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CONVOLUTION PROPERTIES FOR SUBCLASSES OF UNIVALENT FUNCTIONS USING SALAGEAN INTEGRAL OPERATOR

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Abstract. Making use of the Salagean integral operator I^n, we defined subclasses of univalent functions and investigated some convolution properties for these subclasses.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and S is the subclass of A which are univalent.

Let Ω be the class of functions w analytic in U, satisfying $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$.

If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$ if there exists a Schwarz function $w \in \Omega$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence, (cf., e.g., [8]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.2}$$

the Hadamard product or convolution of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

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For $f(z) \in A$, Salagean [11] introduced the following differential operator:

$D^0 f(z) = f(z),\ D^1 f(z) = zf'(z), ...,\ D^n f(z) = D(D^{n-1} f(z))\ (n \in \mathbb{N} = \{1, 2, ...\})$.

We note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k = (h_n * f)(z) \quad (f \in A; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \tag{1.4}$$

where

$$h_n(z) = z + \sum_{k=2}^{\infty} k^n z^k \quad (n \in \mathbb{N}_{0;} z \in U). \tag{1.5}$$

Also, Salagean [11] introduced the following integral operator:

$$I^0 f(z) = f(z),\ I^1 f(z) = \int_0^z \frac{f(t)}{t} dt, ...,\ I^n f(z) = I(I^{n-1} f(z))\ (n \in \mathbb{N}).$$

We note that

$$I^n f(z) = z + \sum_{k=2}^{\infty} k^{-n} a_k z^k = (\lambda_n * f)(z) \quad (n \in \mathbb{N}_0), \tag{1.6}$$

where

$$\lambda_n(z) = z + \sum_{k=2}^{\infty} k^{-n} z^k \quad (n \in \mathbb{N}_{0;} z \in U). \tag{1.7}$$

We note that
(i) $I^{-n} f(z) = D^n f(z)\ (n \in \mathbb{N}_0)$ (see [11]) and $I^{-1} f(z) = Df(z)$;
(ii) $((h_n * \lambda_n)(z)) * f(z) = f(z)\ (n \in \mathbb{N}_0)$;
(iii) $z(I^{n+1} f(z))' = I^n f(z)\ (n \in \mathbb{N}_0)$.

With the help of the Salagean integral operator I^n, we say that a function $f \in A$ is in the class $S^n(A, B)$ $(-1 \le B < A \le 1)$ if it satisfying the subordination condition:

$$\frac{I^n f(z)}{I^{n+1} f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (n \in \mathbb{N}_0). \tag{1.8}$$

Let $C^n(A, B)$ denote the class of the functions $f \in A$ satisfying $zf'(z) \in S^n(A, B)$. We note that $S^{-1}(A, B) = S^*(A, B)$ and $C^{-1}(A, B) = C(A, B)$ (see [4], [6], [7] and [12]).

Denote by $S_\lambda^n(A, B)$ the class of functions $f \in A$ satisfying the subordination condition:

$$\frac{1}{\cos \lambda} \left\{ e^{i\lambda} \frac{I^n f(z)}{I^{n+1} f(z)} - i \sin \lambda \right\} \prec \frac{1 + Az}{1 + Bz} \quad (|\lambda| < \frac{\pi}{2}; n \in \mathbb{N}_0), \tag{1.9}$$

and let $C_\lambda^n(A, B)$ be the class of functions $f \in A$ satisfying $zf' \in S_\lambda^n(A, B)$. We note that $S_\lambda^{-1}(A, B) = S^\lambda(A, B)$ (see Nikitin [9] and Aouf [1] with $\alpha = 0$) and $C_\lambda^{-1}(A, B) = C^\lambda(A, B)$ (see Bhoosnurmath and Devadas [2]).

