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COEFFICIENT BOUND FOR A NEW CLASS OF ANALYTIC AND BI-UNIVALENT FUNCTIONS

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Abstract. In the present investigation we consider a new class of bi-univalent functions in the unit disk Δ using subordination and obtain estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and $\mathcal{S}$ denote the subclass of class $\mathcal{A}$ consisting functions in $\mathcal{A}$ which are also univalent in Δ. A domain $D \subset \mathbb{C}$ is convex if the line segment joining any two points in D lies entirely in D, while a domain is starlike with respect to a point $w_0 \in D$ if the line segment joining any point of D to w_0 lies inside D. A function $f \in \mathcal{A}$ is starlike if $f(\Delta)$ is a starlike domain with respect to origin, and convex if $f(\Delta)$ is convex. Analytically, $f \in \mathcal{A}$ is starlike if and only if $\mathfrak{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0$ in Δ, whereas $f \in \mathcal{A}$ is convex if and only if $\mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$. The classes consisting of starlike and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{K}$ respectively. The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order α, $0 \le \alpha < 1$, are respectively characterized by $\mathfrak{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha$ and $\mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$ in Δ, $f \in \mathcal{A}$. Also let $\mathcal{P}$ denote the family of analytic functions $p(z)$ in Δ such that $p(0) = 1$ and $\mathfrak{Re}\,(p(z)) > 0$ in Δ.

An analytic function f is subordinate to an analytic function g, written as $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there is an analytic function w defined on Δ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \Delta$ such that $f(z) = g(w(z))$. In particular, if g is univalent in Δ then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).$$

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It is well known by the Koebe one quarter theorem [5] that the image of Δ under every function $f \in \mathcal{S}$ contains a disk of radius 1/4. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z,\ (z \in \Delta)$ and

$$f(f^{-1}(w)) = w\ (|w| < r_0(f),\ r_0(f) \geq 1/4).$$

The inverse of $f(z)$ has a series expansion in some disk about the origin of the form

$$f^{-1}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \dots. \tag{2}$$

It was shown early [11, 14] that the inverse of the Koebe function provides the best bound for all $|\gamma_k|$. New proofs of the latter along with unexpected and unusual behavior of the coefficients γ_k for various subclasses of $\mathcal{S}$ have generated further interest in this problem [7, 8, 9, 16].

A function $f(z)$ univalent in a neighborhood of the origin and its inverse satisfy the condition $f(f^{-1}(w)) = w$. Using (1), we have

$$w = f^{-1}(w) + a_2\big(f^{-1}(w)\big)^2 + a_3\big(f^{-1}(w)\big)^3 + \dots \tag{3}$$

Now using (2) we get the following result

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 + \cdots \tag{4}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ denote the class of all bi-univalent functions defined in the unit disk Δ given by the Taylor-Maclaurin series expansion (1). Note the familiar Koebe function is not a member of Σ because it maps unit disk univalently onto entire complex plane minus slit along $-1/4$ to $-\infty$. Hence the image domain does not contain unit disk.

Lewin [10] investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu [13], on the other hand, showed that $\max_{f\in\Sigma} |a_2| = 4/3$. The coefficient estimate problem i.e. bound of $|a_n|$ $(n \in \mathbb{N} \setminus \{1, 2\})$ for each $f \in \Sigma$ given by (1) is still an open problem. Several authors have subsequently studied similar problems in this direction. In [3] (see also [4, 18, 19]), certain subclasses of the bi-univalent function class Σ were introduced, and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ were found. In the present investigation, estimates on the initial coefficients of a new class of bi-univalent functions are obtained. Several related classes are also considered and a connection to earlier known result are made. The classes introduced in this paper are motivated by the corresponding classes investigated in [6, 12, 15]

Let φ be an analytic function with positive real part on Δ, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$, and $\varphi(\Delta)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots (B_1 > 0)\,. \tag{5}$$

We now introduce the following class of functions:

Definition 1.1. Let $0 \leq \gamma \leq 1$, $\tau \in \mathbb{C} \setminus \{0\}$. A function $f \in \Sigma$ is in the class $\Sigma S_\gamma^\tau(\varphi)$, if the following subordinations hold:

$$1 + \frac{1}{\tau}\left[(1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - 1\right] \prec \varphi(z)\ (z \in \Delta) \tag{6}$$

and

$$1+\frac{1}{\tau}\left[(1-\gamma)\frac{g(w)}{w}+\gamma g'(w)-1\right]\prec\varphi(w)\ (w\in\Delta), \tag{7}$$

where $g(w)=f^{-1}(w)$.
We list few particular cases of this class discussed in the literature

