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SOME SANDWICH RESULTS FOR HIGHER-ORDER DERIVATIVES OF MULTIVALENT FUNCTIONS INVOLVING A GENERALIZED DIFFERENTIAL OPERATOR

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Abstract. In this paper, we obtain some applications of first order differential subordination, superordination and sandwich results for higher-order derivatives of $p-$valent functions involving a generalized differential operator. Some of our results improve and generalize previously known results.

1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, p]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} ... \; (a \in \mathbb{C};\; p \in \mathbb{N} = \{1, 2, ...\}).$$

For simplicity $H[a] = H[a, 1]$. Also, let $\mathcal{A}(p)$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}), \tag{1}$$

which are $p-$valent in U. We write $\mathcal{A}(1) = \mathcal{A}$.

If $f,\ g \in H(U)$, we say that f is subordinate to g or g is superordinate to f, written $f(z) \prec g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)),\ z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g., [10], [17] and [18]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h be univalent function in U. If β is analytic function in U and satisfies the first order differential subordination:

$$\phi\left(\beta(z), z\beta'(z); z\right) \prec h(z), \tag{2}$$

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then β is a solution of the differential subordination (2). The univalent function q is called a dominant of the solutions of the differential subordination (2) if $\beta(z) \prec q(z)$ for all β satisfying (2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (2) is called the best dominant. If β and ϕ are univalent functions in U and if satisfies first order differential superordination:

$$h(z) \prec \phi\left(\beta(z), z\beta'(z); z\right), \tag{3}$$

then β is a solution of the differential superordination (3). An analytic function q is called a subordinant of the solutions of the differential superordination (3) if $q(z) \prec \beta(z)$ for all β satisfying (3). A univalent subordinant $\tilde{q}$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants of (3) is called the best subordinant.

Using the results of Miller and Mocanu [18], Bulboaca [9] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [10]. Ali et al. [1], have used the results of Bulboaca [9] to obtain sufficient conditions for normalized analytic functions $f \in \mathcal{A}$ to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [23] obtained a sufficient condition for starlikeness of $f \in \mathcal{A}$ in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [22] obtained sufficient conditions for the normalized analytic function $f \in \mathcal{A}$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

For functions $f \in \mathcal{A}(p)$ given by (1) and $g \in \mathcal{A}(p)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \quad (p \in \mathbb{N}), \tag{4}$$

the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z^p +_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{5}$$

Upon differentiating both sides of (5) $j-$times with respect to z, we have

$$(f * g)^{(j)}(z) = \delta(p; j) z^{p-j} + \sum_{k=p+1}^{\infty} \delta(k; j) a_k b_k z^{k-j}, \tag{6}$$

where

$$\delta(p; j) = \frac{p!}{(p-j)!} \quad (p > j; p \in \mathbb{N}; j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{7}$$

For functions $f, g \in \mathcal{A}(p)$, we define the linear operator $D_{\lambda,p}^n (f * g)^{(j)} : \mathcal{A}(p) \to \mathcal{A}(p)$ by:

$$D_{\lambda,p}^0 (f * g)^{(j)}(z) = (f * g)^{(j)}(z),$$

$$\begin{aligned} D^1_{\lambda,p}(f*g)^{(j)}(z) &= D_{\lambda,p}(f*g)^{(j)}(z) \\ &= (1-\lambda)(f*g)^{(j)}(z)+\frac{\lambda}{p-j}z\left((f*g)^{(j)}\right)'(z) \\ &= \delta(p;j)z^{p-j}+\sum_{k=p+1}^{\infty}\left(\frac{p-j+\lambda(k-p)}{p-j}\right)\delta(k;j)a_kb_kz^{k-j}, \end{aligned}$$

$$\begin{aligned} D^2_{\lambda,p}(f*g)^{(j)}(z) &= D\left(D^1_p(f*g)^{(j)}(z)\right) \\ &= \delta(p;j)z^{p-j}+\sum_{k=p+1}^{\infty}\left(\frac{p-j+\lambda(k-p)}{p-j}\right)^2\delta(k;j)a_kb_kz^{k-j}, \end{aligned}$$

and (in general)

$$\begin{aligned} D^n_{\lambda,p}(f*g)^{(j)}(z) &= D(D^{n-1}_p(f*g)^{(j)}(z)) \\ &= \delta(p;j)z^{p-j}+\sum_{k=p+1}^{\infty}\left(\frac{p-j+\lambda(k-p)}{p-j}\right)^n\delta(k;j)a_kb_kz^{k-j} \\ &\quad (\lambda\geq 0;p>j;p\in\mathbb{N};j,n\in\mathbb{N}_0;z\in U). \end{aligned} \tag{8}$$