Further, let $M^n(A, B)$ be the class of functions $f \in A$ satisfying the subordination condition:

$$\frac{I^n f(z)}{z} \prec \frac{1 + Az}{1 + Bz} \quad (n \in \mathbb{N}_0), \tag{1.10}$$

and $M_{\sigma}^{n}(A,B)$ $(\sigma \geq 0)$ be the class of functions $f \in A$ satisfying the subordination condition:

$$(1-\sigma)\frac{I^n f(z)}{z} + \sigma\frac{I^{n-1} f(z)}{z} \prec \frac{1+Az}{1+Bz} \quad (n \in \mathbb{N}_0). \tag{1.11}$$

Evidently, $M_0^0(A,B) = M(A,B)$ (see Goel and Mehrok [5]).

Also, we note that

(i) $M_{\sigma}^{n}(1-2\beta,-1) = M_{\sigma}^{n}(\beta)(0 \leq \beta < 1)$ the class of functions $f \in A$ satisfying the condition:

$$Re\left\{(1-\sigma)\frac{I^n f(z)}{z} + \sigma\frac{I^{n-1} f(z)}{z}\right\} > \beta;$$

(ii) $M_{\sigma}^{0}(1-2\beta,-1) = M_{\sigma}(\beta)(0 \leq \beta < 1)$ the class of functions $f \in A$ satisfying the condition:

$$Re\left\{(1-\sigma)\frac{f(z)}{z} + \sigma f'(z)\right\} > \beta.$$

Convolution properties for various subclasses of analytic functions have been obtained by several researchers (see [2], [3], [10], [12], [13]). In this paper, we investigate convolution properties of the classes $S^n(A,B)$, $C^n(A,B)$, $S_{\lambda}^{n}(A,B)$, $C_{\lambda}^{n}(A,B)$, $M^n(A,B)$ and $M_{\sigma}^{n}(A,B)$, respectively, associated with the Salagean integral operator.

2. Main Results

Unless otherwise mentioned, we assume throughout this section that $0 \leq \theta < 2\pi, n \in \mathbb{N}_0, \sigma \geq 0, -1 \leq B < A \leq 1$ and $\lambda_n(z)$ given by (1.7).

Theorem 1. The function $f(z)$ defined by (1.1) is in the class $S^n(A,B)$ if and only if

$$\frac{1}{z}\left[(f * \lambda_{n+1})(z) * \frac{z+Cz^2}{(1-z)^2}\right] \neq 0 \ \ (z \in \mathbb{U}) \tag{2.1}$$

for all $C = C_{\theta} = \dfrac{e^{-i\theta}+A}{(B-A)}$, $\theta \in [0,2\pi)$, and also for $C = -1$.

Proof. First suppose $f(z)$ defined by (1.1) is in the class $S^n(A,B)$, we have

$$\frac{I^n f(z)}{I^{n+1} f(z)} \prec \frac{1+Az}{1+Bz}, \tag{2.2}$$

since the function from the left-hand side of the subordination is analytic in $\mathbb{U}$, it follows $I^{n+1}f(z) \neq 0, z \in \mathbb{U}^* = U\backslash\{0\}$, *i.e.* $\frac{1}{z}I^{n+1}f(z) \neq 0, z \in \mathbb{U}$, this is equivalent to the fact that (2.1) holds for $C = -1$.

From (2.2) according to the subordination of two functions we say that there exists a function $w(z) \in \Omega$, such that

$$\frac{I^n f(z)}{I^{n+1} f(z)} = \frac{1+Aw(z)}{1+Bw(z)} \quad (z \in \mathbb{U}),$$

which is equivalent to

$$\frac{I^n f(z)}{I^{n+1} f(z)} \neq \frac{1+Ae^{i\theta}}{1+Be^{i\theta}} \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi),$$

or

$$\frac{1}{z}\{I^n f(z)(1+Be^{i\theta}) - I^{n+1} f(z)(1+Ae^{i\theta})\} \neq 0. \tag{2.3}$$

Since

$$I^{n+1} f(z) * \frac{z}{(1-z)} = I^{n+1} f(z) \tag{2.4}$$

and

$$I^{n+1} f(z) * \left[\frac{z}{(1-z)^2}\right] = I^n f(z)\ . \tag{2.5}$$

Now from (2.3),(2.4) and (2.5), we obtain

$$= \frac{1}{z}\left[(f * \lambda_{n+1})(z) * \frac{z+Cz^2}{(1-z)^2}\right] \neq 0\ (z \in \mathbb{U}; 0 \leq \theta < 2\pi),$$

which leads to (2.1), which proves the necessary part of Theorem 1.

