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ASYMPTOTICS OF SOLUTIONS OF NONLINEAR ABEL-VOLTERRA q-INTEGRAL EQUATIONS NEAR ZERO

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Abstract. This paper is devoted to studying the nonlinear Abel-Volterra q-integral equations of the form

$$\phi^m(x) = \frac{\lambda(x)}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}\phi(t)\, d_qt + f(x) \quad (0 < x < a \leq \infty)$$

with $\alpha > 0$ and $m \in \mathbb{R}$ $(m \neq 0, -1, -2, \ldots)$. The asymptotic behavior of $\phi(x)$ as $x \to 0$ is obtained, provided that the functions $\lambda(x)$ and $f(x)$ have special power asymptotic near zero. The solution $\phi(x)$ in closed form is given in some cases.

1. Introduction

In [8], Mansour proved the existence and uniqueness of positive continuous solutions of the nonlinear Fredholm q-integral equations

$$\phi(x) = \lambda(x) \int_0^1 (qt/x;q)_{\alpha-1}\phi^p(t)\, d_qt \quad (0 \leq x \leq 1) \tag{1}$$

and

$$\phi(x) = f(x) + \lambda(x) \int_0^1 (qt/x;q)_{\alpha-1}\phi^p(t)\, d_qt \quad (0 \leq x \leq 1) \tag{2}$$

where both of λ and f are positive continuous functions on $[0,1]$ and $0 < |p| < 1$. Replace p, and ϕ by $\frac{1}{m}$, and ϕ^m on (1) and (2), respectively, where $m \notin \{0,-1,-2,\ldots\}$. This yields the Fredholm q-integral equations

$$\phi^m(x) = \lambda(x) \int_0^1 (qt/x;q)_{\alpha-1}\phi(t)\, d_qt \quad (0 \leq x \leq 1) \tag{3}$$

and

$$\phi^m(x) = f(x) + \lambda(x) \int_0^1 (qt/x;q)_{\alpha-1}\phi(t)\, d_qt \quad (0 \leq x \leq 1) \tag{4}$$

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In this paper, we investigate the asymptotics of solutions of (3) and (4) when f and λ have the following power asymptotic near zero

$$\lambda(x) \sim x^{\alpha\, pm} \sum_{k=-l}^{\infty} \lambda_k x^{\alpha\, k}, \tag{5}$$

with $\lambda_{-l} \neq 0$ and

$$f(x) \sim x^{\alpha\, pm} \sum_{k=-n}^{\infty} f_k x^{\alpha\, k}. \tag{6}$$

2. Preliminaries and q-notations

Let q be a positive number which is less than 1, $\mathbb{N}$ be the set of all positive integers, and $\mathbb{N}_0$ be the set of all nonnegative integers. In the following, we follow the notations and notions of q-hypergeometric functions, the q-gamma function $\Gamma_q(x)$, Jackson q-exponential functions $e_q(x)$, and the q-shifted factorial as in [3, 4]. By a q-geometric set A we mean a set that satisfies if $x \in A$ then $qx \in A$. Let f be a function defined on a q-geometric set A. The $q-$difference operator is defined by

$$D_q f(x) := \frac{f(x) - f(qx)}{x - qx}, \quad x \neq 0. \tag{7}$$

Jackson [5] introduced an integral denoted by

$$\int_a^b f(x)\, d_q x$$

as a right inverse of the q-derivative. It is defined by

$$\int_a^b f(t)\, d_q t := \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t,\ a, b \in \mathbb{C}, \tag{8}$$

where

$$\int_0^x f(t)\, d_q t := (1-q) \sum_{n=0}^{\infty} x q^n f(xq^n),\ x \in \mathbb{C}, \tag{9}$$

provided that the series at the right-hand side of (9) converges at $x = a$ and b. A q-analogue of the Riemann-Liouville fractional integral operator is introduced in [1] by Al-Salam through

$$I_q^{\alpha} f(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \left(qt/x; q\right)_{\alpha-1} f(t)\, d_q t, \tag{10}$$

$\alpha \notin \{-1, -2, \ldots\}$. Using (9), we obtain

$$I_q^{\alpha} f(x) := x^{\alpha} (1-q)^{\alpha} \sum_{k=0}^{\infty} \frac{(q^{\alpha}; q)_k}{(q; q)_k} f(xq^k), \tag{11}$$

which is valid for all α. The q-translation operator is introduced by Ismail in [4] and is defined on monomials by

$$\varepsilon^y x^n := x^n (-y/x; q)_n, \tag{12}$$

and it is extended to polynomials as a linear operator.

3. Asymptotic Solutions Near Zero

The following theorem is proved in [7]. Therefore we introduce it without a proof.

Theorem 3.1. *Let $p \in \mathbb{Z}, \alpha \in \mathbb{R}$ and $\{\varphi_k\}_{k=p}^{\infty}$ be a sequence of real numbers. If the function $\varphi(x)$ has the asymptotic relation*

$$\phi(x) \sim \sum_{k=p}^{\infty} \varphi_k x^{\alpha k} \ (x \to 0), \tag{13}$$

then for $m \in \mathbb{R}$ with $m \neq 0, -1, -2, \ldots\ldots$, there holds the asymptotic

$$\phi^m(x) \sim x^{\alpha p m} \sum_{k=0}^{\infty} \Phi_{p,k} x^{\alpha k} \ (x \to 0) \tag{14}$$

where the coefficients $\Phi_{p,k}$ are expressed in terms of the coefficients φ_k:

$$\Phi_{p,0} = \binom{m}{0} \varphi_p^m,$$

$$\Phi_{p,1} = \binom{m}{1} \varphi_p^{m-1} \varphi_{p+1},$$

$$\Phi_{p,2} = \binom{m}{1} \varphi_p^{m-1} \varphi_{p+2} + \binom{m}{2} \varphi_p^{m-2} \varphi_{p+1}^2,$$

$$\Phi_{p,3} = \binom{m}{1} \varphi_p^{m-1} \varphi_{p+3} + \binom{m}{2}\binom{2}{1} \varphi_p^{m-2} \varphi_{p+1}\varphi_{p+2} + \binom{m}{3} \varphi_p^{m-3} \varphi_{p+1}^3, \tag{15}$$

$$\Phi_{p,4} = \binom{m}{1} \varphi_p^{m-1} \varphi_{p+4} + \binom{m}{2} \varphi_p^{m-2} \left[\varphi_{p+2}^2 + \binom{2}{1} \varphi_{p+1}\varphi_{p+3}\right]$$
$$+ \binom{m}{3}\binom{3}{1} \varphi_p^{m-3} \varphi_{p+1}^2 \varphi_{p+2} + \binom{m}{4} \varphi_p^{m-4} \varphi_{p+1}^4,$$

etc.