From (8), we can easily deduce that

$$\begin{aligned} \frac{\lambda z}{p-j}\left(D^n_{\lambda,p}(f*g)^{(j)}(z)\right)' &= D^{n+1}_{\lambda,p}(f*g)^{(j)}(z)\ -(1-\lambda)D^n_{\lambda,p}(f*g)^{(j)}(z) \\ &\quad (\lambda>0;p>j;p\in\mathbb{N};n,j\in\mathbb{N}_0;z\in U). \end{aligned} \tag{9}$$

We observe that the linear operator $D^n_{\lambda,p}(f*g)^{(j)}(z)$ reduces to several interesting many other linear operators considered earlier for different choices of j,n,λ and the function g:

(i) For $j=0$, $D^n_{\lambda,p}(f*g)^{(0)}(z)=D^n_{\lambda,p}(f*g)(z)$, where the operator $D^n_{\lambda,p}(f*g)$ $(\lambda\geq 0,\ p\in\mathbb{N},n\in\mathbb{N}_0)$ was introduced and studied by Selvaraj et al. [21] (see also [8]) and $D^n_{\lambda,1}(f*g)(z)=D^n_{\lambda}(f*g)(z)$, where the operator $D^n_{\lambda}\ (f*g)$ was introduced by Aouf and Mostafa [6];

(ii) For

$$g(z)=\frac{z^p}{1-z}\ \ (p\in\mathbb{N};z\in U\ ) \tag{10}$$

we have $D^n_{\lambda,p}(f*g)^{(j)}(z)=D^n_{\lambda,p}f^{(j)}(z)$, $D^n_{\lambda,p}f^{(0)}(z)=D^n_{\lambda,p}f(z)$, where the operator $D^n_{\lambda,p}$ is the $p-$valent Al-Oboudi operator which was introduced by El-Ashwah and Aouf [13], $D^n_{1,p}f^{(j)}(z)=D^n_pf^{(j)}(z)$, where the operator $D^n_pf^{(j)}$ $(p>j,p\in\mathbb{N},n,j\in\mathbb{N}_0)$ was introduced and studied by Aouf [[3], [4]] (see also [7]) and $D^n_{1,p}f^{(0)}(z)=D^n_pf(z)$, where the operator D^n_p is the $p-$valent Sălăgean operator which was introduced and studied by Kamali and Orhan [14] (see also [5]);

(iii) For

$$g(z)=z^p+\sum_{k=p+1}^{\infty}\frac{(\alpha_1)_{k-p}...(\alpha_q)_{k-p}}{(\beta_1)_{k-p}...(\beta_s)_{k-p}}\frac{z^k}{(1)_{k-p}}\qquad (z\in U), \tag{11}$$

(for complex parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ ($\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}$, $j = 1, ..., s$); $q \leq s+1; p \in \mathbb{N}$; $q,\ s \in \mathbb{N}_0$) where $(\nu)_k$ is the Pochhammer symbol defined in terms to the Gamma function Γ, by

$$(\nu)_k = \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1, & (k=0), \\ \nu(\nu+1)(\nu+2)...(\nu+k-1), & (k \in \mathbb{N}), \end{cases}$$

we have $D_{\lambda,p}^n (f * g)^{(j)}(z) = D_{\lambda,p}^n (H_{p,q,s}(\alpha_1)f)^{(j)}(z)$ and $D_{\lambda,p}^0 (f * g)^{(0)}(z) = H_{p,q,s}(\alpha_1)f(z)$, where the operator $H_{p,q,s}(\alpha_1) = H_{p,q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)$ is the Dziok-Srivastava operator which was introduced and studied by Dziok and Srivastava [12] and which contains in turn many interesting operators;