(ii) Reversely, because the assumption (2.1) holds for $C = -1$, it follows that $\frac{1}{z} I^{n+1} f(z) \neq 0$ for all $z \in \mathbb{U}$, hence the function $\varphi(z) = \frac{I^n f(z)}{I^{n+1} f(z)}$ is analytic in $\mathbb{U}$ (i.e. it is regular at $z_0 = 0$, with $\varphi(0) = 1$).

Since it was shown in the first part of the proof that the assumption (2.1) is equivalent to (2.3), we obtain that

$$\frac{I^n f(z)}{I^{n+1} f(z)} \neq \frac{1+Ae^{i\theta}}{1+Be^{i\theta}} \quad (z \in \mathbb{U}; \theta \in [0, 2\pi)), \tag{2.6}$$

if we denote

$$\psi(z) = \frac{1+Az}{1+Bz},$$

the relation (2.6) shows that $\varphi(\mathbb{U}) \cap \psi(\partial\mathbb{U}) = \emptyset$. Thus, the simply-connected domain $\varphi(\mathbb{U})$ is included in a connected component of $\mathbb{C} \setminus \psi(\partial\mathbb{U})$. From here, using the fact that $\varphi(0) = \psi(0)$ together with the univalence of the function ψ, it follows that $\varphi(z) \prec \psi(z)$, which represents in fact the subordination (2.2), i.e. $f \in S^n(A, B)$.

Theorem 2. The function $f(z)$ defined by (1.1) is in the class $C^n(A, B)$ if and only if

$$\frac{1}{z}\left[(f * \lambda_{n+1})(z) * \frac{z+(1+2C)z^2}{(1-z)^3}\right] \neq 0 \quad (z \in \mathbb{U}) \tag{2.7}$$

for all $C = C_\theta = \frac{e^{-i\theta}+A}{(B-A)}$, $\theta \in [0, 2\pi)$, and also for $C = -1$.

Proof. Set

$$g(z) = \frac{z+Cz^2}{(1-z)^2}$$

and we note that

$$zg'(z) = \frac{z+(1+2C)z^2}{(1-z)^3}. \tag{2.8}$$

From the identity $zf'(z) * g(z) = f(z) * zg'(z)$ $(f, g \in \mathcal{A})$ and the fact that

$$f(z) \in C^n(A, B) \Leftrightarrow zf'(z) \in S^n(A, B).$$

The result follows from Theorem 1.

Remark 1. (i) Putting $n = -1$ and $\mathrm{e}^{i\theta} = \varkappa (0 \leq \theta < 2\pi)$ in Theorem 1 , we obtain the result obtained by Padmanabhan and Ganesan [10, Theorem 2];

(ii) Putting $n=-1, A=1-2\alpha\ (0\leq\alpha<1), B=-1$ and $\mathrm{e}^{-i\theta}=-\varkappa\,(0\leq\theta<2\pi)$ in Theorem 1, we obtain the result obtained by Silverman et al. [13, Theorem 2];

(iii) Putting $n=-1$ and $\mathrm{e}^{i\theta}=\varkappa\,(0\leq\theta<2\pi)$ in Theorem 2, we obtain the result obtained by Padmanabhan and Ganesan [10, Theorem 1;

(iv) Putting $n=-1, A=1-2\alpha\ (0\leq\alpha<1), B=-1$ and $\mathrm{e}^{-i\theta}=-\varkappa\,(0\leq\theta<2\pi)$ in Theorem 2, we obtain the result obtained by Silverman et al. [13, Theorem 1].

Theorem 3. The function $f(z)$ defined by (1.1) is in the class $S_\lambda^n(A,B)$ if and only if

$$\frac{1}{z}\left[(f*\lambda_{n+1})(z)*\frac{z+Ez^2}{(1-z)^2}\right]\neq 0\ \ (z\in\mathbb{U}), \tag{2.9}$$

for all $E=E_\theta=\dfrac{e^{-i\theta}+e^{-i\lambda}(A\cos\lambda+iB\sin\lambda)}{(B-e^{-i\lambda}(A\cos\lambda+iB\sin\lambda)}$, $\theta\in[0,2\pi)$, and also for $E=-1$.

