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WEAK SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACE

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In recent years the study of ordinary differential equations in a Banach space has been developed extensively. However almost all of the work was done using the strong topology (see, for example, *Deimling* [8], *Szufla* [42]) while the study of Cauchy problems involving the weak topology is lagging behind. In [41] *Szep* proved a *Peano* type theorem for O.D.E. defined in a reflexive Banach space and having a weakly continuous vector field. His main tools were the *Eberlein-Smulian* theorem and the well known fact that in a reflexive Banach space a set is weakly compact if and only if it is weakly closed and norm bounded (a simple consequence of *Alaoglu*'s theorem and the fact that in a reflexive Banach space weak and weak topologies coincide) . The result of *Szep* was extended to nonreflexive Banach spaces by *Boundourides* [6] and *Cramer-Lakshmikantham-* Mitchell [34]. Both papers based their existence result on a compactness type condition, involving the weak measure of noncompactness introduced by *De Blasi* [2]. It should be noted however that the result of *Cramer- Lakshmikantham-Mitchell* [34] is more general than that of *Boudourides* [6]. Furthermore the proof of the theorem in [6] has a mistake. Specifically, when the author interprets the notion of weak uniform continuity, he claims that the corresponding inequality holds for all elements of the dual space simultaneously (see p. 460) . This is not true. The proper way to define weak uniform continuity can be found in ([34], p. 170). The purpose of this note is to prove a more general existence theorem for weak vector fields that includes the above mentioned as well as some earlier ones obtained by *Chow-Shur* [12] and *Kato* [31]. We will use a compactness type condition introduced by *Pianigiani* [35] in connection with the strong (norm) topology.

In **1971**, *Szep* [41] discussed the abstract Cauchy problem

$$y' = f(t, y), \text{ on } [0, T], \ y(0) = y_0 \in E; \tag{1}$$

here $f : [0, T] \times E \to E$ is weakly-weakly continuous and E is a reflexive Banach space. *Szep* considered a *Peano* type theorem of ordinary differential equations in reflexive Banach spaces and the result of *CramerLakshmikanthamMitchell* [13] is

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stronger than that of *Szep* [41].

The nonreflexive case was examined by Cramer, Lakshmikantham and Mitchell [13] in **1978** and more recently by *Sugajewski* [9], *Cichon* [16], *Cichon* and *Kubiaczyk* [17], Mitchell and Smith [34], and *O'Regan* [37]- [39]. Motivated by the paper of *Cichon* [16]

In **1976**, *Shigeo Kato* [31] gave sufficient conditions for both local and global existence of strongly continuous, once weakly continuously differentiable solutions to (1). Throughout this paper, whenever the author spoke of a solution to ([31]), he defined a strongly continuous, once weakly continuously differentiable function y on some interval $[0, T]$.

Theorem 1. *Let f be a weakly continuous mapping from $[0, T] \times S(y_0, r)$ into E. Suppose further that the range $f([0, T] \times S(y_0, r))$ is relatively compact in E_ω . Then (1) has at least one solution u defined on some interval $[0, T]$.*

If E is a reflexive Banach space then we have the following result similar to that of Theorem 7 in *F. E. Browder* [7].

Theorem 2. *Let E be a reflexive Banach space and let f be a weakly continuous mapping from $[0, \infty) \times E$ into E. Then for each $r > 0$ there exists $T(r) > 0$ such that, for each y_0 in E with $||y_0|| \leq r$, (1) has at least one solution y defined on $[0, T(r)]$.*

REMARK :1

In Theorem 2 if E is a Hilbert space, then *F. E. Browder* proved that (1) has a strongly C^1 solution defined on $[0, T(r)]$ (see [7]).
For the global existence and uniqueness of solutions to (1 the author define $< . , . >$: $E \times E \to R$ by

$$< v, w >= \frac{1}{2} \lim_{h \to +0} (||v - hw|| - ||v + hw||).$$

Theorem 3. *Let E be a reflexive Banach space and let f be a weakly continuous mapping from $[0, \infty) \times E$ into E. Suppose further that*

$$< v - w, f(t, v) - f(t, w) > \leq \beta(t) ||v - w|| \tag{2}$$

for all v, w in E and a.e. $t \in (0, \infty)$, where $\beta \in L^1_{loc}(0, \infty)$. Then for each y_0 in E (1) has a unique solution y defined on $[0, \infty)$.