(iv) For

$$g(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+l+\alpha(k-p)}{p+l} \right)^m z^k \tag{12}$$
$$(\alpha \geq 0;\ l \geq 0;\ p \in \mathbb{N};\ m \in \mathbb{N}_0; z \in U\ ),$$

we have $D_{\lambda,p}^n (f * g)^{(j)}(z) = D_{\lambda,p}^n (I_p(m,\alpha,l)f)^{(j)}(z)$ and $D_{\lambda,p}^0 (f * g)^{(0)}(z) = I_p(m,\alpha,l)f(z)$, where the operator $I_p(m,\alpha,l)$ was introduced and studied by Cătas [11] and which contains in turn many interesting operators such as, $I_p(m,1,l) = I_p(m,l)$, where the operator $I_p(m,l)$ was investigated by Kumar et al. [15];

(v) For

$$g(z) = z^p + \frac{\Gamma(p+\alpha+\beta)}{\Gamma(p+\beta)} {}_{k=p+1}^{\infty} \frac{\Gamma(k+\beta)}{\Gamma(k+\alpha+\beta)} z^k \tag{13}$$
$$(\ \alpha \geq 0;\ p \in \mathbb{N};\ \beta > -1; z \in U\ )$$

we have $D_{\lambda,p}^n (f * g)^{(j)}(z) = D_{\lambda,p}^n \left(Q_{\beta,p}^{\alpha} f \right)^{(j)}(z)$ and $D_{\lambda,p}^0 (f * g)^{(0)}(z) = Q_{\beta,p}^{\alpha} f(z)$, where the operator $Q_{\beta,p}^{\alpha}$ was introduced and studied by Liu and Owa [16];

(vi) For $j = 0$ and g of the form (11) with $p = 1$, we have $D_{\lambda,1}^n (f * g)(z) = D_\lambda^n(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)(z)$, where the operator $D_\lambda^n(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)$ was introduced and studied by Selvaraj and Karthikeyan [20];

(vii) For $j = 0$, $p = 1$ and

$$g(z) = z +_{k=2}^{\infty} \left[\frac{\Gamma(k+1)\Gamma(2-m)}{\Gamma(k+1-m)} \right]^n z^k \tag{14}$$
$$(n \in \mathbb{N}_0; 0 \leq\ m < 1; z \in U\ )$$

we have $D_{\lambda,1}^n (f * g)(z) = D_\lambda^{n,m} f(z)$, where the operator $D_\lambda^{n,m}$ was introduced and studied by Al-Oboudi and Al-Amoudi [2].

In this paper, we will derive several subordination, superordination and sandwich results involving the operator $D_{\lambda,p}^n (f * g)^{(j)}$.

2. Definitions and preliminaries

In order to prove our results, we need the following definition and lemmas.

Definition 2.1 [18]. Denote by Q, the set of all functions f that are analytic and injective on $\overline{U}\backslash E(f)$, where

$$E(f)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}f\left(z\right)=\infty\right\},$$

and are such that $f^{'}\left(\zeta\right)\neq 0$ for $\zeta\in\partial U\backslash E\left(f\right)$.

Lemma 2.1 [22]. Let q be univalent function in U with $q(0)=1$. Let $\gamma_i\in\mathbb{C}(i=1,2)$, $\gamma_2\neq 0$, further assume that

$$\Re\left\{1+\frac{zq^{''}\left(z\right)}{q^{'}\left(z\right)}\right\}>\max\left\{0,-\Re\left(\frac{\gamma_1}{\gamma_2}\right)\right\}. \tag{15}$$

If β is analytic function in U, and

$$\gamma_1\beta\left(z\right)+\gamma_2 z\beta^{'}\left(z\right)\prec\gamma_1 q\left(z\right)+\gamma_2 zq^{'}\left(z\right),$$

then $\beta\prec q$ and q is the best dominant.

Lemma 2.2 [22]. Let q be convex univalent function in U, $q(0)=1$. Let $\gamma_i\in\mathbb{C}(i=1,2)$, $\gamma_2\neq 0$ and $\Re\left(\frac{\gamma_1}{\gamma_2}\right)>0$. If $\beta\in H[q(0),1]\cap Q$, $\gamma_1\beta\left(z\right)+\gamma_2 z\beta^{'}\left(z\right)$ is univalent in U and

$$\gamma_1 q\left(z\right)+\gamma_2 zq^{'}\left(z\right)\prec\gamma_1\beta\left(z\right)+\gamma_2 z\beta^{'}\left(z\right), \tag{16}$$

then $q\prec\beta$ and q is the best subordinant.