In [7], Kilbas and Saigo proved that if in Theorem 3.1 $m \in \{2, 3, \ldots\}$, then

$$\Phi_{p,k} = \sum_{i_0=0}^{m-1} \sum_{i_1, i_2, \ldots, i_j} \frac{m!}{i_0! i_1! i_2! \ldots i_j!} \phi_p^{i_0} \phi_{p+1}^{i_1} \ldots \phi_{p+j}^{i_j}$$

where the summation is taken over all non-negative integers $i_1, i_2, \ldots, i_j$ such that

$$0 \leq i_1 \leq i_2 \leq \ldots \leq i_j \leq k,$$

$$i_0 + i_1 + \ldots + i_j = m, \ i_1 + 2i_2 + \ldots + j i_j = k.$$

Theorem 3.2. *Let $p \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$ be such that $\alpha p > -1$. If*

$$\phi(x) \sim \sum_{j=p}^{\infty} \varphi_j x^{\alpha j} \quad (x \to 0), \tag{16}$$

then

$$I_q^{\alpha} \phi(x) \sim x^{\alpha} \sum_{j=p}^{\infty} \varphi_j x^{j\alpha} \frac{\Gamma_q(j\alpha + 1)}{\Gamma_q(j\alpha + \alpha + 1)} \ (x \to 0).$$

Proof. First we consider (4), where $\lambda(x)$ and $f(x)$ have the asymptotics (5) and (6), respectively. We will seek an asymptotic solution $\varphi(x)$ of (4) in the form (16). From (11)

$$I_q^{\alpha}\phi(x) = x^{\alpha}(1-q)^{\alpha}\sum_{k=0}^{\infty} q^k \frac{(q^{\alpha};q)_k}{(q;q)_k}\phi(xq^k).$$

Hence, applying (13) we have

$$\begin{aligned} I_q^{\alpha}\phi(x) &\sim x^{\alpha}(1-q)^{\alpha}\sum_{k=0}^{\infty} q^k \frac{(q^{\alpha};q)_k}{(q;q)_k}\sum_{j=p}^{\infty}\varphi_j x^{\alpha j} q^{k\alpha j} \\ &= x^{\alpha}(1-q)^{\alpha}\sum_{j=p}^{\infty}\varphi_j x^{j\alpha}\sum_{k=0}^{\infty} q^{k(\alpha j+1)}\frac{(q^{\alpha};q)_k}{(q;q)_k} \\ &= x^{\alpha}(1-q)^{\alpha}\sum_{j=p}^{\infty}\varphi_j x^{j\alpha}\frac{(q^{\alpha j+\alpha+1})}{(q^{\alpha j}+1)}, \end{aligned}$$

where we applied the q-binomial theorem in the last step, cf. [3, xvii]. Consequently,

$$I_q^{\alpha}\phi(x) \sim x^{\alpha}\sum_{j=p}^{\infty}\varphi_j x^{j\alpha}\frac{\Gamma_q(j\alpha+1)}{\Gamma_q(j\alpha+\alpha+1)},$$

and the theorem follows. □

In view of the asymptotic (5) and the general properties of asymptotic expansions, see [9, Chapter 1], we have

$$\frac{\lambda(x)x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}\phi(t)\,d_qt$$
$$\sim x^{\alpha pm}\sum_{k=-n}^{\infty}\left(\sum_{i=l-n-1}^{l+k-1}\frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)}\lambda_{k-i-1}\varphi_i\right)x^{\alpha k}\ (x\to 0).$$

Then, taking into account Theorem 3.1 and Lemma 6, we obtain

$$x^{\alpha(l-n-1)m}\sum_{k=0}^{\infty}\Phi_{l-n-1,k}x^{\alpha k} \sim$$
$$x^{\alpha pm}\sum_{k=-n}^{\infty}\left(\sum_{i=l-n-1}^{l+k-1}\frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)}\lambda_{k-i-1}\varphi_i\right)x^{\alpha k} + x^{\alpha pm}\sum_{k=-n}^{\infty} f_k x^{\alpha k}\ (x\to 0). \tag{17}$$

Theorem 3.3. *Let $\alpha>0$, p, $m\in\mathbb{R}$ $(m\neq 0,-1,-2,.....)$ be such that $\alpha p>-1$. Assume that as well as $l,n,r:=(l-n-p-1)m\in\mathbb{Z}$ such that $l-n-1>-1/\alpha$ and $r\geq -n$. Let $\lambda(x)$ and $f(x)$ have the asymptotics* (5) *and* (6) *and the coefficients φ_k satisfy*

$$\sum_{i=l-n-1}^{l+k-1}\frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)}\lambda_{k-i-1}\varphi_i + f_k = 0 \tag{18}$$
$$(k=-n,-n+1,...,r-1),$$

and

$$\Phi_{l-n-1,k-r} = \sum_{i=l-n-1}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1}\varphi_i + f_k \tag{19}$$

$$(k = r, r+1, ...).$$

Then (4) *is asymptotically solvable in* $\mathcal{L}^1_q(0,a)$ *for some* $a > 0$ *and its asymptotic solution near zero has the form* (13).

Proof. Suppose $r = (l-n-p-1)m \in \mathbb{Z}$ for $m \in \mathbb{R}, (m \neq 0, -1, -2, ...)$ such that $r \geq -n$. Then (17) is equivalent to

$$x^{\alpha(r+pm)} \sum_{k=0}^{\infty} \Phi_{l-n-1,k} x^{\alpha k} \sim x^{\alpha pm} \sum_{k=-n}^{\infty} \left(\sum_{i=l-n-1}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1}\varphi_i \right) x^{\alpha k} + x^{\alpha pm} \sum_{k=-n}^{\infty} f_k x^{\alpha k}. \tag{20}$$

Make the substation $\mu = k+r$ on the left hand side of the last relation and then replace μ by k, this gives

$$x^{\alpha pm} \sum_{k=r}^{\infty} \Phi_{l-n-1,k-r} x^{\alpha k} \sim x^{\alpha pm} \sum_{k=-n}^{\infty} \left(\sum_{i=l-n-1}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1}\varphi_i \right) x^{\alpha k} + x^{\alpha pm} \sum_{k=-n}^{\infty} f_k x^{\alpha k}. \tag{21}$$

Hence, it follows from (21) that the coefficients φ_k satisfy (18) and (19). Then (4) is asymptotically solvable and its asymptotic solution near zero is given by (13). □

Theorem 3.4. *Let* $\alpha > 0, m > 0$ *and let* $p, l, n \in \mathbb{Z}$ *be such that* $n = l-p-1 < 0$ *and* $(p-l+1)/m \in \mathbb{Z}$ *and set* $r = p+(p-l+1)/m > -1/\alpha$. *Let* $\lambda(x)$ *and* $f(x)$ *have the asymptotics* (5) *and* (6), *respectively. Moreover, let the coefficients* φ_k *satisfy the relations*

$$\Phi_{r,k+l-p-1} = f_k \tag{22}$$

$$(k = p-l+1, p-l+2, ..., r-l),$$

and

$$\Phi_{r,k+l-p-1} = \sum_{i=r}^{l+k-1} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)} \lambda_{k-i-1}\varphi_i + f_k \tag{23}$$

$$(k = r-l+1, r-l+2, ...).$$

Then (4) *is asymptotically solvable in* $\mathcal{L}^1_q(0,a)$ *and its asymptotic solution near zero has the form*

$$\phi(x) \sim \sum_{k=r}^{\infty} \varphi_k x^{\alpha k} \ (x \to 0). \tag{24}$$

Proof. We assume

$$\phi(x) \sim \sum_{k=r}^{\infty} \varphi_k x^{\alpha k} \ (x \to 0),$$

and we shall prove that the coefficients φ_k are given by (22), (23). Applying the same arguments as in Theorem 3.3, we came to the asymptotic relation

$$\begin{aligned} x^{\alpha r m} \sum_{k=0}^{\infty} \Phi_{r,k} x^{\alpha k} \sim x^{\alpha p m} \sum_{k=-n}^{\infty} \left(\sum_{i=r}^{l+k-1} \frac{\Gamma_q(\alpha i + 1)}{\Gamma_q(\alpha i + \alpha + 1)} \lambda_{k-i-1} \varphi_i \right) x^{\alpha k} \\ + x^{\alpha p m} \sum_{k=p-l-1}^{\infty} f_k x^{\alpha k}. \end{aligned} \tag{25}$$

Since $m(r-p) \in \mathbb{N}$, the left hand side of (25) can be written as

$$\begin{aligned} x^{\alpha r m} \sum_{k=0}^{\infty} \Phi_{r,k} x^{\alpha k} &= x^{\alpha p m + \alpha (r-p) m} \sum_{k=0}^{\infty} \Phi_{r,k} x^{\alpha k} \\ &= x^{\alpha p m} \sum_{k=0}^{\infty} \Phi_{r,k} x^{\alpha (k + (r-p) m)}. \end{aligned}$$