In **1999**, *O'Regan* [40] proved the existence of weak solutions for (1) by using the fixed point theorem :

Theorem 4. *Let E be a Banach space with Q a nonempty, bounded, closed, convex, equi-continuous subset of $C([0,T],E)$. Suppose $F : Q \to Q$ is wk-sequentially continuous and assume*

$$FQ(t) \text{ is weakly relatively compact in } E \text{ for each } t \in [0,T] \tag{3}$$

holds. Then $x(t) = Fx(t)$ has a solution in Q.

where the operator F defined by

$$Fx(t) = xo + \int_0^t f(s, x(s))ds.$$

Under the assumptions:

(i) for each $t \in [0,T], f_t = f(t,.)$ is weakly sequentially continuous (i.e., for each $t \in [0,T]$, for each weakly convergent sequence (x_n), the sequence $f_t(x_n)$ is weakly convergent),
(ii) for each continuous $y : [0,T] \to E,\ f(., y(.))$ is Pettis integrable on $[0,T]$,
(iii) for any $r > 0$, there exists $h_r \in L_1[0,T]$ with $|f(t,y)| \le h_r(t)$ for a.e. $t \in [0,T]$ and all $y \in E$ with $|y| \le r$.

REMARK :2
If E is reflexive, then (3) is automatically satisfied since a subset of a reflexive Banach space is weakly compact iff it is closed in the weak topology and bounded in the norm topology. (Now the result follows since $F : Q \to Q$ and Q is a bounded subset of $C([0,T],E)$. Alternatively, we know that there exists an $M > 0$ with $|Y|_0 \le M$ for each $y \in Q$. Also (iii) implies that there exists $h_M \in L_1[0,T]$ with $|f(t,y(t))| \le h_M(t)$ for a.e. $t \in [0,T]$ and all $y \in Q$. Fix $t \in [0,T]$, $y \in Q$, and without loss of generality, assume $Fy(t) \neq 0$. Then there exists $\phi \in E^*$ with $|\phi| = 1$ and $|Fy(t)| = \phi(Fy(t))$. Consequently, $|Fy(t)| = \phi(Fy(t)) \le |x_0| + \int_0^T h_M(S)ds.$)

In **2003**, *A. M. Gomaa* [25] used a measure of weak (strong) noncompactness and there were three existence theorems for solution of the Cauchy problem (1) defined in an infinite dimensional Banach space with the vector field f, generalize many previous theorems such as the results of IbrahimGomaa [28], *Pianigiani* [35] and *Szufla* [42] while it is well known that in view of Peano theorem this problem, as the vector field f is continuous and bounded, has always at least one solution in the space X of finite dimension. Since the *Kuratowski* measure of noncompactness and the ball measure of noncompactness are measures of strong noncompactness, moreover the *Hausdorff* measures are measures of weak noncompactness and he constructed many measures such as in (see [15]), then his results generalized results of *Szufla* [42] and *Pianigiani* [35].

Definition 1. *By a* Kamke *function we mean a function $w : I \times R \to R$ such that:*

(i) *w is a* Caratheodry *function,*
(ii) *for all $t \in I$; $w(t,0) = 0$,*
(iii) *for any $c \in (0,b]$, $y \equiv 0$ is the only absolutely continuous function on $[0,c]$ which satisfies $\dot{y} \le w(t,y(t))$ a.e. on $[0,c]$ and such that $y(0) = 0$.*

Definition 2. *A function $y : I \times E$ is called weak solution of the Cauchy problem (1) if y is strongly continuous function, y is weakly differentiable and its weak derivative $\dot{y}$ satisfies (1).*

In the following theorem *Gomaa* considered f as a weakly continuous vector field, f is bounded by an integrable function and under a generalization of the compactness assumptions (by using a measure of weak noncompactness) he had at least one weak solution for the Cauchy problem (1).