3. Subordination resuts

Unless otherwise mentioned, we assume throughout this paper that $\gamma\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}$, $\lambda\geq 0$, $p>j$, $p\in\mathbb{N}$, $n,j\in\mathbb{N}_0$ and $\delta\left(p;j\right)$ is given by (7).

Theorem 3.1. Let q be univalent in U with $q(0)=1$ and assume that

$$\Re\left\{1+\frac{zq^{''}\left(z\right)}{q^{'}\left(z\right)}\right\}>\max\left\{0,-\Re\left(\frac{1}{\gamma}\right)\right\}. \tag{17}$$

If $f\in\mathcal{A}\left(p\right)$ satisfy the following subordination condition:

$$\frac{D_{\lambda,p}^{n}\left(f*g\right)^{(j)}\left(z\right)}{D_{\lambda,p}^{n+1}\left(f*g\right)^{(j)}\left(z\right)}+\gamma\frac{(p-j)}{\lambda}\left\{1-\frac{D_{\lambda,p}^{n}\left(f*g\right)^{(j)}\left(z\right)D_{\lambda,p}^{n+2}\left(f*g\right)^{(j)}\left(z\right)}{\left[D_{\lambda,p}^{n+1}\left(f*g\right)^{(j)}\left(z\right)\right]^2}\right\}$$
$$\prec\quad q\left(z\right)+\gamma zq^{'}\left(z\right), \tag{18}$$

then

$$\frac{D_{\lambda,p}^{n}\left(f*g\right)^{(j)}\left(z\right)}{D_{\lambda,p}^{n+1}\left(f*g\right)^{(j)}\left(z\right)}\prec q\left(z\right)$$

and q is the best dominant.

Proof. Define a function β by

$$\beta\left(z\right)=\frac{D_{\lambda,p}^{n}\left(f*g\right)^{(j)}\left(z\right)}{D_{\lambda,p}^{n+1}\left(f*g\right)^{(j)}\left(z\right)}\qquad\left(z\in U\right). \tag{19}$$

Then the function β is analytic in U and $\beta(0) = 1$. Therefore, differentiating (19) logarithmically with respect to z and using the identity (9) in the resulting equation, we have

$$\begin{aligned} & \frac{D_{\lambda,p}^{n}\left(f*g\right)^{(j)}(z)}{D_{\lambda,p}^{n+1}\left(f*g\right)^{(j)}(z)}+\gamma\frac{(p-j)}{\lambda}\left\{1-\frac{D_{\lambda,p}^{n}\left(f*g\right)^{(j)}(z)D_{\lambda,p}^{n+2}\left(f*g\right)^{(j)}(z)}{\left[D_{\lambda,p}^{n+1}\left(f*g\right)^{(j)}(z)\right]^{2}}\right\} \\ = & \quad \beta(z)+\gamma z\beta^{'}(z), \end{aligned}$$

that is,

$$\beta(z)+\gamma z\beta^{'}(z)\prec q(z)+\gamma zq^{'}(z).$$

Therefore, Theorem 3.1 now follows by applying Lemma 2.1. □

Putting $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3.1, it easy to check that the assumption (17) holds whenever $-1 \leq B < A \leq 1$, hence we obtain the following corollary.

Corollary 3.1. Let $-1 \leq B < A \leq 1$ and assume that

$$\Re\left\{\frac{1-Bz}{1+Bz}\right\}>\max\left\{0,-\Re\left(\frac{1}{\gamma}\right)\right\}.$$

If $f \in \mathcal{A}(p)$ satisfy the following subordination condition:

$$\begin{aligned} & \frac{D_{\lambda,p}^{n}\left(f*g\right)^{(j)}(z)}{D_{\lambda,p}^{n+1}\left(f*g\right)^{(j)}(z)}+\gamma\frac{(p-j)}{\lambda}\left\{1-\frac{D_{\lambda,p}^{n}\left(f*g\right)^{(j)}(z)D_{\lambda,p}^{n+2}\left(f*g\right)^{(j)}(z)}{\left[D_{\lambda,p}^{n+1}\left(f*g\right)^{(j)}(z)\right]^{2}}\right\} \\ \prec & \quad \frac{1+Az}{1+Bz}+\gamma\frac{(A-B)z}{(1+Bz)^{2}}, \end{aligned}$$

then

$$\frac{D_{\lambda,p}^{n}\left(f*g\right)^{(j)}(z)}{D_{\lambda,p}^{n+1}\left(f*g\right)^{(j)}(z)}\prec\frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominant.