Make the substation $\mu = k + (r-p)m$ on the last series and then replace μ by k, this gives

$$\begin{aligned} x^{\alpha p m} \sum_{k=(r-p)m}^{\infty} \Phi_{r,k-(r-p)m} x^{\alpha k} \sim x^{\alpha p m} \sum_{k=-n}^{\infty} \left(\sum_{i=r}^{l+k-1} \frac{\Gamma_q(\alpha i + 1)}{\Gamma_q(\alpha i + \alpha + 1)} \lambda_{k-i-1} \varphi_i \right) x^{\alpha k} \\ + x^{\alpha p m} \sum_{k=p-l+1}^{\infty} f_k x^{\alpha k}. \end{aligned} \tag{26}$$

Equating the coefficients of $x^{\alpha k}$ on (26) gives (22) and (23). Then (4) is asymptotically solvable in $\mathcal{L}_q^1(0,a)$ for some $a > 0$ and its asymptotic solution is in the form (24). □

Corollary 3.5. *Let $\alpha > 0$, $m > 0$ and let l be a positive integer such that $l/m \in \mathbb{Z}$ and set $r = -1 + l/m$. Let*

$$\lambda(x) \sim x^{-\alpha m} \sum_{k=l}^{\infty} \lambda_k x^{\alpha k}, \ f(x) \sim x^{-\alpha m} \sum_{k=l}^{\infty} f_k x^{\alpha k} \ (x \to 0),$$

with $\lambda_l \neq 0$, $f_l \neq 0$. Then (4) *is asymptotically solvable and its asymptotic solution near zero has form* (24).

Proof. The proof follows from Theorem 3.4 by substituting with $p = -1$, $n = l$ and $f_k = 0$ $(k = -l, -l+1, ..., -l+r)$ in equations (5) and (6). Hence, the coefficients φ_k satisfy

$$\begin{gathered} \Phi_{r,k+l} = f_k \\ (k = -l, -l+1, ..., -l+r), \end{gathered} \tag{27}$$

and

$$\Phi_{r,k+l} = \sum_{i=r}^{l+k-1} \frac{\Gamma_q(\alpha i + 1)}{\Gamma_q(\alpha i + \alpha + 1)} \lambda_{k-i-1} \varphi_i + f_k \tag{28}$$

$$(k = r - l + 1, r - l + 2, ...).$$

Then (4) is asymptotically solvable in $\mathcal{L}_q^1(0, a)$ for some $a > 0$ and its asymptotic solution near zero has the form (24). □

Theorem 3.6. *Let $\alpha > 0, m < 1\,(m \neq 0, -1, -2, ...)$ and let $p, l, n \in \mathbb{Z}$ be such that $n = l-p-1 < 0$ and $(p-l+1)/(1-m) \in \mathbb{Z}$ and let $r = p-(p-l+1)/(1-m) > -1/\alpha$. Let $\lambda(x)$and $f(x)$ have asymptotics (5) and (6) and the coefficients φ_k satisfy*

$$\Phi_{r,k+l-r-1} = \sum_{i=r}^{l+k-1} \frac{\Gamma_q(\alpha i + 1)}{\Gamma_q(\alpha i + \alpha + 1)} \lambda_{k-i-1} \varphi_i \tag{29}$$

$$(k = r - l + 1, r - l + 2, ..., p - l),$$

and

$$\Phi_{r,k+l-r-1} = \sum_{i=r}^{l+k-1} \frac{\Gamma_q(\alpha i + 1)}{\Gamma_q(\alpha i + \alpha + 1)} \lambda_{k-i-1} \varphi_i + f_k \tag{30}$$

$$(k = p - l + 1, p - l + 2, ...).$$

Then (4) is asymptotically solvable and its asymptotic solution near zero has the form (24).

Proof. Applying Theorem (3.4) we have

$$x^{\alpha pm} \sum_{k=r-l+1}^{\infty} \Phi_{r,k-(r-l+1)} x^{\alpha k} \sim x^{\alpha pm} \sum_{k=-n}^{\infty} \left(\sum_{i=r}^{l+k-1} \frac{\Gamma_q(\alpha i + 1)}{\Gamma_q(\alpha i + \alpha + 1)} \lambda_{k-i-1} \varphi_i \right) x^{\alpha k} + x^{\alpha pm} \sum_{k=p-l+1}^{\infty} f_k x^{\alpha k}.$$

Then the coefficients φ_k satisfy (29) and (30). Then (4) is asymptotically solvable and its asymptotic solution near zero has the form (24) □

4. Asymptotic of the Solution in Some Special Cases

In this section we give asymptotic solutions of (4) when $\lambda(x)$ and $f(x)$ have the special case:

$$\lambda(x) = \lambda x^{\alpha(pm-l)} \text{ and } f(x) = -x^{\alpha pm} \sum_{k=-n}^{N} f_k x^{\alpha k}.$$

Hence (4) takes the form

$$\phi^m(x) = \frac{\lambda x^{\alpha(pm-l)}}{\Gamma_q(\alpha)} x^{\alpha-1} \int_0^x (qt/x; q)_{\alpha-1}\, \phi(t)\, d_q t \, - x^{\alpha pm} \sum_{k=-n}^{N} f_k x^{\alpha k}, \tag{31}$$

where $0 < x < a \leq \infty$, $\lambda \neq 0$ and $N \geq -n$.

Theorem 4.1. *Let $\alpha > 0, p, m \in \mathbb{R}\,(m \neq 0, -1, -2, ...)$ and let l, n, $(l - n - p - 1)m \in \mathbb{Z}$ be such that $l - n - 1 > -1/\alpha$ and $(l - n - p - 1)m \geq -n$.*

(1) *When* $r = (l-n-p-1)m > N$ *and the coefficients* φ_k *satisfy*

$$\varphi_k = 0 \ (k = N+l, N+l+1, ..., r+l-2), \tag{32}$$

$$\varphi_k = \frac{-f_{k-l+1}\Gamma_q(\alpha k+\alpha+1)}{\lambda\Gamma_q(\alpha k+1)} \tag{33}$$

$$(k = -n+l-1, -n+l, ..., N+l-1),$$

and

$$\Phi_{l-n-1,k-l+1-r} = \frac{\lambda\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\varphi_k \tag{34}$$

$$(k = r+l-1, r+l, ...).$$

Then (31) *is asymptotically solvable in* $\mathcal{L}_q^1(0,a)$ *for some* $a > 0$ *and its asymptotic solution near zero has the form*

$$\phi(x) \sim \sum_{k=l-n-1}^{l+N-1} \frac{\Gamma_q(\alpha k+\alpha+1)}{\lambda\Gamma_q(\alpha k+1)} f_{k-l+1}x^{\alpha k} + \sum_{k=r+l-1}^{\infty} \varphi_k x^{\alpha k}. \tag{35}$$

(2) *When* $-n < r \leq N$ *and the coefficients* φ_k *satisfy*

$$\Phi_{l-n-1,k-l+1-r} = \frac{\lambda\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\varphi_k - f_{k-l+1} \tag{36}$$

$$(k = r+l-1, r+l, ..., N+l-1),$$

$$\Phi_{l-n-1,k-l+1-r} = \frac{\lambda\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\varphi_k \tag{37}$$

$$(k = N+l, N+l+1, ...).$$

Then (31) *is asymptotically solvable in* $\mathcal{L}_q^1(0,a)$ *for some* $a > 0$ *and its asymptotic solution near zero has the form*

$$\phi(x) \sim \sum_{k=l-n-1}^{r+l-2} \frac{\Gamma_q(\alpha k+\alpha+1)}{\lambda\Gamma_q(\alpha k+1)} f_{k-l+1}x^{\alpha k} + \sum_{k=r+l-1}^{\infty} \varphi_k x^{\alpha k}. \tag{38}$$

(3) *When* $r = -n$ *and the coefficients* φ_k *satisfy*

$$\Phi_{l-n-1,k-l+1+n} = \frac{\lambda\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\varphi_k - f_{k-l+1} \tag{39}$$

$$(k = -n+l-1, -n+l, ..., N+l-1),$$

$$\Phi_{l-n-1,k-l+1+n} = \frac{\lambda\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\varphi_k \tag{40}$$

$$(k = N+l, N+l+1, ...).$$

Then (31) *is asymptotically solvable in* $\mathcal{L}_q^1(0,a)$ *and its asymptotic solution near zero has the form* (13).