Proof. First suppose $f(z)$ defined by (1.1) is in the class $S_\lambda^n(A,B)$, we have

$$\frac{1}{\cos\lambda}\left\{e^{i\lambda}\frac{I^nf(z)}{I^{n+1}f(z)}-i\sin\lambda\right\}\prec\frac{1+Az}{1+Bz}\qquad(|\lambda|<\frac{\pi}{2};n\in\mathbb{N}_0), \tag{2.10}$$

since the function from the left-hand side of the subordination is analytic in $\mathbb{U}$, it follows $I^{n+1}f(z)\neq 0, z\in\mathbb{U}^*=U\backslash\{0\}$, *i.e.* $\frac{1}{z}I^{n+1}f(z)\neq 0, z\in\mathbb{U}$, this is equivalent to the fact that (2.9) holds for $E=-1$.

From (2.10) according to the subordination of two functions we say that there exists a function $w(z)\in\Omega$, such that

$$\frac{1}{\cos\lambda}\left\{e^{i\lambda}\frac{I^nf(z)}{I^{n+1}f(z)}-i\sin\lambda\right\}=\frac{1+Aw(z)}{1+Bw(z)}\qquad(z\in\mathbb{U}),$$

which is equivalent to

$$\frac{1}{\cos\lambda}\left\{e^{i\lambda}\frac{I^nf(z)}{I^{n+1}f(z)}-i\sin\lambda\right\}\neq\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}\qquad(z\in\mathbb{U};0\leq\theta<2\pi),$$

or

$$\frac{1}{z}\{e^{i\lambda}I^nf(z)(1+Be^{i\theta})-I^{n+1}f(z)[(1+Ae^{i\theta})\cos\lambda+i\sin\lambda(1+Be^{i\theta})]\}\neq 0. \tag{2.11}$$

By simplifying (2.11), we obtain (2.9). This completes the proof of Theorem3.

Theorem 4. The function $f(z)$ defined by (1.1) is in the class $C_\lambda^n(A,B)$ if and only if

$$\frac{1}{z}\left[(f*\lambda_{n+1})(z)*\frac{z+(1+2E)z^2}{(1-z)^3}\right]\neq 0\ \ (z\in\mathbb{U}) \tag{2.12}$$

for all $E=E_\theta=\dfrac{e^{-i\theta}+e^{-i\lambda}(A\cos\lambda+iB\sin\lambda)}{(B-e^{-i\lambda}(A\cos\lambda+iB\sin\lambda)}$, $\theta\in[0,2\pi)$, and also for $E=-1$.

Proof. Set

$$g(z)=\frac{z+Ez^2}{(1-z)^2},$$

and we note that

$$zg'(z)=\frac{z+(1+2E)z^2}{(1-z)^3}.$$

From the identity $zf\prime(z)*g(z)=f(z)*zg'(z)\ \ (f,g\in\mathcal{A})$ and the fact that

$$f(z)\in Q_\lambda^n(A,B)\ \Leftrightarrow zf'(z)\in S_\lambda^n(A,B)\ .$$

The result follows from Theorem 3.

Remark 2. (i) Putting $n=-1$ and $e^{i\theta}=\varkappa\,(0\leq\theta<2\pi)$ in Theorem 3, we obtain the result obtained by Padmanabhan and Ganesan [10, Theorem 4];

(ii) Putting $n=-1, A=1-2\alpha\ (0\leq\alpha<1)\,, B=-1$ and $e^{-i\theta}=-\varkappa\,(0\leq\theta<2\pi)$ in Theorem 3, we obtain the result obtained by Silverman et al. [13, Theorem 4];

(iii) Putting $n=-1$ and $e^{i\theta}=\varkappa\,(0\leq\theta<2\pi)$ in Theorem 4, we obtain the result obtained by Padmanabhan and Ganesan [10, Theorem 3];

(iv) Putting $n=-1, A=1-2\alpha\ (0\leq\alpha<1)\,, B=-1$ and $e^{-i\theta}=-\varkappa\,(0\leq\theta<2\pi)$ in Theorem 4, we obtain the result obtained by Silverman et al. [13, Theorem 3].