Taking $g = \frac{z^p}{1-z}$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.2. Let q be univalent in U with $q(0) = 1$ and assume that (17) holds. If $f \in \mathcal{A}(p)$ satisfies the following subordination condition:

$$\begin{aligned} & \frac{D_{\lambda,p}^{n}f^{(j)}(z)}{D_{\lambda,p}^{n+1}f^{(j)}(z)}+\gamma\frac{(p-j)}{\lambda}\left\{1-\frac{D_{\lambda,p}^{n}f^{(j)}(z)D_{\lambda,p}^{n+2}f^{(j)}(z)}{\left[D_{\lambda,p}^{n+1}f^{(j)}(z)\right]^{2}}\right\} \\ \prec & \quad q(z)+\gamma zq^{'}(z), \end{aligned}$$

then

$$\frac{D_{\lambda,p}^{n}f^{(j)}(z)}{D_{\lambda,p}^{n+1}f^{(j)}(z)}\prec q(z)$$

and q is the best dominant.

Remark 3.1. Taking $\lambda = 1$ in Corollary 3.2, we obtain the result obtained by Aouf and Seoudy [[7], Theorem 1].

Taking $p=\lambda=1$, $j=0$ and $g=\frac{z}{1-z}$ in Theorem 3.1, we obtain the following corollary which improves the result obtained by Shanmugam et al. [[22], Theorem 5.1] and also obtained by Nechita [[19], Corollary 7].

Corollary 3.3. Let q be univalent in U with $q(0)=1$ and assume that (17) holds. If $f\in\mathcal{A}$ satisfies the following subordination condition:

$$\frac{D^n f(z)}{D^{n+1}f(z)}+\gamma\left\{1-\frac{D^n f(z)D^{n+2}f(z)}{\left[D^{n+1}f(z)\right]^2}\right\}\prec q\left(z\right)+\gamma zq^{'}\left(z\right),$$

then

$$\frac{D^n f(z)}{D^{n+1}f(z)}\prec q\left(z\right)$$

and q is the best dominant.

Remark 3.2. Taking $n=0$ in Corollary 3.3, we obtain the result obtained by Shanmugam et al. [[22], Theorem 3.1].

4. SUPERORDINATION RESULTS

Now, by appealing to Lemma 2.2 it can be easily prove the following theorem.

Theorem 4.1. Let q be convex univalent in U with $q\left(0\right)=1$ and $\Re\left(\frac{1}{\gamma}\right)>0$. If $f\in\mathcal{A}\left(p\right)$ such that $\frac{D^n_{\lambda,p}(f*g)^{(j)}(z)}{D^{n+1}_{\lambda,p}(f*g)^{(j)}(z)}\in H\left[q\left(0\right),1\right]\cap Q$,

$$\frac{D^n_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)}{D^{n+1}_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)}+\gamma\frac{(p-j)}{\lambda}\left\{1-\frac{D^n_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)D^{n+2}_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)}{\left[D^{n+1}_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)\right]^2}\right\}$$

is univalent in U and the following superordination condition

$$\begin{aligned}&q\left(z\right)+\gamma zq^{'}\left(z\right)\\ \prec\ &\frac{D^n_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)}{D^{n+1}_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)}+\gamma\frac{(p-j)}{\lambda}\left\{1-\frac{D^n_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)D^{n+2}_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)}{\left[D^{n+1}_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)\right]^2}\right\}\end{aligned}$$

holds, then

$$q\left(z\right)\prec\frac{D^n_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)}{D^{n+1}_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)}$$

and q is the best subordinant.

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in Theorem 4.1, we have the following corollary.