Theorem 5. *Let γ be a weak measure of noncompactness and f be a continuous function from $I \times E_w$ to E_w such that:*

- (C_1) *for each $\epsilon > 0$ and for any nonempty bounded subset U of E there exists a closed subset I_ϵ of I with $\lambda(I - I_\epsilon) < \epsilon$ and $\gamma(f(J \times U)) \leq \sup_{t \in J} w(t, \gamma(U))$, for any compact subset J of I_ϵ,*
- (C_2) *there exists $\mu \in L^1(I, R^+)$ such that $||f(t, y)|| \leq \mu(t)\ \forall (t, y) \in I \times E$. Then, for all $y_0 \in E$, the Cauchy problem (1) has at least one weak solution.*

Theorem 6. *Let $f : I \times B_T \to E$ be a* Caratheeodory *function such that f satisfies conditions (C_1) and (C_2) in Theorem 5 and let, for each $y \in B_T$, $f(\{y\} \times I)$ be separable. Then problem (1) has a solution.*

In Theorem 6 he assumed that $f : I \times B_T \to E$ is a *Caratheodory* vector field, f is bounded by an integrable function and define an iterative scheme y_n in S defined by

$$y_n(t) = \begin{cases} y_0 & \text{if } 0 \leq t \leq \frac{T}{n} \\ \\ y_0 + \int_0^{t - \frac{T}{n}} f(s, y_n(s))\, ds & \text{if } \frac{T}{n} \leq t \leq T. \end{cases}$$

also, for all $y \in E$, $f(\{y\} \times I)$ is separable and with a generalization of the compactness assumptions (by using a measure of strong noncompactness) we obtain a solution of (1).

Lastly, in the third theorem , he supposed f is bounded and continuous with a generalization of the compactness assumptions in Theorem 6, then there exists at least one solution of problem (1).

In **2007**, *A. M. Gomaa* [26] proved an existence theorem for bounded weak solution of the differential equation

$$y(t) = A(t)y(t) + f(t, y(t)),\ t \geq 0. \tag{4}$$

Where $\{A(t): t \in R^+\}$ is a family of linear operators from a Banach space E into itself, $B_r = \{x \in E: ||x|| \le r\}$ and $f: R^+ \times B_r \to E$ is weakly-weakly continuous.
In fact, in the case $A(t) = 0$ we have, as a special case, some improvement to the existence theorem of *Cramer-Lakshmikantham-Mitchell* [13], , *Boundourides* [6], *Ibrahim-Gomaa* [28], *Szep* [41] and *Papageorgiou* [35]. *Cramer- Lakshmikantham-Mitchell* [13] studied the special case of Problem (4) in a nonreflexive Banach space, *Boundourides* [6] and *Papageorgiou* [35], found weak solutions for the special case of Problem (4) on a finite interval $[0, T]$ with $0 < T < 1$. *Szep* in [41] studied the special case of Problem (4) in a reflexive Banach space, while the author used in this paper more general compactness assumptions. *Ibrahim-Gomaa* [28] proved the existence of weak solutions for the special case of Problem (4) on a finite interval $[0, T]$.
If $A(t) \neq 0$ the result of this paper is a generalization to that of *Cichon* [15] , since he was able to reduce the compactness assumptions. Moreover *Cichon* [14], found a weak solution for the problem (4) and *Gomaa* in [27] studied the nonlinear differential equation with delay.

In **2005**, H. A. H. Salem and et al. [45] discussed the existence of pseudo-solution for the Cauchy problem

$$\frac{dx}{dt} = f(t, x(t)), \; t \in I = [0,1], \; x(0) = x_0.$$

Also, discussed the existence of solutions for the Cauchy problem

$$\frac{dx}{dt} = f(t, D^\beta x(t)), \; t \in I, \; 0 < \beta < 1, \; x(0) = x_0,$$

with x taking values in E. By using the fixed point theorem.

Theorem 7. *Let E be a Banach space and let Q be a nonempty, bounded, closed and convex subset of the space E and let $T : Q \to Q$ be a weakly sequentially continuous and assume that $TQ(t)$ is relatively weakly compact in E for each $t \in [0, 1]$. Then, T has a fixed point in the set Q.*

In **2008**, *A. M. A. El-Sayed and et al.* [23] proved the existence of solutions (based on *O'Regan* fixed point theorem 7), in the Banach space $C[I, E]$, of the nonlocal boundary value problem

$$\begin{cases} D^\beta y(t) + f(t, y(t)) = 0 & \text{for } 0 < t < 1, \; \beta \in (1, 2) \\ I^\gamma y(t)|_{t=0}, & \text{for } \gamma \in (0,1], \; \alpha \, y(\eta) = y(1), \; 0 < \eta < 1, \; 0 < \alpha \, \eta^{\beta-1} < 1. \end{cases}$$

where D^β is the *Riemann Liouville* fractional order derivative, $\beta \in (1, 2)$.

