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K-NEW GENERALIZED MITTAG-LEFFLER FUNCTION

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Abstract. This paper deals with the k-new generalized Mittag-Leffler function. Some of its properties including differentiation, fractional Fourier transform, Laplace transform and k-Beta transform are presented. Also k-Riemann-Liouville fractional integral and differentiation are determined.

1. Introduction

Diaz et al.[1, 2, 3] have introduced k-generalized gamma, k-beta, k-zeta functions and k-Pochhammer symbols. They proved a number of their properties and inequalities for the above k-generalized functions. They have also studied k-hypergeometric functions based on k-Pochhammer symbols for factorial functions. Mansour [8], Kokologiannaki [4], Krasniqi [5] and Merovci [7] extended the study of k-gamma and k-beta functions. Romero et al. [10, 11, 12] introduced k-functions and a new k-Weyl fractional integral operator. Recently Romero, Luque, Dorrego and Cerutti [14] investigated k-Riemann-Liouville fractional derivative and derive some of its properties. Musbeen and Habibullah [9] introduced k version of the classical Riemann-Liouville fractional integral.

The integral representation of k-gamma function is given by

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k}) = \int_0^\infty t^{x-1}e^{\frac{t^k}{k}}dt, Re(x) > 0, k > 0 \tag{1}$$

and k-beta function defined as

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt, x > 0, y > 0 \tag{2}$$

So that

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$$B_k(x,y) = \frac{1}{k}B(\frac{x}{k}, \frac{y}{k}) and B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} \tag{3}$$

Most of the functions like k-Zeta, k-Mittag-Leffler function for two and three parameter, k-Wright and k-hypergeometric function could be defined by the following formulae

$$\xi_k(z,p) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(z+nk)^p}, k, z > 0, p > 1. \tag{4}$$

$$E_k^{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(z)^n}{\Gamma_k(\alpha n+\beta)}, \alpha, \beta > 0. \tag{5}$$

$$E_{k,\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)n!}(z)^n, k \in R, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0. \tag{6}$$

$$W_{k,\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)(n!)^2}(z)^n, k \in R, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0. \tag{7}$$

$$F_k((\beta,k);(\gamma,k);z) = \sum_{n=0}^{\infty} \frac{(\beta)_{n,k}}{(\gamma)_{n,k}n!}(z)^n, k \in R, \alpha, \beta, \gamma \in C; Re(\beta) > 0, Re(\gamma) > 0. \tag{8}$$

In this paper ,we introduce a k-new generalized Mittag-Leffler function which is a generalization of the Mittag-Leffler function given by Salim [16] and it is denoted by $E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(z)$.Some elementary definitions concerning to the fractional calculus are present and derive properties of above function.Finally determine k-Riemann-Liouville fractional integral and differentiation of investigated function.

Definition 1. Let f be a sufficiently well-behaved function with support in $R^+$and let α be a real number,$\alpha > 0$ .The k-Riemann-Liouville fractional integral of order α,I_+^{α} f is given by

$$(I^{\alpha}_{k,a}f)(z)=\frac{1}{k\Gamma_k(\alpha)}\int\limits_a^z (z-t)^{\frac{\alpha}{k}-1}f(t)dt \tag{9}$$

as special case can be obtained the definition due to Mubeen and G. Habibullah [9]

$$(I^{\alpha}_{k}f)(z)=\frac{1}{k\Gamma_k(\alpha)}\int\limits_0^z (z-t)^{\frac{\alpha}{k}-1}f(t)dt \tag{10}$$

when $k\to 1$, it then reduces to the classical Riemann-Liouville fractional integral

$$(I^{\alpha}f)(z)=\frac{1}{\Gamma(\alpha)}\int\limits_0^z (z-t)^{\alpha-1}f(t)dt \tag{11}$$

Definition 2. Let β be a real number, $0<\beta\leq 1$. The k-Riemann-Liouville fractional derivative is given by

$$(D^{\beta}_{k}f)(t)=(\frac{d}{dt})I^{1-\beta}_{k}f(t)dt \tag{12}$$

$$(I^{1-\beta}_{k}f)(x)=\frac{1}{k\Gamma_k(1-\beta)}\int\limits_0^z (z-t)^{\frac{1-\beta}{k}-1}f(t)dt \tag{13}$$

Definition 3 Let u be a function belonging to $\varphi(R)$. The Fractional Fourier transform (FFT) of order α, $0<\alpha\leq 1$, is defined as [13]

$$\hat{u}_a(\omega)=\mathcal{F}_a[u](\omega)=\int\limits_R e^{i\omega^{\frac{1}{\alpha}}t}u(t)dt \tag{14}$$

It may by observed that if $\alpha=1$, (14) reduces at the conventionally Fourier transform given by

$$\mathcal{F}[\varphi](z)=\int\limits_{-\infty}^{+\infty} e^{izt}\varphi(t)dt \tag{15}$$

when $\omega>0$ it reduce to the FFT introduced by Luchko, Martinez and Trujillo [6].