Theorem 5. The function $f(z)$ defined by (1.1) is in the class $M^{n}(A,B)$ if and only if

$$\frac{1}{z}\left[(f*\lambda_{n+1})(z)*\frac{z+C(2z^2-z^3)}{(1-z)^2}\right]\neq 0\ \ (z\in\mathbb{U})\,, \tag{2.13}$$

for all $C=C_\theta=\dfrac{e^{-i\theta}+A}{(B-A)}$, $\theta\in[0,2\pi)$, and also for $C=-1$.

Proof. First suppose $f(z)$ defined by (1.1) is in the class $M^{n}(A,B)\,,$ we have

$$\frac{I^n f(z)}{z}\prec\frac{1+Az}{1+Bz}. \tag{2.14}$$

From (2.14) according to the subordination of two functions we say that there exists a function $w(z)\in\Omega$, such that

$$\frac{I^n f(z)}{z}=\frac{1+Aw(z)}{1+Bw(z)}\qquad(z\in\mathbb{U}),$$

which is equivalent to

$$\frac{I^n f(z)}{z}\neq\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}\qquad(z\in\mathbb{U};0\leq\theta<2\pi),$$

or

$$\frac{1}{z}\{I^n f(z)(1+Be^{i\theta})-z(1+Ae^{i\theta})\}\neq 0.$$

Since

$$\frac{1}{z}I^{n+1}f(z)*\left\{(1+Be^{i\theta})\frac{z}{(1-z)^2}-z(1+Ae^{i\theta})\frac{(1-z)^2}{(1-z)^2}\right\}\neq 0$$

then

$$=\frac{1}{z}\left[I^{n+1}f(z)*\frac{z+C(2z^2-z^3)}{(1-z)^2}\right]\neq 0\ (z\in\mathbb{U};0\leq\theta<2\pi),$$

which proves Theorem 5

Theorem 6. The function $f(z)$ defined by (1.1) is in the class $M_{\sigma}^{n}(A,B)$ if and only if

$$\frac{1}{z}\left[(f*\lambda_{n+1})(z)*\frac{z[1-(1-2\sigma)z](1+Be^{i\theta})-z(1-z)^3(1+Ae^{i\theta})}{(1-z)^3}\right]\neq 0\ \ (z\in\mathbb{U})\,. \tag{2.15}$$

Proof. First suppose $f(z)$ defined by (1.1) is in the class $M_{\sigma}^{n}(A,B)\,,$ we have

$$(1-\sigma)\frac{I^n f(z)}{z}+\sigma\frac{I^{n-1}f(z)}{z}\prec\frac{1+Az}{1+Bz}\ \ (\sigma\geq 0;n\in\mathbb{N}_0). \tag{2.16}$$

From (2.16) according to the subordination of two functions we say that there exists a function $w(z) \in \Omega$, such that

$$(1-\sigma)\frac{I^n f(z)}{z} + \sigma\frac{I^{n-1} f(z)}{z} = \frac{1+Aw(z)}{1+Bw(z)} \quad (z \in \mathbb{U})$$

which is equivalent to

$$(1-\sigma)\frac{I^n f(z)}{z} + \sigma\frac{I^{n-1} f(z)}{z} \neq \frac{1+Ae^{i\theta}}{1+Be^{i\theta}} \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi),$$

or

$$\frac{1}{z}\{[(1-\sigma)I^n f(z) + \sigma I^{n-1} f(z) f(z)](1+Be^{i\theta}) - z(1+Ae^{i\theta})\} \neq 0.$$

Since

$$\frac{1}{z}\left(I^{n+1} f(z) * \left\{(1+Be^{i\theta})\left[\frac{(1-\sigma)z}{(1-z)^2} + \frac{\sigma z(1+z)}{(1-z)^3}\right] - z(1+Ae^{i\theta})\frac{(1-z)^3}{(1-z)^3}\right\}\right) \neq 0$$

which proves Theorem 6.

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