Corollary 4.1. Let $\Re\left(\frac{1}{\gamma}\right)>0$ and $f\in\mathcal{A}\left(p\right)$ such that $\frac{D^n_{\lambda,p}(f*g)^{(j)}(z)}{D^{n+1}_{\lambda,p}(f*g)^{(j)}(z)}\in H\left[q\left(0\right),1\right]\cap Q$,

$$\frac{D^n_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)}{D^{n+1}_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)}+\gamma\frac{(p-j)}{\lambda}\left\{1-\frac{D^n_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)D^{n+2}_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)}{\left[D^{n+1}_{\lambda,p}\left(f*g\right)^{(j)}\left(z\right)\right]^2}\right\}$$

is univalent in U and the following superordination condition

$$\frac{1+Az}{1+Bz}+\gamma\frac{(A-B)\,z}{(1+Bz)^2}$$
$$\prec \frac{D_{\lambda,p}^{n}(f*g)^{(j)}(z)}{D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}+\gamma\frac{(p-j)}{\lambda}\left\{1-\frac{D_{\lambda,p}^{n}(f*g)^{(j)}(z)D_{\lambda,p}^{n+2}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)\right]^2}\right\}$$

holds, then

$$\frac{1+Az}{1+Bz}\prec\frac{D_{\lambda,p}^{n}(f*g)^{(j)}(z)}{D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

Taking $g=\frac{z^p}{1-z}$ in Theorem 4.1, we obtain the following corollary.

Corollary 4.2. Let q be convex univalent in U with $q(0)=1$ and $\Re\left(\frac{1}{\gamma}\right)>0$. If $f\in\mathcal{A}(p)$ such that $\frac{D_{\lambda,p}^{n}f^{(j)}(z)}{D_{\lambda,p}^{n+1}f^{(j)}(z)}\in H[q(0),1]\cap Q$,

$$\frac{D_{\lambda,p}^{n}f^{(j)}(z)}{D_{\lambda,p}^{n+1}f^{(j)}(z)}+\gamma\frac{(p-j)}{\lambda}\left\{1-\frac{D_{\lambda,p}^{n}f^{(j)}(z)D_{\lambda,p}^{n+2}f^{(j)}(z)}{\left[D_{\lambda,p}^{n+1}f^{(j)}(z)\right]^2}\right\}$$

is univalent in U and the following superordination condition

$$q(z)+\gamma zq'(z)$$
$$\prec \frac{D_{\lambda,p}^{n}f^{(j)}(z)}{D_{\lambda,p}^{n+1}f^{(j)}(z)}+\gamma\frac{(p-j)}{\lambda}\left\{1-\frac{D_{\lambda,p}^{n}f^{(j)}(z)D_{\lambda,p}^{n+2}f^{(j)}(z)}{\left[D_{\lambda,p}^{n+1}f^{(j)}(z)\right]^2}\right\}$$

holds, then

$$q(z)\prec\frac{D_{\lambda,p}^{n}f^{(j)}(z)}{D_{\lambda,p}^{n+1}f^{(j)}(z)}$$

and q is the best subordinant.

Remark 4.1. Taking $\lambda=1$ in Corollary 4.2, we obtain the result obtained by Aouf and Seoudy [[7], Theorem 2].

Taking $p=\lambda=1$, $j=0$ and $g=\frac{z}{1-z}$ in Theorem 4.1, we obtain the following result which improves the result obtained by Shanmugam et al. [[22], Theorem 5.2] and also obtained by Nechita [[19], Corollary 12].

Corollary 4.3 . Let q be convex univalent in U with $q(0)=1$ and $\Re\left(\frac{1}{\gamma}\right)>0$. If $f\in\mathcal{A}$ such that $\frac{D^nf(z)}{D^{n+1}f(z)}\in H[q(0),1]\cap Q$,

$$\frac{D^nf(z)}{D^{n+1}f(z)}+\gamma\left\{1-\frac{D^nf(z).D^{n+2}f(z)}{\left[D^{n+1}f(z)\right]^2}\right\}$$

is univalent in U and the following superordination condition

$$q(z)+\gamma zq'(z)\prec\frac{D^nf(z)}{D^{n+1}f(z)}+\gamma\left\{1-\frac{D^nf(z).D^{n+2}f(z)}{\left[D^{n+1}f(z)\right]^2}\right\}$$

holds, then

$$q(z) \prec \frac{D^n f(z)}{D^{n+1} f(z)}$$

and q is the best subordinant.