Definition 4 Let $k \in R$;$\alpha, \beta, \gamma, \delta \in C$,$Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0, p > 0, q > 0$ and $q \leq Re(\alpha) + p$.The k-new generalized Mittag-Leffler function is defined as

$$E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n + \beta)(\delta)_{pn,k}}(z)^n, \tag{16}$$

where $(\gamma)_{qn,k}$is the k-Pochhammer symbol given by

$$(\gamma)_{qn,k} = q^{qn}(\frac{\gamma}{q})_{n,k}(\frac{\gamma+k}{q})_{n,k}...(\frac{\gamma+(n-1)k}{q})_{n,k}; \tag{17}$$

$$(\gamma)_{qn,k} = \frac{\Gamma_k(\gamma + qnk)}{\Gamma_k(\gamma)} \tag{18}$$

Most of the function based on gamma function can be redefined by applying k-gamma function. For example the function investigated by Shukla and Prajapati[15] and Salim [17] could be defined in terms of k-gamma function as

$$E^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n + \beta)n!}(z)^n, k \in R, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, q > 0. \tag{19}$$

$$E^{\gamma,\delta}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta)(\delta)_{n,k}}(z)^n, k \in R, \alpha, \beta, \gamma, \delta \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0. \tag{20}$$

Proposition1. Let $\gamma \in C$,$k \in R$ and $n \in N$.Then the following identity holds

$$(\frac{\gamma}{k})_{qn} = q^{qn}k^{-n}(\gamma)_{qn,k} \tag{21}$$

as particular case

$$(\gamma)_{kn} = k^{n(k-1)}(\gamma)_{n,k} \tag{22}$$

Proof: On applying results

$$(\gamma)_{kn} = k^{kn}\prod_{r=1}^{n}(\frac{\gamma+r-1}{k})_n = k^{kn}(\frac{\gamma}{k})_n = k^k(\gamma)_n$$

and

$$(\gamma)_{n,k} = (\gamma)(\gamma+k)(\gamma+2k)....(\gamma+(n-1)k; n = 1,2,3.......$$

$$(\frac{\gamma}{k})_{qn} = q^{qn}(\frac{\gamma/k}{q})(\frac{\gamma/k+1}{q})(\frac{\gamma/k+2}{q})......$$

$$(\frac{\gamma}{k})_{qn} = q^{qn}k^{-n}(\frac{\gamma}{q})(\frac{\gamma+k}{q})(\frac{\gamma+2k}{q})....$$

$$(\frac{\gamma}{k})_{qn} = q^{qn}k^{-n}(\gamma)_{qn,k}$$

Identity(22) may be obtain by using above definition of $(\gamma)_{n,k}$and pochhammer symbol

$$(\gamma)_n = (\gamma)(\gamma+1)(\gamma+2)....(\gamma+(n-1)); n = 1,2,3........$$

Functional relationship between a new generalization of Mittag-Leffler function and the k-new generalized Mittag-Leffler function can be obtained by using proposition 1 and (1)

$$E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z) = k^{1-\frac{\beta}{k}} E^{\frac{\gamma}{k},\frac{\delta}{k},q}_{\frac{\alpha}{k},\frac{\beta}{k},p}(k^{-\frac{\alpha}{k}}(\frac{p^p}{q^q})z) \tag{23}$$

or equivalantely

$$k^{\frac{\beta}{k}-1} E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(k^{\frac{\alpha}{k}}(\frac{q^q}{p^p})z) = E^{\frac{\gamma}{k},\frac{\delta}{k},q}_{\frac{\alpha}{k},\frac{\beta}{k},p}(z) \tag{24}$$

2. Elementary Properties

Theorem 2.1 If the condition (16) is satisfied, then there holds

$$E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z) = \beta E^{\gamma,\delta,q}_{k,\alpha,\beta+k,p}(z) + \alpha z \frac{d}{dz} E^{\gamma,\delta,q}_{k,\alpha,\beta+k,p}(z)$$