Under the assumptions:
$f: I \times D \to E$ satisfies the following:

(i) For each $t \in I$, $f_t = f(t, .)$ is weakly sequentially continuous;

(ii) For each $x \in D$, $f(.,x(.))$ is weakly measurable on I ;

(iii) The weak closure of the range of $f(I \times D)$ is weakly compact in E (or equivalently: there exists an M such that $||f(t,x)|| \leq M \quad (t,x) \in I \times D$);

In **2010**, *M. Benchohra and et al.* [11] investigated the existence of weak solutions in reflexive Banach Space, for the boundary value problem of fractional differential equations

$$\begin{aligned} D^{\alpha}x(t) &= f(t,x(t)), \ t \in I = [0,T] \\ x(0) - x'(0) &= \int_0^T g(s,x(s))\ ds \\ x(T) - x'(T) &= \int_0^T h(s,x(s))\ ds, \end{aligned} \tag{5}$$

where D^{α}, $1 < \alpha \leq 2$ is the *Caputo* fractional derivative, $f, g, \ h : I \times E \to E$ be a given functions satisfy:

(H_1) For each $t \in I$, $f(t,.)$, $g(t,.)$ and $h(t,.)$ are weakly sequentially continuous.

(H_2) For each $x \in C(I, I \times E)$, $f(.,x(.))$, $g(.,x(.))$ and $h(.,x(.))$ are *Pettis* integrable on I.

(H_3) There exist p_g, $p_h \in L^1(I,R^+)$ and $p_f \in L^{\infty}(I,R^+)$ such that:

$$||f(t,x(t))|| \leq p_f(t)||x||, \ a.e. \ t \in I \text{ and each } x \in E,$$
$$||g(t,x(t))|| \leq p_g(t)||x||, \ a.e. \ t \in I \text{ and each } x \in E,$$
$$||h(t,x(t))|| \leq p_h(t)||x||, \ a.e. \ t \in I \text{ and each } x \in E,$$

The authors used *Mönch*'s fixed point theorem combined with the technique of measures of weak noncompactness.

Theorem 8. *Let Q be a closed, convex and equi-continuous subset of a metrizable locally convex vector $C(I,E)$ such that $0 \in Q$. Assume that $F : Q \to Q$ is weakly sequentially continuous. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup F(V))) \Rightarrow V \ \textit{is relativlecompact,} \tag{6}$$

holds for every subset $V \subset Q$, then F has a fixed point.

In **2011**, *Zhi-Wei Lv* and et al. [50] used the monotone iterative technique combined with cone theory to investigate the existence of solutions to the Cauchy problem for *Caputo* fractional differential equations in Banach spaces.

$$\begin{aligned} D^{\alpha}x(t) &= f(t,x(t)), \ t \in I = [0,1] \\ x(0) &= x_0 \end{aligned} \tag{7}$$

where D is the standard *Caputo*'s derivative of order $0 < \alpha < 1$,
$t \in J = [0,1], f \in C(J \times E, E)$.
New existence theorems are obtained for the case of a cone P being normal and

fully regular respectively.

In **2012**, *M. Benchohra* and et al. [10] investigated the existence of weak solutions, for the boundary value problem of fractional differential equations

$$\begin{aligned} D^{\alpha}x(t) &= f(t,x(t)),\ t \in I=[0,1] \\ x(1) &= x(0)+\mu \int_0^1 x(s)\ ds, \end{aligned} \tag{8}$$

where D^{α}, $0<\alpha\leq 1$ is the *Caputo* fractional derivative, $f: I\times E \to E$ be a given function and μ is a positive real number.
The authors used *Mönch*'s fixed point theorem 8 combined with the technique of measures of weak noncompactness.

Under the following assumptions:

(i) For each $t\in I$, the function $f(t,.)$ is weakly sequentially continuous.

(ii) For each $x\in C(I,E)$, the function $f(.,x(.))$ is Pettis integrable on I.

(iii) There exist $p\in L^{\infty}(I,E)$ and a continuous nondecreasing function $\psi:[0,\infty)\to(0,\infty)$ such that

$$||f(t,x)||\leq p(t)\psi(||x||)$$

(iv) There exists a constant $r>0$ such that

$$\frac{r}{||p||_{L^{\infty}}\ \tilde{G}\ \psi(r)}>1.$$

(v) For each bounded set $Q\subset E$, and each $t\in I$, the following inequality holds

$$\beta(f(t,Q))\leq p(t)\beta(Q)$$

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