Remark 4.2. Taking $n = 0$ in Corollary 4.3, we obtain the result obtained by Shanmugam et al. [[22], Theorem 3.2].

5. SANDWICH RESUTS

Combining Theorem 3.1 and Theorem 4.1, we get the following sandwich theorem for the linear operator $D^n_{\lambda,p}(f*g)^{(j)}$.

Theorem 5.1. Let q_1 be convex univalent in U with $q_1(0) = 1$, $\Re\left(\frac{1}{\gamma}\right) > 0$, q_2 be univalent in U with $q_2(0) = 1$ and satisfies (17). If $f \in \mathcal{A}(p)$ such that $\frac{D^n_{\lambda,p}(f*g)^{(j)}(z)}{D^{n+1}_{\lambda,p}(f*g)^{(j)}(z)} \in H[q(0), 1] \cap Q$,

$$\frac{D^n_{\lambda,p}(f*g)^{(j)}(z)}{D^{n+1}_{\lambda,p}(f*g)^{(j)}(z)} + \gamma\frac{(p-j)}{\lambda}\left\{1 - \frac{D^n_{\lambda,p}(f*g)^{(j)}(z) D^{n+2}_{\lambda,p}(f*g)^{(j)}(z)}{\left[D^{n+1}_{\lambda,p}(f*g)^{(j)}(z)\right]^2}\right\}$$

is univalent in U and

$$\begin{aligned} & q_1(z) + \gamma z q_1'(z) \\ \prec\ & \frac{D^n_{\lambda,p}(f*g)^{(j)}(z)}{D^{n+1}_{\lambda,p}(f*g)^{(j)}(z)} + \gamma\frac{(p-j)}{\lambda}\left\{1 - \frac{D^n_{\lambda,p}(f*g)^{(j)}(z) D^{n+2}_{\lambda,p}(f*g)^{(j)}(z)}{\left[D^{n+1}_{\lambda,p}(f*g)^{(j)}(z)\right]^2}\right\} \\ \prec\ & q_2(z) + \gamma z q_2'(z) \end{aligned}$$

holds, then

$$q_1(z) \prec \frac{D^n_{\lambda,p}(f*g)^{(j)}(z)}{D^{n+1}_{\lambda,p}(f*g)^{(j)}(z)} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Taking $q_i(z) = \frac{1+A_i z}{1+B_i z}$ $(i = 1, 2; -1 \le B_2 \le B_1 < A_1 \le A_2 \le 1)$ in Theorem 5.1, we obtain the following corollary.

Corollary 5.1. Let $\Re\left(\frac{1}{\gamma}\right) > 0$ and $f \in \mathcal{A}(p)$ such that $\frac{D^n_{\lambda,p}(f*g)^{(j)}(z)}{D^{n+1}_{\lambda,p}(f*g)^{(j)}(z)} \in H[q(0), 1] \cap Q$,

$$\frac{D^n_{\lambda,p}(f*g)^{(j)}(z)}{D^{n+1}_{\lambda,p}(f*g)^{(j)}(z)} + \gamma\frac{(p-j)}{\lambda}\left\{1 - \frac{D^n_{\lambda,p}(f*g)^{(j)}(z) D^{n+2}_{\lambda,p}(f*g)^{(j)}(z)}{\left[D^{n+1}_{\lambda,p}(f*g)^{(j)}(z)\right]^2}\right\}$$

is univalent in U and

$$\begin{aligned} & \frac{1+A_1z}{1+B_1z}+\gamma\frac{(A_1-B_1)\,z}{(1+B_1z)^2} \\ \prec\ & \frac{D_{\lambda,p}^{n}\,(f*g)^{(j)}\,(z)}{D_{\lambda,p}^{n+1}\,(f*g)^{(j)}\,(z)}+\gamma\frac{(p-j)}{\lambda}\left\{1-\frac{D_{\lambda,p}^{n}\,(f*g)^{(j)}\,(z)D_{\lambda,p}^{n+2}\,(f*g)^{(j)}\,(z)}{\left[D_{\lambda,p}^{n+1}\,(f*g)^{(j)}\,(z)\right]^2}\right\} \\ \prec\ & \frac{1+A_2z}{1+B_2z}+\gamma\frac{(A_2-B_2)\,z}{(1+B_2z)^2} \end{aligned}$$

holds, then

$$\frac{1+A_1z}{1+B_1z}\prec\frac{D_{\lambda,p}^{n}\,(f*g)^{(j)}\,(z)}{D_{\lambda,p}^{n+1}\,(f*g)^{(j)}\,(z)}\prec\frac{1+A_2z}{1+B_2z},$$

$\frac{1+A_1z}{1+B_1z}$ and $\frac{1+A_2z}{1+B_2z}$ are, respectively, the best subordinant and the best dominant.