Proof: On applying (16)we have

$$\beta E^{\gamma,\delta,q}_{k,\alpha,\beta+k,p}(z) + \alpha z \frac{d}{dz} E^{\gamma,\delta,q}_{k,\alpha,\beta+k,p}(z) = \beta \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta+k)(\delta)_{pn,k}}(z)^n + \alpha z \frac{d}{dz} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta+k)(\delta)_{pn,k}}(z)^n$$

$$= \beta \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta+k)(\delta)_{pn,k}}(z)^n + \alpha \sum_{n=0}^{\infty} \frac{n(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta+k)(\delta)_{pn,k}}(z)^n$$

$$= (\alpha n+\beta) \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta+k)(\delta)_{pn,k}}(z)^n \tag{25}$$

Now $\Gamma_k(z+k) = z\Gamma_k(z)$,(25) becomes

$$\beta E^{\gamma,\delta,q}_{k,\alpha,\beta+k,p}(z) + \alpha z \frac{d}{dz} E^{\gamma,\delta,q}_{k,\alpha,\beta+k,p}(z) = E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z)$$

Theorem 2.2 If the condition (16) is satisfied, then there holds

$$E^{\gamma+k,\delta+k,q}_{k,\alpha,\beta,p}(z) - E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z) = z\frac{q\delta(\gamma)_{q,k}}{p\gamma(\delta)_{p,k}} E^{\gamma+qk,\delta+pk,q}_{k,\alpha,\alpha+\beta,p}(z)$$

Proof: Using (16) and the relationship$(\gamma+k)_{qn,k} - (\gamma)_{qn,k} = qn(k/\gamma)(\gamma)_{qn,k}$

$$\sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(\alpha n+\beta)}[\frac{(\gamma+k)_{qn,k} - (\gamma)_{qn,k}}{(\delta+k)_{pn,k} - (\delta)_{pn,k}}]$$

$$= \sum_{n=1}^{\infty} \frac{z^n}{\Gamma_k(\alpha n+\beta)}[\frac{\delta qnk(\gamma)_{qn,k}}{\gamma pnk(\delta)_{pn,k}}]$$

$$= \sum_{n=0}^{\infty} \frac{z^{n+1}}{\Gamma_k(\alpha n+\alpha+\beta)}[\frac{\delta q(\gamma)_{q(n+1),k}}{\gamma p(\delta)_{p(n+1),k}}]$$

on applying the result

$$(\gamma+qjk)_{qn,k}(\gamma)_{qj,k} = (\gamma)_{q(n+j),k}$$

$$= z\frac{q\delta}{p\gamma}\sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(\alpha n+\alpha+\beta)}[\frac{(\gamma)_{q,k}(\gamma+qk)_{qn,k}}{(\delta)_{p,k}(\delta+pk)_{pn,k}}]$$

$$= z\frac{q\delta(\gamma)_{q,k}}{p\gamma(\delta)_{p,k}}\sum_{n\geq 0} \frac{z^n(\gamma+qk)_{qn,k}}{\Gamma_k(\alpha n+\alpha+\beta)(\delta+pk)_{pn,k}} = z\frac{q\delta(\gamma)_{q,k}}{p\gamma(\delta)_{p,k}} E^{\gamma+qk,\delta+pk,q}_{k,\alpha,\alpha+\beta,p}(z)$$

Theorem 2.3 If the condition (16) is satisfied, then for m$\in N$

$$(i)\frac{d}{dz}E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z) = (\frac{(\gamma)_{q,k}}{(\delta)_{p,k}}(n+1)E^{\gamma+qk,\delta+pk,q}_{k,\alpha,\alpha+\beta,p}(z)$$

$$(ii)(\frac{d}{dz})^m E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z) = (\frac{(\gamma)_{qm,k}}{(\delta)_{pm,k}}(n+1)_m E^{\gamma+qmk,\delta+pmk,q}_{k,\alpha,\alpha m+\beta,p}(z)$$

$$(iii)(\frac{d}{dz})^m [z^{\frac{\beta}{k}-1}E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(\omega z^{\frac{\alpha}{k}})] = k^{-m} z^{\frac{\beta}{k}-m-1} E^{\gamma,\delta,q}_{k,\alpha,\beta-mk,p}(\omega z^{\frac{\alpha}{k}})$$

