifferential equations with fractional order and nonlocal conditions, *Abstract and Applied Analysis*, **2011** (2011), pp. 20, article id, 952782.
[23] Fang Li, Jin Liang and Hong-Kun Xu, Existence of mild solutions for fractional integrodifferential equations of Sobolev type with nonlocal conditions, *Journal of Mathematical Analysis and Applications*, **391** (2012), 510-525.
[24] Fang Li, Nonlocal Cauchy problem for delay fractional integrodifferential equations of neutral type, *Advances in Difference Equations*, **2012** (2012):47, pp. 23.
[25] Lanping Zhu and Gang Li, Existence results of semilinear differential equations with nonlocal initial conditions in Banach spaces, *Nonlinear Analysis: TMA*, **74** (2011), 5133-5140.
[26] E. Hernández and D. O' Regan, On a new class of abstract impulsive differential equations, *Proc. Amer. Math. Soc.*, **141**, pp. 1641-1649, 2012.
[27] Qixiang Dong, Zhenbin Fan and Gang Li, Existence of solutions to nonlocal neutral functional differential and integrodifferential equations, *International Journal of Nonlinear Science*, **5** (2008), 140-151.
[28] Shaochun Ji and Gang Li, A unified approach to nonlocal impulsive differential equations with the measure of noncompactness, *Advances in Difference Equations*, **2012** (2012), pp. 14.
[29] Xingmei Xue, Nonlinear differential equations with nonlocal conditions in Banach spaces, *Nonlinear Analysis: TMA*, **63** (2005), 575-586.
[30] Xingmei Xue, Existence of solutions for semilinear nonlocal Cauchy problems in Banach space, *Electronic Journal of Differential Equations*, **64** (2005), 1-7.
[31] Xingmei Xue, Nonlocal nonlinear differential equations with measure of noncompactness in Banach space, *Nonlinear Analysis: TMA*, **70** (2009), 2593-2601.
[32] H.P. Heinz, On the behavior of measure of noncompactness with respect to differentiation and integration of vector-valued functions, *Nonlinear Analysis: TMA*, **7** (1983), 1351-1371.
[33] M. Kamenskii, V. Obukhovskii and P. Zecca, Condensing multivalued maps and semilinear differential inclusions in Banach spaces, De Gruyter Ser. *Nonlinear Analysis and Applications*, vol. **7**, de Gruyter, Berlin (2001).
[34] M. Kisielewicz, Multivalued differential equations in separable Banach spaces, *Journal of Optimization Theory and Applications*, **37** (1982), 231-249.
[35] R. R. Akhmerov, M. I. Kamenskiǐ, A. S. Potapov, A. E. Rodkina and B. N. Sadovskiǐ, Measures of noncompactness and Condensing operators, *Birkhäuser, Boston-Basel*, Berlin, Germany, 1992.
[36] Józef Banas and Kazimierz Goebel, Measure of noncompactness in Banach spaces, *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, New York, USA, 1980.
[37] Ravi P. Agarwal, Mouffak Benchohra and Djamila Seba, On the application of the measure of noncompactness to the existence of solutions for fractional differential equations, *Results Mathematics*, **55** (2009), 221-230.
[38] Runping Ye, Existence of solutions for impulsive partial neutral functional differential equation with infinite delay, *Nonlinear Analysis: TMA*, **73** (2010), 155-162.
[39] M. Muslim, Approximation of solutions to history-valued neutral functional differential equations, *Computers and Mathematics with Applications*, **51** (2006), 537-550.
[40] E. Bazhlekova, Fractional evolution equations in Banach spaces, Ph.D. Thesis, *Eindhoven University of Technology*, 2001.
[41] A. Pazy, Semigroup of Linear Operators and Applications to Partial Differential Equations, *Springer*, Berlin, Germany, 1983.

Alka Chadha
Department of Mathematics, Indian Institute of Technology, Roorkee, India
E-mail address: alkachadda23@gmail.com, alkachaddha03@gmail.com

Dwijendra N. Pandey
Department of Mathematics, Indian Institute of Technology, Roorkee, India
E-mail address: dwij.iitk@gmail.com