

MOUNTAIN PASS SOLUTION FOR A FRACTIONAL BOUNDARY VALUE PROBLEM

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ABSTRACT. In this work we prove the existence of mountain pass solution for a fractional boundary value problem given by

$$\begin{aligned} {}_t D_T^\alpha ({}_0 D_t^\alpha u(t)) &= f(t, u(t)), \quad t \in [0, T] \\ u(0) &= u(T) = 0. \end{aligned}$$

1. INTRODUCTION

Fractional order models can be found to be more adequate than integer order models in some real world problems as fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order. As a consequence, the subject of fractional differential equations is gaining more importance and attention. There has been significant development in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives. For details and examples, one can see the monographs [15], [24], [26] and the papers [2], [3], [5], [7], [11], [17], [21], [23], [29], [31]. Moreover the existence of almost periodic, asymptotically almost periodic, almost automorphic, asymptotically almost automorphic, and pseudo-almost periodic solutions have been great attention in the qualitative theory of fractional differential equations, due to its mathematical interest and applications. Some recent contributions on the existence of such solutions for abstract differential equations and fractional differential equations have been made, see [1], [3], [4], [7], [12], [13], [20], [25] for details.

Recently, also equations including both left and right fractional derivatives are discussed. Apart from their possible applications, equations with left and right derivatives is an interesting and new field in fractional differential equations theory. In this topic, many results are obtained dealing with the existence and multiplicity

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of solutions of nonlinear fractional differential equations by using techniques of nonlinear analysis, such as fixed point theory [9] (including Leray-Schauder nonlinear alternative), topological degree theory [18] (including co-incidence degree theory) and comparison method [32] (including upper and lower solutions and monotone iterative method) and so on.

It should be noted that critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions for integer order differential equations. The idea behind them is trying to find solutions of a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space. In the last 30 years, the critical point theory has become to a wonderful tool in studying the existence of solutions to differential equations with variational structures, we refer the reader to the books due to Mawhin and Willem [22], Rabinowitz [28] and the references listed therein.

Motivated by the above classical works, in recent paper [19], for the first time, the authors showed that the critical point theory is an effective approach to tackle the existence of solutions for the following fractional boundary value problem

$$\begin{aligned} {}_t D_T^\alpha ({}_0 D_t^\alpha u(t)) &= \nabla F(t, u(t)), \text{ a.e. } t \in [0, T], \\ u(0) &= u(T) = 0. \end{aligned} \quad (1)$$

and obtained the existence of at least one nontrivial solution. We note that it is not easy to use the critical point theory to study (1), since it is often very difficult to establish a suitable space and variational functional for the fractional boundary value problem.

In this paper we want to contribute with the development of this new area on fractional differential equations theory. More precisely we study the fractional nonlinear Dirichlet problem given by

$$\begin{aligned} {}_t D_T^\alpha ({}_0 D_t^\alpha u(t)) &= f(t, u(t)), \quad t \in [0, T], \\ u(0) &= u(T) = 0. \end{aligned} \quad (2)$$

where $\alpha \in (1/2, 1)$ and $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following condition

- (f₁) $f \in C([0, T] \times \mathbb{R})$.
- (f₂) There is a constant $\mu > 2$ such that

$$0 < \mu F(t, u) \leq u f(t, u) \text{ for every } t \in [0, T] \text{ and } u \in \mathbb{R} \setminus \{0\}.$$

Before stating our results, let us introduce the main ingredients involved in our approach. We define

$$\|u\|_\alpha^2 = \int_0^T |u(t)|^2 dt + \int_0^T |{}_0 D_t^\alpha u(t)|^2 dt,$$

and the space $E^\alpha = \overline{C_0^\infty[a, b]}^{\|\cdot\|_\alpha}$. For $u \in E^\alpha$ we may define the functional

$$I(u) = \frac{1}{2} \int_0^T |{}_0 D_t^\alpha u(t)|^2 dt - \int_0^T F(t, u(t)) dt, \quad (3)$$

which is of class C^1 . We say that $u \in E^\alpha$ is a weak solution of (2) if u is a critical point of I . Now we are in a position to state our main existence theorem.

Theorem 1 Suppose that f satisfies (f₁)-(f₂), then (2) has at least one nonzero weak solution on E^α .

The main ingredient in the proof of Theorem 1, is the mountain pass theorem due to Ambrosetti-Rabinowitz [6]. We recall this result.

Theorem 2 (Mountain pass theorem) Let X be a real Banach space and $\phi \in C^1(X, \mathbb{R})$ satisfying PS condition. Suppose that

- (i) $\phi(0) = 0$,
- (ii) there is $\rho > 0$ and $\sigma > 0$ such that $\phi(z) \geq \sigma$ for all $z \in X$ with $\|z\| = \rho$,
- (iii) there exists z_1 in X with $\|z_1\| \geq \rho$ such that $\phi(z_1) < \sigma$.