[1] If we set $\gamma=1$ and $\tau=1$ in $\Sigma S_\gamma^\tau(\varphi)$ we obtain the class introduced in [1].
[2] If we set $\gamma=1$ and $\tau=1$ and $\varphi(z)=\frac{1+(1-2\beta)z}{1-z}$ $(0\le\beta<1)$ we obtain the class introduced in [17, p. 1191].
[3] If we set $\gamma=1$ and $\tau=1$ and $\varphi(z)=\left(\frac{1+z}{1-z}\right)^\alpha$ $(0<\alpha\le 1)$ we obtain the class introduced in [17, p. 1190].
For more details about these classes see the corresponding references.

Further if we set $\tau=1$, $\gamma=0$ and $\varphi(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1; z\in\Delta)$ in Definition 1.1, we obtain a new class $\Sigma S(A,B)$ defined in the following way.
A function $f\in\Sigma$ is in the class $\Sigma S(A,B)$, if the following subordinations hold:

$$\frac{f(z)}{z}\prec\frac{1+Az}{1+Bz}\ \ \textit{and}\ \ \frac{g(w)}{w}\prec\frac{1+Aw}{1+Bw}\ (z,w\in\Delta), \tag{8}$$

where $g(w)=f^{-1}(w)$.
To prove our main result we need following Lemma:
Lemma 1.1 (see [5]). *Let the function $p\in\mathcal{P}$ be given by the series*

$$p(z)=1+c_1z+c_2z^2+c_3z^3+\cdots\ (z\in\Delta), \tag{9}$$

then, the sharp estimate

$$|c_n|\le 2\ (n\in\mathbb{N}), \tag{10}$$

holds.

2. Main Results

For functions in the class $\Sigma S_\gamma^\tau(\varphi)$, the following result is obtained.

Theorem 2.1. *Let $f(z)\in\Sigma S_\gamma^\tau(\varphi)$ is of the form (1), then*

$$|a_2|\le\frac{|\tau|\,B_1^{3/2}}{\sqrt{\left|\tau B_1^2(1+2\gamma)+(1+\gamma)^2(B_1-B_2)\right|}} \tag{11}$$

and

$$|a_3|\le B_1\,|\tau|\left(\frac{1}{1+2\gamma}+\frac{B_1\,|\tau|}{(1+\gamma)^2}\right). \tag{12}$$

Proof. Let $f\in\Sigma S_\gamma^\tau(\varphi)$ and $g=f^{-1}$. Then there are analytic functions u, v : $\Delta\to\Delta$, with $u(0)=v(0)=0$, satisfying

$$1+\frac{1}{\tau}\left[(1-\gamma)\frac{f(z)}{z}+\gamma f'(z)-1\right]=\varphi(u(z))\ (z\in\Delta) \tag{13}$$

and

$$1+\frac{1}{\tau}\left[(1-\gamma)\frac{g(w)}{w}+\gamma g'(w)-1\right]=\varphi(v(w))\ (w\in\Delta). \tag{14}$$

Define the functions p_1 and p_2 by

$$p_1(z)=\frac{1+u(z)}{1-u(z)}=1+c_1z+c_2z^2+\cdots \text{ and } p_2(z)=\frac{1+v(z)}{1-v(z)}=1+b_1z+b_2z^2+\cdots.$$

Then p_1 and p_2 are analytic in Δ with $p_1(0)=1=p_2(0)$. Since $u,\ v:\Delta\to\Delta$, the functions p_1 and p_2 have a positive real part in Δ, and in view of Lemma 1.1

$$|b_n|\le 2 \quad \textit{and} \quad |c_n|\le 2\ (n\in\mathbb{N}). \tag{15}$$

Solving for $u(z)$ and $v(z)$ we have

$$u(z)=\frac{p_1(z)-1}{p_1(z)+1}=\frac{1}{2}\left(c_1z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\cdots\right)(z\in\Delta). \tag{16}$$

and

$$v(z)=\frac{p_2(z)-1}{p_2(z)+1}=\frac{1}{2}\left(b_1z+\left(b_2-\frac{b_1^2}{2}\right)z^2+\cdots\right)(z\in\Delta). \tag{17}$$