Proof (i): From (16) and applying the relationship, we have

$$(\gamma)_{q(n+j),k} = (\gamma)_{qj,k}(\gamma+qjk)_{qn,k}$$

$$\frac{d}{dz}E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z) = \frac{d}{dz}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}(z)^n$$

$$\sum_{n=1}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}n(z)^{n-1}$$

$$\sum_{n=0}^{\infty}\frac{(\gamma)_{q(n+1),k}}{\Gamma_k(\alpha(n+1)+\beta)(\delta)_{p(n+1),k}}(n+1)(z)^{n}$$

$$\frac{(\gamma)_{q,k}}{(\delta)_{p,k}}(n+1)\sum_{n=0}^{\infty}\frac{(\gamma+qk)_{qn,k}}{\Gamma_k(\alpha(n+1)+\beta)(\delta+pk)_{pn,k}}(z)^{n}$$

$$\frac{d}{dz}E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(z)=\frac{(\gamma)_{q,k}}{(\delta)_{p,k}}(n+1)E_{k,\alpha,\alpha+\beta,p}^{\gamma+qk,\delta+pk,q}(z)$$

(ii)

$$(\frac{d}{dz})^{m}E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(z)=(\frac{d}{dz})^{m}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}(z)^{n}$$

$$\sum_{n=m}^{\infty}\frac{(\gamma)_{qn,k}n(n-1)(n-2)....(n-m+1)}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}z^{(n-m)}$$

$$\sum_{n=0}^{\infty}\frac{(\gamma)_{q(n+m),k}(n+m)(n+m-1)(n+m-2)....(n+1)}{\Gamma_k(\alpha(n+m)+\beta)(\delta)_{p(n+m),k}}z^{n}$$

$$\frac{(\gamma)_{qm,k}}{(\delta)_{pm,k}}(n+1)_m\sum_{n=0}^{\infty}\frac{(\gamma+qmk)_{qn,k}}{\Gamma_k(\alpha(n+m)+\beta)(\delta+pmk)_{pn,k}}z^{n}$$

$$(\frac{d}{dz})^{m}E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(z)=(\frac{(\gamma)_{qm,k}}{(\delta)_{pm,k}}(n+1)_mE_{k,\alpha,\alpha m+\beta,p}^{\gamma+qmk,\delta+pmk,q}(z)$$

(iii)In view of definition (16), we have

$$(\frac{d}{dz})^{m}[z^{\frac{\beta}{k}-1}E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(\omega z^{\frac{\alpha}{k}})]=(\frac{d}{dz})^{m}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}(\omega)^{n}(z)^{\frac{\alpha n+\beta}{k}-1}$$

Using the formula

$$\frac{d^m}{dz^m}z^n=\frac{\Gamma(n+1)}{\Gamma(n-m+1)}z^{n-m},n\geq m,wehave$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}(\omega)^{n}\frac{\Gamma(\frac{\alpha n+\beta}{k}-1+1)}{\Gamma(\frac{\alpha n+\beta}{k}-1-m+1)}(z)^{\frac{\alpha n+\beta}{k}-(m+1)}$$

Finally using the formula (1),the above expression becomes

$$(\frac{d}{dz})^{m}[z^{\frac{\beta}{k}-1}E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(\omega z^{\frac{\alpha}{k}})]=k^{-m}z^{\frac{\beta}{k}-m-1}E_{k,\alpha,\beta-mk,p}^{\gamma,\delta,q}(\omega z^{\frac{\alpha}{k}})$$

3. Integral Transform of $E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z)$

Theorem 3.1(Laplace Transform) Let $k \in R$;$\alpha, \beta, \gamma, \delta \in C$,$Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0, Re(s) > 0, s \neq 0, p > 0, q > 0$ and $q \leq R(\alpha) + p$.Then

$$L[E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z)] = \frac{n!}{s} E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(s^{-1})$$

Proof: From (16) and definition of Laplace transform

$$L(f)(s) = \int_0^\infty e^{-st} f(t)dt$$

$$L[E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z)] = \int_0^\infty e^{-sz} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n + \beta)(\delta)_{pn,k}} z^n dz$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n + \beta)(\delta)_{pn,k}} \int_0^\infty e^{-sz} z^n dz \tag{26}$$

we know that

$$\int_0^\infty e^{-sz}(z)^n dz = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}} \tag{27}$$

From (26) and (27), we have

$$L[E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z)] = \frac{1}{s} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n + \beta)(\delta)_{pn,k}} n! s^{-n} \tag{28}$$

$$L[E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z)] = \frac{n!}{s} E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(s^{-1})$$