Proof. From (5) and (6) we obtain $\lambda_{-l} = \lambda, \lambda_j = 0$ for $j > l,$ and $f_k = 0$ for $k > N$. Hence, conditions (32)-(34) imply that the conditions (18) and (19) of Theorem 3.3 are satisfied. Hence ϕ has the asymptotic (13) where the coefficients are given by (32)- (34). That is ϕ has the asymptotic (35). This proves (1) of the Theorem. The proofs of the points (2) and (3) are similar to the proof of (1) and so they are omitted. □

Corollary 4.2. *Under the assumptions* (32)- (34) *of Theorem 4.1(1), the solution* $\phi(x)$ *of* (31) *has the asymptotic*

$$\phi(x) = \sum_{k=l-n-1}^{l+N-1} \frac{\Gamma_q(\alpha k+\alpha+1)}{\lambda\Gamma_q(\alpha k+1)} f_{k-l+1} x^{\alpha k} + \varphi_{r+l-1} x^{\alpha(r+l-1)} + O(x^{\alpha(r+l)})\ (x \to 0), \tag{41}$$

where

$$\varphi_{r+l-1} = \frac{\Gamma_q(\alpha(r+l)+1)}{\lambda\Gamma_q(\alpha(r+l-1)+1)} \times \left(\frac{\Gamma_q(\alpha(l-n)+1) f_{-n}}{\lambda\Gamma_q(\alpha(l-n-1)+1)}\right)^m. \tag{42}$$

Furthermore, if $N > -n$, *we have*

$$\phi(x) = \sum_{k=l-n-1}^{l+N-1} \frac{\Gamma_q(\alpha k+\alpha+1)}{\lambda\Gamma_q(\alpha k+1)} f_{k-l+1} x^{\alpha k} + \varphi_{r+l-1} x^{\alpha(r+l-1)} + \varphi_{r+l} x^{\alpha(r+l)} + O(x^{\alpha(r+l+1)})\ (x \to 0), \tag{43}$$

where φ_{r+l-1} *is given by* (42) *and*

$$\varphi_{r+l} = \frac{m\Gamma_q(\alpha(r+l+1)+1)}{\lambda\Gamma_q(\alpha(r+l)+1)} \left(\frac{\Gamma_q(\alpha(l-n)+1) f_{-n}}{\lambda\Gamma_q(\alpha(l-n-1)+1)}\right)^{m-1} \times \frac{\Gamma_q(\alpha(l-n+1)+1) f_{-n+1}}{\lambda\Gamma_q(\alpha(l-n)+1)}. \tag{44}$$

Proof. Substitute with $k = r+l-1$ in (34). This gives

$$\begin{aligned} \varphi_{r+l-1} &= \frac{\Gamma_q(\alpha(r+l)+1)}{\lambda\Gamma_q(\alpha(r+l-1)+1)} \Phi_{l-n-1,0} \\ &= \frac{\Gamma_q(\alpha(r+l)+1)}{\lambda\Gamma_q(\alpha(r+l-1)+1)} \varphi_{l-n-1}^m. \end{aligned}$$

Then put $k = -n$ in (18) we get

$$\varphi_{l-n-1} = \frac{\Gamma_q(\alpha(l-n)+1)}{\lambda\Gamma_q(\alpha(l-n-1)+1)} f_{-n}.$$

Then we are done. Similarly if $N > -n$. □

Corollary 4.3. *Under the assumptions of Theorem 4.1(2), the solution* $\phi(x)$ *of* (31) *has the asymptotic*

$$\phi(x) = \sum_{k=l-n-1}^{r+l-2} \frac{\Gamma_q(\alpha k+\alpha+1)}{\lambda\Gamma_q(\alpha k+1)} f_{k-l+1} x^{\alpha k} + \varphi_{r+l-1} x^{\alpha(r+l-1)} + O(x^{\alpha(r+l)})\ (x \to 0), \tag{45}$$

where

$$\varphi_{r+l-1} = \frac{\Gamma_q(\alpha(r+l)+1)}{\lambda\Gamma_q(\alpha(r+l-1)+1)} \times \left\{ f_r + \left(\frac{\Gamma_q(\alpha(l-n)+1) f_{-n}}{\lambda\Gamma_q(\alpha(l-n-1)+1)}\right)^m \right\}. \tag{46}$$

Furthermore, if $N \geq r > -n+1$, *we have*

$$\phi(x) = \sum_{k=l-n-1}^{r+l-2} \frac{\Gamma_q(\alpha k+\alpha+1)}{\lambda\Gamma_q(\alpha k+1)} f_{k-l+1} x^{\alpha k}$$

$$+\varphi_{r+l-1}x^{\alpha(r+l-1)}+\varphi_{r+l}x^{\alpha(r+l)}+O(x^{\alpha(r+l+1)})\ (x\to 0), \tag{47}$$

where φ_{r+l-1} is given by(46) *and*

$$\begin{aligned}\varphi_{r+l}&=\frac{m\Gamma_q(\alpha(r+l+1)+1)}{\lambda\Gamma_q(\alpha(r+l)+1)}\times\left(\frac{\Gamma_q(\alpha(l-n)+1)f_{-n}}{\lambda\Gamma_q(\alpha(l-n-1)+1)}\right)^{m-1}\\&\times\frac{\Gamma_q(\alpha(l-n+1)+1)f_{-n+1}}{\lambda\Gamma_q(\alpha(l-n)+1)}.\end{aligned} \tag{48}$$

Proof. Substitute with $k=r+l-1$ in (37). This gives

$$\begin{aligned}\varphi_{r+l-1}&=\frac{\Gamma_q(\alpha(r+l)+1)}{\lambda\Gamma_q(\alpha(r+l-1)-1)}\Phi_{l-n-1,0}\\&=\frac{\Gamma_q(\alpha(r+l)+1)}{\lambda\Gamma_q(\alpha(r+l-1)+1)}\varphi^m_{l-n-1}\\&=\frac{\Gamma_q(\alpha(r+l)+1)}{\lambda\Gamma_q(\alpha(r+l-1)+1)}\left\{f_r+\left(\frac{\Gamma_q(\alpha(l-n)+1)f_{-n}}{\lambda\Gamma_q(\alpha(l-n-1)+1)}\right)^m\right\}.\end{aligned}$$

Similarly if $N\geq r>-n+1$ □

Corollary 4.4. *Under the assumptions of Theorem 4.1(3), the solution $\phi(x)$ of*(31) *has the asymptotic*

$$\phi(x)=Ax^{\alpha(l-n-1)}+O(x^{\alpha(l-n)})\ (x\to 0), \tag{49}$$

where $\xi=A$ is a solution of the equation

$$\xi^m-\frac{\lambda\Gamma_q(\alpha(l-n-1)+1)}{\Gamma_q(\alpha(l-n)+1)}\xi+f_{-n}=0. \tag{50}$$