Taking $g=\frac{z^p}{1-z}$ in Theorem 5.1, we obtain the following corollary.

Corollary 5.2. Let q_1 be convex univalent in U with $q_1\,(0)=1$, $\Re\left(\frac{1}{\gamma}\right)>0$, q_2 be univalent in U with $q_2\,(0)=1$ and satisfies (17). If $f\in\mathcal{A}\,(p)$ such that $\frac{D_{\lambda,p}^{n}f^{(j)}(z)}{D_{\lambda,p}^{n+1}f^{(j)}(z)}\in H\,[q\,(0)\,,1]\cap Q$,

$$\frac{D_{\lambda,p}^{n}f^{(j)}(z)}{D_{\lambda,p}^{n+1}f^{(j)}(z)}+\gamma\frac{(p-j)}{\lambda}\left\{1-\frac{D_{\lambda,p}^{n}f^{(j)}(z)D_{\lambda,p}^{n+2}f^{(j)}(z)}{\left[D_{\lambda,p}^{n+1}f^{(j)}(z)\right]^2}\right\}$$

is univalent in U and

$$\begin{aligned} & q_1\,(z)+\gamma zq_1^{'}\,(z) \\ \prec\ & \frac{D_{\lambda,p}^{n}f^{(j)}(z)}{D_{\lambda,p}^{n+1}f^{(j)}(z)}+\gamma\frac{(p-j)}{\lambda}\left\{1-\frac{D_{\lambda,p}^{n}f^{(j)}(z)D_{\lambda,p}^{n+2}f^{(j)}(z)}{\left[D_{\lambda,p}^{n+1}f^{(j)}(z)\right]^2}\right\} \\ \prec\ & q_2\,(z)+\gamma zq_2^{'}\,(z) \end{aligned}$$

holds, then

$$q_1\,(z)\prec\frac{D_{\lambda,p}^{n}f^{(j)}(z)}{D_{\lambda,p}^{n+1}f^{(j)}(z)}\prec q_2\,(z)\,,$$

q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Remark 5.1. Taking $\lambda=1$ in Corollary 5.2, we obtain the sandwich result obtained by Aouf and Seoudy [[7], Theorem 3].

Taking $p=\lambda=1$, $j=0$ and $g=\frac{z}{1-z}$ in Theorem 5.1, we obtain the following sandwich result which improves the result obtained by Shanmugam et al. [[22], Theorem 5.3].

Corollary 5.3. Let q_1 be convex univalent in U with $q_1\,(0)=1$, $\Re\left(\frac{1}{\gamma}\right)>0$, q_2 be univalent in U with $q_2\,(0)=1$ and satisfies (17). If $f\in\mathcal{A}$ such that $\frac{D^nf(z)}{D^{n+1}f(z)}\in H\,[q\,(0)\,,1]\cap Q$,

$$\frac{D^nf(z)}{D^{n+1}f(z)}+\gamma\left\{1-\frac{D^nf(z).D^{n+2}f(z)}{[D^{n+1}f(z)]^2}\right\}$$

is univalent in U and

$$q_1(z) + \gamma z q_1^{'}(z) \prec \frac{D^n f(z)}{D^{n+1} f(z)} + \gamma \left\{ 1 - \frac{D^n f(z).D^{n+2} f(z)}{\left[D^{n+1} f(z)\right]^2} \right\} \prec q_2(z) + \gamma z q_2^{'}(z)$$

holds, then

$$q_1(z) \prec \frac{D^n f(z)}{D^{n+1} f(z)} \prec q_2(z),$$

q_1 and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Remark 5.2. Taking $n = 0$ in Corollary 5.3, we obtain the sandwich result obtained by Shanmugam et al. [[22], Corollary 3.3].

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