Theorem 3.2 (k Beta Transform) Let $k \in R$;$\alpha, \beta, \gamma, \delta \in C$,$Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0, p > 0, q > 0$ and $q \leq Re(\alpha) + p$. Then

$$\frac{1}{\Gamma_k(\delta)} \int_0^1 x^{\frac{\beta}{k}} - 1(1-x)^{\frac{\delta}{k}-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(zx^{\frac{\alpha}{k}})dx = kE^{\gamma,\delta,q}_{k,\alpha,\beta+\delta,p}(z)$$

Proof: From (2)and (16)

$$= \frac{k}{k\Gamma_k(\delta)} \sum_{n\geq 0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}} z^n \int_0^1 x^{\frac{\alpha n+\beta}{k}-1}(1-x)^{\frac{\delta}{k}-1}dx$$

$$= \frac{k}{\Gamma_k(\delta)} \sum_{n\geq 0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}} z^n B_k(\alpha n+\beta,\delta)$$

Now on using (3) we have

$$\frac{1}{\Gamma_k(\delta)} \int_0^1 x^{\frac{\beta}{k}-1}(1-x)^{\frac{\delta}{k}-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(zx^{\frac{\alpha}{k}})dx = kE^{\gamma,\delta,q}_{k,\alpha,\beta+\delta,p}(z)$$

Theorem 3.3(Fractional Fourier Transform) The FFT of order of the new k-Mittag-Leffler function for $t<0$, we have

$$\mathcal{F}_\alpha[E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z)] = \sum_{n=0}^{\infty} \frac{n!(\gamma)_{qn,k}(-1)^{-n} i^{-n-1}\omega^{\frac{-(n+1)}{\alpha}}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}$$

Proof: From (14) and definition (16)

$$\mathcal{F}_\alpha[E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z)] = \int_R e^{i\omega^{\frac{1}{\alpha}}z} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}} z^n dz$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}} \int_{-\infty}^{0} (e^{i\omega^{\frac{1}{\alpha}}z}) z^n dz$$

on changing variables

$$i\omega^{\frac{1}{\alpha}}z = -t, i\omega^{\frac{1}{\alpha}}dz = -dt$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}(-1)^{-n} i^{-n-1}\omega^{\frac{-(n+1)}{\alpha}}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}} \int_0^{\infty} e^{-t}t^n dt$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}(-1)^{-n} i^{-n-1}\omega^{\frac{-(n+1)}{\alpha}}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}} \Gamma(n+1)$$

$$\mathcal{F}_\alpha[E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z)] = \sum_{n=0}^{\infty} \frac{n!(\gamma)_{qn,k}(-1)^{-n} i^{-n-1}\omega^{\frac{-(n+1)}{\alpha}}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}$$

4. K-Fractional Integration and K-Fractional Differentiation of $E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z)$

Theorem 4.1 (k-Fractional Integration) Let $k,\nu \in R$;$\alpha,\beta,\gamma,\delta \in C$,$Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0, p > 0, q > 0$then

$$I^{\nu}_{k}[z^{\frac{\beta}{k}-1}E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z^{\frac{\alpha}{k}})] = z^{\frac{\beta+\nu}{k}-1}E^{\gamma,\delta,q}_{k,\alpha,\beta+\nu,p}(z^{\frac{\alpha}{k}})$$

Proof: From (10) and definition (16)

$$I^{\nu}_{k}[z^{\frac{\beta}{k}-1}E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z^{\frac{\alpha}{k}})] = \frac{1}{k\Gamma_k(\nu)}\int_0^z (z-t)^{\frac{\nu}{k}-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}t^{\frac{\alpha n+\beta}{k}-1}dt$$

$$= \frac{1}{k\Gamma_k(\nu)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}\int_0^z (z-t)^{\frac{\nu}{k}-1}t^{\frac{\alpha n+\beta}{k}-1}dt \tag{30}$$

Set $t = zx, dt = zdx$ and replacing in (30)we have

$$= \frac{1}{k\Gamma_k(\nu)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}z^{\frac{\alpha n+\beta+\nu}{k}-1}\int_0^1 (1-x)^{\frac{\nu}{k}-1}x^{\frac{\alpha n+\beta}{k}-1}dx \tag{31}$$

the integral in (31) result

$$\frac{1}{k}\int_0^1 x^{\frac{\alpha n+\beta}{k}-1}(1-x)^{\frac{\nu}{k}-1}dx = \frac{1}{k}B(\frac{\nu}{k},\frac{\alpha n+\beta}{k})$$

$$= \frac{1}{k\Gamma_k(\nu)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}z^{\frac{\alpha n+\beta+\nu}{k}-1}B(\frac{\nu}{k},\frac{\alpha n+\beta}{k})$$

where B(x,y) is the Beta function. Then

$$= \frac{1}{k\Gamma_k(\nu)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}z^{\frac{\alpha n+\beta+\nu}{k}-1}\frac{\Gamma\frac{\nu}{k}\Gamma\frac{\alpha n+\beta}{k}}{\frac{\Gamma\alpha n+\beta+\nu}{k}}$$