Furthermore, if $N\geq -n+1$ and

$$\frac{\lambda\Gamma_q(\alpha(l-n)+1)}{\Gamma_q(\alpha(l-n-1)+1)}\neq mA^{m-1},$$

we have

$$\phi(x)=Ax^{\alpha(l-n-1)}+Bx^{\alpha(l-n)}+O(x^{\alpha(l-n+1)})\ (x\to 0), \tag{51}$$

where

$$B=\left[\frac{\lambda\Gamma_q(\alpha(l-n)+1)}{\Gamma_q(\alpha(l-n-1)+1)}-mA^{m-1}\right]^{-1}f_{-n+1}.$$

Proof. Substitute with $k=l-n-1$ in (39). This gives

$$\Phi_{l-n-1,0}=\frac{\lambda\Gamma_q(\alpha(l-n-1)+1)}{\Gamma_q(\alpha(l-n)+1)}\varphi_{l-n-1}-f_{-n},$$

$$\varphi^m_{l-n-1}-\frac{\lambda\Gamma_q(\alpha(l-n-1)+1)}{\Gamma_q(\alpha(l-n)+1)}\varphi_{l-n-1}+f_{-n}=0.$$

Then $\varphi_{l-n-1}=A$ is a solution of equation (50). Now we prove (51) if $N\geq -n+1$ and

$$\frac{\lambda\Gamma_q(\alpha(l-n)+1)}{\Gamma_q(\alpha(l-n-1)+1)}\neq mA^{m-1},$$

put $k=l-n$ in (39). We obtain

$$\Phi_{l-n-1,1}=\frac{\lambda\Gamma_q(\alpha(l-n)+1)}{\Gamma_q(\alpha(l-n+1)+1)}\varphi_{l-n}-f_{-n+1},$$

$$m\varphi_{l-n-1}^{m-1}\varphi_{l-n} = \frac{\lambda\Gamma_q(\alpha(l-n)+1)}{\Gamma_q(\alpha(l-n+1)+1)}\varphi_{l-n} - f_{-n+1},$$

where we used (15). Since $A = \varphi_{l-n-1}$ and $B = \varphi_{l-n}$, then we have

$$B = \left[\frac{\lambda\Gamma_q(\alpha(l-n)+1)}{\Gamma_q(\alpha(l-n-1)+1)} - mA^{m-1}\right]^{-1} f_{-n+1}.$$

□

In the remaining of this section we derive an exact solution of the equation

$$\phi^m(x) = \frac{\lambda x^{\alpha(pm-l)}}{\Gamma_q(\alpha)} x^{\alpha-1}\int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt\; - bx^{\alpha(pm-n)} \tag{52}$$

$$(0 < x < a \leq \infty).$$

From Corollary 4.4, the solution $\phi(x)$ of (52) has the asymptotic (49) near zero, where $\xi = A$ is a solution of the equation

$$\xi^m - \frac{\lambda\Gamma_q(\alpha(l-n-1)+1)}{\Gamma_q(\alpha(l-n)+1)}\xi + b = 0.$$

Theorem 4.5. *Let $\alpha > 0$, $\beta > -1$ ($\beta \neq 0$), and $l \in \mathbb{R}$ with $l \neq -\alpha$ and $l \neq -\alpha-\beta$. For $a, b \in \mathbb{R}$ ($a \neq 0$) let the equation*

$$\xi^{1+(l+\alpha)/\beta} - \frac{\lambda\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}\xi - b = 0, \tag{53}$$

be solvable and let $\xi = c$ be its solution. Then the nonlinear integral equation

$$\phi^{1+(l+\alpha)/\beta}(x) = \frac{\lambda x^l}{\Gamma_q(\alpha)} x^{\alpha-1}\int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt + bx^{\alpha+\beta+l} \tag{54}$$

$$(0 < x < a \leq \infty)$$

is solvable and its solution is given by

$$\phi(x) = cx^\beta. \tag{55}$$

Proof. We apply Corollary 4.4 with $m = 1+\frac{\alpha+\gamma}{\beta}$, $\gamma = \alpha(pm-l)$, and $\alpha(pm-n) = \alpha+\beta+\gamma$. Then the solution is given by $\phi(x) = cx^\beta + O(x^{\beta+\alpha})$. But a direct substitution verifies that $\phi(x) = cx^\beta$ is a solution of

$$\xi^m - \frac{\lambda\Gamma_q(l-n-\alpha+1)}{\Gamma_q(n-l+1)}\xi - b = 0.$$

□

Now we consider the homogeneous equation associated with (54) which is

$$\phi^{1+(l+\alpha)/\beta}(x) = \frac{\lambda x^l}{\Gamma_q(\alpha)} x^{\alpha-1}\int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt\ (0 < x < a \leq \infty). \tag{56}$$

According to Theorems 3.1 and (**??**) we obtain the following result

Theorem 4.6. *Let the conditions of Theorem 4.5 are satisfied and let $\xi = c$ be the unique solution of* (53).

(i) *If* $-1 < (l+\alpha)/\beta < 0$, *then* (55) *is the unique solution of* (56) *in the space* $C[0, a]$ *for some* $a > 0$. *If in additionally,* $\lambda, b,$ *and* C *are positive numbers, then this solution belongs to* $C^+[0, d]$.

(ii) *If* $(\alpha + l)/\beta > 0$, $a, b,$ *and* c *are positive numbers, then* (55) *is the unique solution of* (56) *in* $C^+[0, 1]$.

Remark 4.7. In [6, PP. 441–442] Karapetyants et al. studied the existence of positive solutions of the algebraic equation

$$\xi^m - d\xi - b = 0, \tag{57}$$

with $m > 0$, $m \neq 1$ and a, $b \in \mathbb{R} - \{0\}$. They investigated the positive solvability of (57) by using the properties of the function

$$f(\xi) = \xi^m - d\xi - b.$$

Set

$$c_0 := \left(\frac{d}{m}\right)^{\frac{1}{m-1}}, \quad E := f(c_0). \tag{58}$$

The authors of [6] obtained the following result which we state without proof.

Theorem 4.8. *Let* $m > 0$, $m \neq 1$ *and* a, $b \in \mathbb{R} - \{0\}$. *Let* E *and* c_0 *be as in* (58). *Equation* (57)

(i) *does not have positive solutions if either* $d < 0$, $b < 0$ *or* $d > 0$, $b < 0$, $m > 1$, $E > 0$ *or* $d > 0$, $b > 0$, $0 < m < 1$, $E < 0$;

(ii) *has a unique positive solution*

ii.1 $\xi = c_1 > 0$ *if* $d < 0$, $b > 0$;

ii.2 $\xi = c_1 > c_0 > 0$ *if either* $d > 0$, $b < 0$, $0 < m < 1$ *or* $d > 0$, $b > 0$, $m = 1$;

ii.3 $\xi = c_0 > 0$ *if* $E = 0$ $c_1 > c_0 > 0$ *if either and either* $d > 0$, $b < 0$, $m > 1$ *or* $d > 0$, $b > 0$, $0 < m < 1$

(iii) *has two positive solutions* $\xi = c_2$ *and* $\xi = c_3$, $0 < c_2 < c_0 < c_3$, *if either* $d > 0$, $b < 0$, $m > 1$, $E < 0$ *or* $d > 0$, $b > 0$, $0 < m < 1$, $E > 0$.