Then ϕ possesses a critical value $c \geq \sigma$. Moreover, c can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{z \in [0,1]} \phi(\gamma(z)),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = z_1\}$

This article is organized as follows. In Section 2 we present preliminaries on fractional calculus. In Section 3 we introduce the functional setting of the problem. In Section 3 we prove the Theorem 1.

2. FRACTIONAL CALCULUS

In this section we introduce some basic definitions of fractional calculus which are used further in this paper. For the proof see [15], [26] and [30].

Definition 1 (Left and Right Riemann-Liouville fractional integral) Let u be a function defined on $[a, b]$. The left (right) Riemann-Liouville fractional integral of order $\alpha > 0$ for function u is defined by

$$\begin{aligned} {}_a I_t^\alpha u(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \quad t \in [a, b], \\ {}_t I_b^\alpha u(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s) ds, \quad t \in [a, b], \end{aligned}$$

provided in both cases that the right-hand side is pointwise defined on $[a, b]$.

Definition 2 (Left and Right Riemman-Liouville fractional derivative) Let u be a function defined on $[a, b]$. The left and right Riemann - Liouville fractional derivatives of order $\alpha > 0$ for function u denoted by ${}_a D_t^\alpha u(t)$ and ${}_t D_b^\alpha u(t)$, respectively, are defined by

$$\begin{aligned} {}_a D_t^\alpha u(t) &= \frac{d^n}{dt^n} {}_a I_t^{n-\alpha} u(t), \\ {}_t D_b^\alpha u(t) &= (-1)^n \frac{d^n}{dt^n} {}_t I_b^{n-\alpha} u(t), \end{aligned}$$

where $t \in [a, b]$, $n - 1 \leq \alpha < n$ and $n \in \mathbb{N}$.

Remark 1 According to definition ?? and definition ??, if α becomes an integer $n \in \mathbb{N}$ we recover the usual definitions, namely

$$\begin{aligned} {}_a I_t^n u(t) &= \frac{1}{\Gamma(n)} \int_a^t (t-s)^{n-1} u(s) ds, \quad t \in [a, b], \quad n \in \mathbb{N}, \\ {}_t I_b^n u(t) &= \frac{1}{\Gamma(n)} \int_t^b (s-t)^{n-1} u(s) ds, \quad t \in [a, b], \quad n \in \mathbb{N}, \end{aligned}$$

and

$$\begin{aligned} {}_a D_t^n u(t) &= u^{(n)}(t), \quad t \in [a, b], \\ {}_t D_b^n u(t) &= (-1)^n u^{(n)}(t), \quad t \in [a, b]. \end{aligned}$$

Remark 2 If $u \in C[a, b]$, it is obvious that the Riemann-Liouville fractional integral of order $\alpha > 0$ is bounded in $[a, b]$. On the other hand, following [15], it is known that the Riemann-Liouville fractional derivative of order $\alpha \in [n - 1, n)$ exists a.e. on $[a, b]$ if $u \in AC^n[a, b]$, where $AC^n[a, b]$ is the space of functions u such that $u \in C^{n-1}([a, b])$ and $u^{(n-1)}$ is absolutely continuous on $[a, b]$.

Now we announce some properties of the Riemann-Liouville fractional integral and derivative operators.

Theorem 1

$$\begin{aligned} {}_a I_t^\alpha ({}_a I_t^\beta u(t)) &= {}_a I_t^{\alpha+\beta} u(t) \quad \text{and} \\ {}_t I_b^\alpha ({}_t I_b^\beta u(t)) &= {}_t I_b^{\alpha+\beta} u(t) \quad \forall \alpha, \beta > 0, \end{aligned}$$

in any point $t \in [a, b]$ for continuous function u and for almost every point in $[a, b]$ if the function $u \in L^1[a, b]$.

Theorem 2 (Left inverse) Let $u \in L^1[a, b]$ and $\alpha > 0$,

$$\begin{aligned} {}_a D_t^\alpha ({}_a I_t^\alpha u(t)) &= u(t), \quad \text{a.e. } t \in [a, b] \quad \text{and} \\ {}_t D_b^\alpha ({}_t I_b^\alpha u(t)) &= u(t), \quad \text{a.e. } t \in [a, b]. \end{aligned}$$

Theorem 3 For $n - 1 \leq \alpha < n$, if the left and right Riemann-Liouville fractional derivatives ${}_a D_t^\alpha u(t)$ and ${}_t D_b^\alpha u(t)$, of the function u are integral on $[a, b]$, then

$$\begin{aligned} {}_a I_t^\alpha ({}_a D_t^\alpha u(t)) &= u(t) - \sum_{k=1}^n [{}_a I_t^{k-\alpha} u(t)]_{t=a} \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k+1)}, \\ {}_t I_b^\alpha ({}_t D_b^\alpha u(t)) &= u(t) - \sum_{k=1}^n [{}_t I_b^{k-\alpha} u(t)]_{t=b} \frac{(-1)^{n-k} (b-t)^{\alpha-k}}{\Gamma(\alpha-k+1)}, \end{aligned}$$

for $t \in [a, b]$.