Now applying (1)

$$= z^{\frac{\alpha n+\beta+\nu}{k}-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}\frac{k^{\frac{\alpha n+\beta+\nu}{k}-1}\Gamma_k(\nu)\Gamma_k(\alpha n+\beta)}{k^{\frac{\alpha n+\beta+\nu}{k}-2}\Gamma_k(\alpha n+\beta+\nu)}$$

$$I^{\nu}_{k}[z^{\frac{\beta}{k}-1}E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z^{\frac{\alpha}{k}})] = z^{\frac{\beta+\nu}{k}-1}E^{\gamma,\delta,q}_{k,\alpha,\beta+\nu,p}(z^{\frac{\alpha}{k}})$$

Theorem 4.2 (k-Fractional Differentiation)Let $k, \nu \in R$;$\alpha, \beta, \gamma, \delta \in C$,$Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0, p > 0, q > 0$then

$$D_k^{\nu}[z^{\frac{\beta}{k}-1}E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(z^{\frac{\alpha}{k}})] = \frac{z^{\frac{\beta-\nu+1}{k}-2}}{k}E_{k,\alpha,\beta-\nu-k+1,p}^{\gamma,\delta,q}(z^{\frac{\alpha}{k}})$$

Proof: From (12),(13) and definition (16)

$$D_k^{\nu}[z^{\frac{\beta}{k}-1}E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(z^{\frac{\alpha}{k}})] = \frac{1}{k\Gamma_k(1-\nu)}(\frac{d}{dz})\int_0^z (z-t)^{\frac{1-\nu}{k}-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}t^{\frac{\alpha n+\beta}{k}-1}dt \tag{32}$$

Set $t = zx, dt = zdx$ and replacing in (32)we have

$$= \frac{1}{k\Gamma_k(1-\nu)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta(\delta)_{pn,k}}(\frac{d}{dz})z^{\frac{\alpha n+\beta-\nu+1}{k}-1}\int_0^1 (1-x)^{\frac{1-\nu}{k}-1}x^{\frac{\alpha n+\beta}{k}-1}dx$$

Now applying definition of Beta function

$$= \frac{1}{k\Gamma_k(1-\nu)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}(\frac{d}{dz})z^{\frac{\alpha n+\beta-\nu+1}{k}-1}B(\frac{1-\nu}{k},\frac{\alpha n+\beta}{k})$$

$$= \frac{1}{k\Gamma_k(1-\nu)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}(\frac{\alpha n+\beta-\nu+1}{k}-1)z^{\frac{\alpha n+\beta-\nu+1}{k}-2}\frac{\Gamma(\frac{1-\nu}{k})\Gamma\frac{\alpha n+\beta}{k}}{\Gamma(\frac{\alpha n+\beta-\nu+1}{k}-1+1)}$$

using (1)and the result $\Gamma(n+1) = n\Gamma n$

$$= \frac{1}{\Gamma_k(1-\nu)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}\frac{(\frac{\alpha n+\beta-\nu+1}{k}-1)}{(\frac{\alpha n+\beta-\nu+1}{k}-1)}z^{\frac{\alpha n+\beta-\nu+1}{k}-2}\frac{k^{\frac{\alpha n+\beta-\nu+1-k}{k}-1}\Gamma_k(1-\nu)\Gamma_k(\alpha n+\beta)}{k^{\frac{\alpha n+\beta-\nu+1}{k}-1}\Gamma_k(\alpha n+\beta-\nu+1-k)}$$

$$= \frac{z^{\frac{\beta-\nu+1}{k}-2}}{k}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n+\beta-\nu-k+1)(\delta)_{pn,k}}z^{\frac{\alpha n}{k}}$$

$$D_k^{\nu}[z^{\frac{\beta}{k}-1}E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(z^{\frac{\alpha}{k}})] = \frac{z^{\frac{\beta-\nu+1}{k}-2}}{k}E_{k,\alpha,\beta-\nu-k+1,p}^{\gamma,\delta,q}(z^{\frac{\alpha}{k}})$$

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