5. Asymptotic solution of linear equation in general case

In this section we investigate the special case $m = 1$ of (4) when $0 < \alpha < 1$. In other words, we give the asymptotic solution of the equation

$$\phi(x) = \frac{\lambda(x)}{\Gamma_q(\alpha)} x^{\alpha-1} \int_0^x (qt/x; q)_{\alpha-1}\, \phi(t)\, d_q t \; + f(x) \tag{59}$$

$$(0 < x < \infty, 0 < \alpha < 1).$$

In [2, P.214], the authors studied (4) for all $\alpha > 0$ when $\lambda(x) = \lambda$ for all $x \in (0, a]$, and $f \in \mathcal{L}_q^1[0, a]$ where a is a positive number satisfying the inequality

$$|\lambda| a^\alpha (1-q)^\alpha < 1.$$

They proved that the q-integral equation (59) under the previous conditions has a unique solution

$$\phi(x) = f(x) + \lambda x^{\alpha-1} \int_0^x (qt/x; q)_{\alpha-1} \varepsilon^{-q^\alpha t} e_{\alpha,\alpha}(\lambda t^\alpha; q) f(t)\, d_q t,$$

in the space $\mathcal{L}_q^1[0, a]$ where ε is the q-translation operator defined in (12).

Theorem 5.1. *Let $f(x)$ and $\lambda(x)$ have the asymptotics as $x \to 0$*

$$f(x) \sim \sum_{k=-1}^{\infty} f_k x^{\alpha k}, \tag{60}$$

and

$$\lambda(x) \sim \sum_{k=-1}^{\infty} \lambda_k x^{\alpha k}. \tag{61}$$

Assume that

$$\lambda_{-1} \neq \frac{\Gamma_q(\alpha k+\alpha+1)}{\Gamma_q(\alpha k+1)} \quad (k=-1,0,1,....). \tag{62}$$

Then the unique power asymptotic solution $\phi(x)$ of (59) near zero in the space of all continuous functions is given by the form $\phi(x) \sim \sum_{k=-1}^{\infty} \varphi_k x^{\alpha k}$, where φ_k is given by

$$\varphi_k = \left[1 - \frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\lambda_{-1}\right]^{-1} \times \left[\sum_{i=-1}^{k-1} \frac{\Gamma_q(\alpha i+1)\lambda_{k-i-1}}{\Gamma_q(\alpha i+\alpha+1)}\varphi_i + f_k\right] \tag{63}$$

$$(k=-1,0,1,2,......).$$

Proof. Using (23) we obtain

$$\Phi_{-1,k+1} = \sum_{i=-1}^{k} \frac{\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha i+\alpha+1)}\lambda_{k-i-1}\varphi_i + f_k \quad (k=-1,0,1,...).$$

Substitute with $p=-1$ in (16) yields

$$\phi(x) \sim \sum_{j=-1}^{\infty} \varphi_j x^{\alpha j}, \tag{64}$$

and from (14) with $p=-1$ and $m=1$

$$\phi(x) \sim x^{-\alpha} \sum_{k=0}^{\infty} \Phi_{-1,k} x^{\alpha k} = \sum_{j=-1}^{\infty} \Phi_{-1,j+1} x^{\alpha j}. \tag{65}$$

Compared to coefficients of $x^{\alpha j}$ in (64) and (65) we obtain

$$\Phi_{-1,j+1} = \varphi_j \quad (j=-1,0,1,....).$$

So, we have the following formulas for the coefficients φ_k

$$\varphi_k = \sum_{i=-1}^{k} \frac{\Gamma_q(\alpha i+1)\lambda_{k-i-1}}{\Gamma_q(\alpha i+\alpha+1)}\varphi_i + f_k \quad (k=-1,0,1,...),$$

equivalently

$$\left[1 - \frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\lambda_{-1}\right]\varphi_k = \sum_{i=-1}^{k-1} \frac{\Gamma_q(\alpha i+1)\lambda_{k-i-1}}{\Gamma_q(\alpha i+\alpha+1)}\varphi_i + f_k \; (k=-1,0,1,...). \tag{66}$$

Hence if (62) satisfied, asymptotic solution $\phi(x)$ of equation (59) is given by the form (64) where φ_k given by (63). □

Theorem 5.2. *Let $f(x)$ and $\lambda(x)$ have the asymptotics* (60) *and* (61)*, respectively as $x \to 0$. Assume that there exists a number $j \in \{-1, 0, 1, ...\}$ such that*

$$\lambda_{-1} = \frac{\Gamma_q(\alpha j + \alpha + 1)}{\Gamma_q(\alpha j + 1)}. \tag{67}$$

If the coefficients $f_k\,(k = -1, 0, 1, ..., j)$ in the asymptotic expansion (60) *satisfy the relation*

$$\sum_{i=-1}^{j-1} \frac{\Gamma_q(\alpha j + 1)\lambda_{j-i-1}}{\Gamma_q(\alpha i + \alpha + 1)}\varphi_i + f_j = 0, \tag{68}$$

then the unique power asymptotic solution $\phi(x)$ of equation (59) *is given by*

$$\phi(x) \sim cx^{\alpha j} \sum_{\substack{k=-1 \\ k\neq j}}^{\infty} \varphi_k x^{\alpha k}, \tag{69}$$

where c is an arbitrary constant. If the condition (68) *is not satisfied, then equation* (59) *does not have any asymptotic solution of the form* (64)*.*

Proof. Using Theorem 5.1 and suppose (62) is not valid. This means there exists a number $j \in \{-1, 0, 1, ...\}$ such that (67) holds. In this case the coefficients $f_k\,(k = -1, 0, 1, ..., j)$ in the asymptotic expansion (60) satisfy the relation (68), where $\varphi_i\,(i = -1, 0, 1, ..., j-1)$ are expressed via $f_i\,(i = -1, 0, 1, ..., j-1)$ by means of (63). For example, when $j = -1, 0, 1$, the relations (67) and (68) have the form

$$\lambda_{-1} = \frac{1}{\Gamma_q(1-\alpha)}\; f_{-1} = 0 \quad \text{for} \quad j = -1$$

$$\lambda_{-1} = \Gamma_q(\alpha + 1)\; \lambda_0\varphi_{-1} + f_0 = 0 \quad \text{for} \quad j = 0$$

$$\lambda_{-1} = \frac{\Gamma_q(2\alpha + 1)}{\Gamma_q(\alpha + 1)}\; \Gamma_q(\alpha + 1)\lambda_1\varphi_{-1} + \lambda_0\varphi_0 + f_1 = 0 \quad \text{for} \quad j = 1.$$

Thus if condition (68) is satisfied, then the asymptotic solution of (59) has the form (69), where c is an arbitrary constant and $\varphi_k\,(k \neq j)$ are given by (63). If condition (68) is not satisfied, equation (59) does not have any asymptotic solution of the form (64) □

Theorem 5.3. *Let*

$$f(x) \sim \sum_{k=0}^{\infty} f_k x^{\alpha k}, \tag{70}$$

and $\lambda(x)$ has the asymptotic (61)*,as $x \to 0$, and let condition* (62) *be satisfied. Then the unique power asymptotic solution $\phi(x)$ of equation* (59) *is given by the form*

$$\phi(x) \sim \sum_{k=0}^{\infty} \varphi_k x^{\alpha k}, \tag{71}$$

where $\varphi_k\,(k \in \mathbb{N}_0)$ are given by

$$\varphi_k = \left[1 - \frac{\Gamma_q(\alpha k + 1)}{\Gamma_q(\alpha k + \alpha + 1)}\lambda_{-1}\right]^{-1} \times \left[\sum_{i=0}^{k-1} \frac{\Gamma_q(\alpha k + 1)\lambda_{k-i-1}}{\Gamma_q(\alpha i + \alpha + 1)}\varphi_i + f_k\right]. \tag{72}$$

Proof. If $\lambda(x)$ has the asymptotic (60), then (66) takes the form

$$(1-\Gamma_q(1-\alpha)\lambda_{-1})\varphi_{-1}=0,$$

$$\left[1-\frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\lambda_{-1}\right]\varphi_k=\sum_{i=-1}^{k-1}\frac{\Gamma_q(\alpha k+1)\lambda_{k-i-1}}{\Gamma_q(\alpha i+\alpha+1)}\varphi_i+f_k\ (k=0,1,...). \quad (73)$$