In view of (5) and (13)-(17), clearly

$$\begin{aligned} & 1+\frac{1}{\tau}\left[(1-\gamma)\frac{f(z)}{z}+\gamma f'(z)-1\right] \\ = \ & \varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) \\ = \ & 1+\frac{1}{2}B_1c_1z+\left(\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right)z^2+\cdots. \end{aligned} \tag{18}$$

and

$$\begin{aligned} & 1+\frac{1}{\tau}\left[(1-\gamma)\frac{g(w)}{w}+\gamma g'(w)-1\right] \\ = \ & \varphi\left(\frac{p_2(w)-1}{p_2(w)+1}\right) \\ = \ & 1+\frac{1}{2}B_1b_1w+\left(\frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2\right)w^2+\cdots. \end{aligned} \tag{19}$$

Since $f\in\Sigma$ has the Maclaurin series given by (1), a computation shows that its inverse $g=f^{-1}$ has the expansion

$$g(w)=f^{-1}(w)=w-a_2w^2+\left(2a_2^2-a_3\right)w^3+\cdots \tag{20}$$

Using (1) and (20) in (18) and (19), we obtain

$$\frac{(1+\gamma)a_2}{\tau}=\frac{B_1c_1}{2}, \tag{21}$$

$$\frac{(1+2\gamma)a_3}{\tau}=\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2, \tag{22}$$

$$\frac{-(1+\gamma)a_2}{\tau}=\frac{B_1b_1}{2} \tag{23}$$

and

$$\frac{(1+2\gamma)\left(2a_2^2-a_3\right)}{\tau}=\frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2. \tag{24}$$

From (21) and (23), it follows that $c_1 = -b_1$. Further computation gives

$$a_2^2 = \frac{\tau^2 B_1^3 (b_2 + c_2)}{4\left[\tau B_1^2 (1+2\gamma) + (1+\gamma)^2 (B_1 - B_2)\right]}. \tag{25}$$

and

$$a_3 = \frac{B_1^2 \tau^2 b_1^2}{4(1+\gamma)^2} + \frac{B_1 \tau}{4(1+2\gamma)} (c_2 - b_2). \tag{26}$$

In view of Lemma 1.1 we get the desired result (11) and (12).

Remark 2.1. If we set $\gamma = 1$ and $\tau = 1$ in Theorem 2.1 we get Theorem 2.1 of [1].

If we set

$$\varphi(z) = \frac{1 + Az}{1 + Bz} \; (-1 \le B < A \le 1; z \in \Delta) \tag{27}$$

in Theorem 2.1, we get the following Corollary:

Corollary 2.1. *Let $f(z) \in \Sigma S_\gamma^\tau(\frac{1+Az}{1+Bz})$ is of the form (1), then*

$$|a_2| \le \frac{|\tau| (A - B)}{\sqrt{\left|\tau(A-B)(1+2\gamma) + (1+\gamma)^2 (1+B)\right|}} \tag{28}$$

and

$$|a_3| \le (A - B) |\tau| \left(\frac{1}{1+2\gamma} + \frac{(A-B)|\tau|}{(1+\gamma)^2} \right). \tag{29}$$

Remark 2.2. If we set $\gamma = 1,\ \tau = 1,\ A = 1 - 2\beta\ (0 \le \beta < 1)$ and $B = -1$ in Corollary 2.1 we get the Theorem 2 of [17].

Corollary 2.2. *Let $f(z) \in \Sigma S_\gamma^\tau(\left(\frac{1+z}{1-z}\right)^\alpha)$ is of the form (1), then*

$$|a_2| \le \frac{2 |\tau| \alpha}{\sqrt{\left|2\alpha\tau(1+2\gamma) + (1+\gamma)^2 (1-\alpha)\right|}} \tag{30}$$

and

$$|a_3| \le 2\alpha |\tau| \left(\frac{1}{1+2\gamma} + \frac{2\alpha |\tau|}{(1+\gamma)^2} \right). \tag{31}$$

Remark 2.3. Further if we set $\gamma = 1$, $\tau = 1$ in Corollary 2.2, we get Theorem 1 of [17].

Finally setting $\tau = 1$, $\gamma = 0$ in Corollary 2.1, we get the following new result:

Corollary 2.3. *Let $f(z) \in \Sigma S(A, B)$ is of the form (1), then*

$$|a_2| \le \frac{A - B}{\sqrt{A+1}} \;\; \textit{and} \;\; |a_3| \le (A - B + 1)(A - B). \tag{32}$$

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