When condition (62) holds, $\varphi_{-1}=0$ and hence the asymptotic solution (64) of equation (59) has the form (71), where $\varphi_k\,(k=0,1,2,...)$ are given by (72). □

Theorem 5.4. *Assume that the functions $f(x)$ and $\lambda(x)$ have the asymptotics (70) and (61), respectively, as $x\to 0$. If $\lambda_{-1}=1$, then the unique power asymptotic solution $\phi(x)$ of equation (59) in the space $\mathcal{L}_q^1[0,a]$ for some $a>0$ is given by*

$$\phi(x)\sim cx^{-\alpha}+\sum_{k=0}^{\infty}\varphi_k x^{\alpha k}, \quad (74)$$

where c is an arbitrary constant and $\varphi_k\,(k\in\mathbb{N}_0)$ are found from (72). Assume there exists a number $j\in\{-1,0,1,....\}$ such that

$$(1-\Gamma_q(1-\alpha)\lambda_{-1})\varphi_{-1}=0\ \ (j=-1)$$

and

$$\sum_{i=-1}^{j-1}\frac{\Gamma_q(\alpha j+1)\lambda_{j-i-1}}{\Gamma_q(\alpha i+\alpha+1)}\varphi_i+f_j=0\quad (j\in\mathbb{N}_0). \quad (75)$$

If the coefficients $f_k\,(k=0,1,....,j)$ in the asymptotic expansion (70) satisfy the relation

$$\sum_{i=0}^{k-1}\frac{\Gamma_q(\alpha k+1)\lambda_{k-i-1}}{\Gamma_q(\alpha i+\alpha+1)}\varphi_i+f_k=0, \quad (76)$$

then the unique power asymptotic solution $\phi(x)$ of equation (59) is given by

$$\phi(x)\sim cx^{\alpha j}+\sum_{\substack{k=0\\k\neq j}}^{\infty}\varphi_k x^{\alpha k}, \quad (77)$$

where c is an arbitrary constant.

Proof. If condition (62) is not valid, then there exists a number $j\in\{-1,0,1,....\}$ such that (67) holds. Then (68) has the form (75). When $\lambda_{-1}=1$, the asymptotic solution $\phi(x)$ of equation (59) has the form (74), where c is an arbitrary constant and $\varphi_k\,(k=0,1,2,...)$ are found from (72). If $\lambda_{-1}\neq 1$ and (75) holds, then $\varphi_{-1}=0$ and coefficients $f_k\,(k=0,1,....,j)$ in the asymptotic expansion (70) satisfy the relation (76), where $\varphi_i\,(i=0,1,2,...,j-1)$ are expressed via $f_i\,(i=0,1,....,j-1)$ by formulas (72). In this case the the asymptotic solution $\phi(x)$ of equation (59) has the form (77), where c is an arbitrary constant. □

Theorem 5.5. *Let the functions $f(x)$ and $\lambda(x)$ have asymptotic of the forms (70) and (61), respectively. Then the unique power asymptotic solution $\phi(x)$ of equation (59) with any $\alpha>0$ is given by (71), where $\varphi_k\ (k=0,1,....)$ are found from*

$$\varphi_{-1}=0,\ \varphi_0=f_0,\ \varphi_k=\sum_{i=0}^{k-1}\frac{\lambda_{k-1-i}\Gamma_q(\alpha i+1)}{\Gamma_q(\alpha(i+1)+1)}\varphi_i+f_k. \quad (78)$$

Proof. This proof according to Theorem 5.4. □

Now, we use Theorem 5.5 to give asymptotics of $\phi(x)$ as $x \to 0$ when $f(x)$ has the asymptotic (70).

Corollary 5.6. *The asymptotic solution of the linear Volterra q-integral equation*

$$\phi(x) = \frac{\lambda x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}\phi(x)\, d_qt + f(x)\ (x>0)$$

is given by

$$\phi(x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{\Gamma_q(\alpha n+1)} \left[\sum_{k=0}^{n} f_k \lambda^{-k}\Gamma_q(\alpha k+1)\right] x^{\alpha n}.$$

Proof. We apply Theorem 5.5 with $\lambda(x) = \lambda$. That is in (61)

$$\lambda_k = 0 \text{ for all } k \neq 0 \text{ and } \lambda_0 = 1.$$

Hence, the coefficients φ_k of the solution (71) satisfy the first order difference equation

$$\varphi_k - \frac{\lambda}{\Gamma_q(\alpha k+1)}\varphi_{k-1} = f_k\ (k \in \mathbb{N}), \quad \varphi_0 = f_0.$$

Set $\psi_k = \varphi_k \lambda^{-k}\Gamma_q(\alpha k+1)$ $(k \geq 1)$. Then ψ_k satisfies the difference equation

$$\psi_k - \psi_{k-1} = f_k\lambda^{-k}\Gamma_q(\alpha k+1).$$

Hence $\psi_n = \sum_{k=0}^{n} f_k\lambda^{-k}\Gamma_q(\alpha k+1)$. Consequently,

$$\varphi_n = \frac{\lambda^n}{\Gamma_q(\alpha n+1)} \sum_{k=0}^{n} f_k\lambda^{-k}\Gamma_q(\alpha k+1),$$

which proves the Corollary. □

Example 5.7. Equation (59) with $\lambda(x) = \lambda x^{\alpha(m-1)}$

$$\phi(x) = \frac{\lambda x^{\alpha m-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}\, \phi(t)\, d_qt\ + f(x) \tag{79}$$

$(0 < x < \infty,\ 0 < \alpha < 1,\ m = 1, 2, ...;\ \lambda \neq 0)$ and $f(x)$ has the asymptotic (60). In this case $f(x)$ has the form

$$f(x) = f_{-1}x^{-\alpha} + f_0(x^{\alpha}),$$

$f_0(z) := \sum_{k=0}^{\infty} f_k z^k$ is an entire function in z^{α}. Hence,

$$\lambda_{m-1} = \lambda,\ \lambda_k = 0 \quad (k = -1, 0, 1, ...; k \neq m-1), \tag{80}$$

in (61) and therefore the relation (78) takes the form

$$\varphi_k = f_k \quad (k = -1, 0, 1, ..., m-2), \tag{81}$$

$$\varphi_k = \frac{\Gamma_q(\alpha k+1)\lambda}{\Gamma_q[\alpha(k-m+1)+1]}\varphi_{k-m} + f_k \quad (k = m-1, m, ...).$$

Thus we obtain for $k = -1, 0, 1, ..., m-2; n = 1, 2, ...,$ that

$$\varphi_{nm+k} = \frac{\Gamma_q[\alpha(nm+k)+1]}{\Gamma_q[\alpha(nm+k-m+1)+1]}\lambda\varphi_{(n-1)m+k} + f_{nm+k}.$$

The asymptotic of solution $\phi(x)$ of equation (79)

$$\phi(x)\sim\sum_{k=-1}^{m-2}\sum_{n=0}^{\infty}\left[\sum_{j=1}^{n}\lambda^{n-j}\left(\prod_{i=j+1}^{n}\frac{\Gamma_q(\alpha(im+K)+1)}{\Gamma_q(\alpha(im-m+K+1)+1)}\right)\right]\times \frac{x^{\alpha(nm+k)}}{\Gamma_q(\alpha(nm+k)+1)}.$$

Example 5.8. The equation

$$\phi(x)=\frac{\lambda x^{\alpha m-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt\,+\frac{d}{x^\alpha}+be_q(x^\alpha(1-q)) \tag{82}$$

$$(0<x^\alpha(1-q)<1; 0<\alpha<1; m=1,2,\ldots).$$

Hence,

$$f_{-1}=d,\ f_k=\frac{b}{\Gamma_q(k+1)}\,(k=0,1,2,...).$$

Consequently, equation (82) has the asymptotic solution, as $x\to 0$,

$$\begin{aligned}\phi(x)\ &\sim\ d\sum_{n=0}^{\infty}\lambda^n\prod_{i=0}^{n-1}\frac{\Gamma_q[\alpha(im-1)+1]}{\Gamma_q(\alpha im+1)}x^{\alpha(mn-1)}\\ &+\ b\sum_{k=-1}^{m-2}\sum_{n=1}^{\infty}\left[\sum_{j=1}^{n}\frac{\lambda^{n-j}}{\Gamma_q(jm+k+1)}\prod_{i=j}^{n-1}\frac{\Gamma_q(\alpha(im+K)+1)}{\Gamma_q(\alpha(im+K+1)+1)}\right]x^{\alpha(nm+k)}.\end{aligned}$$

6. **Exact solutions of linear equations**

In the section we show that in some cases the asymptotic solution $\phi(x)$ of the linear equation (59) with certain conditions on $\lambda(x)$ and $f(x)$ gives the exact solution. This result is a q-analogue of the result introduced by Saigo and Kilbas in [10]. Consider (59) with $\lambda(x)=\lambda x^{-\alpha}$ and

$$f(x)=f_1x^{-\alpha}+f_0(x^\alpha)$$

where $f_0(z):=\sum_{k=0}^{\infty}f_kz^k$ is an analytic function of z^α in a disk around zero, say $|z^\alpha|<R$. In this case we have the integral equation

$$\phi(x)=\frac{\lambda}{x\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt\,+f(x^\alpha) \tag{83}$$

$$(0<x^\alpha<R, 0<\alpha<1, \lambda\neq 0),$$

That is

$$\lambda_{-1}=\lambda,\ \lambda_k=0\ \ (k\in\mathbb{N}_0)$$

in (61) and therefore the relation in (66) can be simplified to

$$\left[1-\frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\lambda\right]\varphi_k=f_k\quad (k=-1,0,1,...). \tag{84}$$

Condition (62) takes the form

$$\lambda\neq\frac{\Gamma_q(\alpha k+\alpha+1)}{\Gamma_q(\alpha k+1)}\quad (k=-1,0,1,....). \tag{85}$$

Let (85) hold. Then from (84) we obtain

$$\varphi_k = \left[1 - \frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\lambda\right]^{-1} f_k \quad (k=-1,0,1,...),$$

and the asymptotic solution (64) has the form

$$\phi(x) \sim \sum_{k=-1}^{\infty} \left[1 - \frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\lambda\right]^{-1} f_k x^{\alpha k} \text{ as } x \to 0. \tag{86}$$

Since

$$\left[1 - \frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\lambda\right]^{-1} \sim (1-\lambda(1-q)^{\alpha})^{-1} \quad (k \to \infty),$$

the power series on the left hand side of (86) is an analytic function in z^α for $|z^\alpha| < R$ and the asymptotic solution give an exact solution. If (85) does not hold and there exists a number $j \in \{-1,0,1,...\}$ such that

$$\lambda = \frac{\Gamma_q(\alpha j+\alpha+1)}{\Gamma_q(\alpha j+1)}, \tag{87}$$

the condition of the asymptotic solvability (68) takes the simple form

$$f_j = 0, \tag{88}$$

and the asymptotic solution of equation (83) has the form

$$\phi(x) \sim c x^{\alpha j} + \sum_{\substack{k=-1\\k\neq j}}^{\infty} \left[1 - \frac{\Gamma_q(\alpha k+1)\Gamma_q(\alpha j+\alpha+1)}{\Gamma_q(\alpha k+\alpha+1)\Gamma_q(\alpha j+1)}\right]^{-1} f_k x^{\alpha k}. \tag{89}$$

Similarly, the power series on the right hand side of (89) represents an analytic function in z^α for $|z^\alpha| < R$ and it is the exact solution in this case.

In the following examples we get exact solutions of (83) for certain choices of the function $f_0(x^\alpha)$.

Example 6.1. The equation

$$\phi(x) = \frac{\lambda}{x\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt \,+ b x^{\alpha l} \tag{90}$$

$$(0 < x < \infty,\ 0 < \alpha < 1,\ \lambda \neq \frac{\Gamma_q(\alpha l+\alpha+1)}{\Gamma_q(\alpha l+1)})$$

and $l \in \{-1,0,1,.....\}$ has the solution

$$\phi(x) = \left[1 - \frac{\Gamma_q(\alpha l+1)}{\Gamma_q(\alpha l+\alpha+1)}\lambda\right]^{-1} b x^{\alpha l}.$$

It is worth noting that the homogeneous equation

$$\frac{\Gamma_q(\alpha l+1)}{\Gamma_q(\alpha l+\alpha+1)}\phi(x) = \frac{1}{x\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}\,\phi(t)\,d_qt$$

$$(0 < x < \infty, 0 < \alpha < 1),$$

for $l = -1,0,1,.....$ has the solution $\phi(x) = c x^{\alpha l}$ with c is an arbitrary constant.

Example 6.2. The equation

$$\phi(x) = \frac{\lambda}{x\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}\, \phi(t)\, d_qt \,+ \frac{d}{x^\alpha} + be_q(x^\alpha(1-q)) \tag{91}$$

$$(0 < x^\alpha(1-q) < 1, 0 < \alpha < 1),$$

$e_q(x^\alpha(1-q)) = \sum_{j=0}^{\infty} \frac{x^{\alpha j}}{\Gamma_q(j+1)}$, has the solution

$$\phi(x) = \frac{d}{1-\Gamma_q(1-\alpha)\lambda} x^{-\alpha} + b\sum_{k=0}^{\infty} \left[1 - \frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)}\lambda\right]^{-1} \frac{x^{\alpha k}}{\Gamma_q(k+1)}.$$

If (85) holds, the equation

$$\phi(x) = \frac{1}{x\Gamma_q(\alpha)\Gamma_q(1-\alpha)} \int_0^x (qt/x;q)_{\alpha-1}\, \phi(t)\, d_qt \,+ be_q(x^\alpha(1-q))$$

$$(0 < x^\alpha(1-q) < 1, 0 < \alpha < 1),$$

and

$$\phi(x) = \frac{\Gamma_q(\alpha j+\alpha+1)}{x\Gamma_q(\alpha j+1)\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}\, \phi(t)\, d_qt + \frac{d}{x^\alpha} + b\left[e_q(x^\alpha(1-q)) - \frac{x^{\alpha j}}{\Gamma_q(j+1)}\right]$$

$(0 < x < \infty,\ 0 < \alpha < 1,\ j \in \{0,1,2,....\})$ have solution

$$\phi(x) = cx^{-\alpha} + b\sum_{k=0}^{\infty} \left[1 - \frac{\Gamma_q(\alpha k+1)}{\Gamma_q(\alpha k+\alpha+1)\Gamma_q(1-\alpha)}\right]^{-1} \frac{x^{\alpha k}}{\Gamma_q(k+1)},$$

and

$$\begin{aligned}\phi(x) = cx^{-\alpha j} \quad &+ \quad \frac{d\,\Gamma_q(\alpha j+1)x^{-\alpha}}{\Gamma_q(\alpha j+1) - \Gamma_q(1-\alpha)\Gamma_q(\alpha j+\alpha+1)} \\ &+ \quad b\sum_{\substack{k=0\\k\neq j}}^{\infty} \left[1 - \frac{\Gamma_q(\alpha k+1)\Gamma_q(\alpha j+\alpha+1)}{\Gamma_q(\alpha k+\alpha+1)\Gamma_q(\alpha j+1)}\right]^{-1} \frac{x^{\alpha k}}{\Gamma_q(k+1)},\end{aligned}$$

respectively, where c is an arbitrary constant.

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