Theorem 4 (Integration by parts)

$$\int_a^b [{}_a I_t^\alpha u(t)] v(t) dt = \int_a^b u(t) {}_t I_b^\alpha v(t) dt, \quad \alpha > 0, \quad (4)$$

provided that $u \in L^p[a, b]$, $v \in L^q[a, b]$ and

$$p \geq 1, \quad q \geq 1 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} < 1 + \alpha \quad \text{or} \quad p \neq 1, \quad q \neq 1 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1 + \alpha.$$

$$\int_a^b [{}_a D_t^\alpha u(t)] v(t) dt = \int_a^b u(t) {}_t D_b^\alpha v(t) dt, \quad 0 < \alpha \leq 1, \quad (5)$$

provided the boundary conditions

$$\begin{aligned} u(a) = u(b) = 0, \quad u' \in L^\infty[a, b], \quad v \in L^1[a, b] \quad \text{or} \\ v(a) = v(b) = 0, \quad v' \in L^\infty[a, b], \quad u \in L^1[a, b], \end{aligned}$$

are fulfilled.

3. FRACTIONAL DERIVATIVE SPACE

In order to establish a variational structure which enables us to reduce the existence of solutions of BVP (1) to the one of finding critical points of corresponding functional, it is necessary to construct appropriate function spaces. For this setting we take some results from [19].

Let us recall that for any fixed $t \in [0, T]$ and $1 \leq p < \infty$,

$$\begin{aligned} \|u\|_{L^p[0,t]} &= \left(\int_0^t |u(s)|^p ds \right)^{1/p}, \\ \|u\|_{L^p} &= \left(\int_0^T |u(s)|^p ds \right)^{1/p} \quad \text{and} \\ \|u\|_{\infty} &= \max_{t \in [0, T]} |u(t)|. \end{aligned}$$

Definition 1 Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative spaces $E_0^{\alpha,p}$ is defined by

$$E_0^{\alpha,p} = \{u \in L^p[0, T] / {}_0D_t^\alpha u \in L^p[0, T] \text{ and } u(0) = u(T) = 0\} = \overline{C_0^\infty[0, T]}^{\|\cdot\|_{\alpha,p}}.$$

where $\|\cdot\|_{\alpha,p}$ is defined by

$$\|u\|_{\alpha,p}^p = \int_0^T |u(t)|^p dt + \int_0^T |{}_0D_t^\alpha u(t)|^p dt. \quad (6)$$

Proposition 1 [19] Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.

Lemma 1 [30] Let $0 < \alpha \leq 1$ and $1 \leq p < \infty$. For any $u \in L^p[0, T]$ we have

$$\|{}_0I_\xi^\alpha u\|_{L^p[0,t]} \leq \frac{t^\alpha}{\Gamma(\alpha+1)} \|u\|_{L^p[0,t]}, \quad \text{for } \xi \in [0, t], \quad t \in [0, T]. \quad (7)$$

Proposition 2 [19] Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, if $\alpha > 1/p$ we have

$${}_0I_t^\alpha ({}_0D_t^\alpha u(t)) = u(t).$$

Moreover, $E_0^{\alpha,p} \in C[0, T]$.

Proposition 3 [19] Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, if $\alpha > 1/p$ we have

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0D_t^\alpha u\|_{L^p}. \quad (8)$$

If $\alpha > 1/p$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_{\infty} \leq \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|{}_0D_t^\alpha u\|_{L^p}. \quad (9)$$

According to (8), we can consider in $E_0^{\alpha,p}$ the following norm

$$\|u\|_{\alpha,p} = \|{}_0D_t^\alpha u\|_{L^p}, \quad (10)$$

and (10) is equivalent to (6).

Proposition 4 [19] Let $0 < \alpha \leq 1$ and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and $\{u_k\} \rightarrow u$ in $E_0^{\alpha,p}$. Then $u_k \rightarrow u$ in $C[0, T]$, i.e.

$$\|u_k - u\|_{\infty} \rightarrow 0, \quad k \rightarrow \infty.$$

We denote by $E^\alpha = E_0^{\alpha,2}$, this is a Hilbert space with respect to the norm $\|u\|_\alpha = \|u\|_{\alpha,2}$ given by (10).

4. MOUNTAIN PASS SOLUTION

In this section we deal with the fractional boundary problem

$$\begin{aligned} {}_tD_T^\alpha({}_0D_t^\alpha u(t)) &= f(t, u(t)), \quad t \in [0, T], \\ u(0) &= u(T) = 0. \end{aligned} \quad (11)$$

where $\alpha \in (1/2, 1)$ and $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following condition

- (f_1) $f \in C([0, T] \times \mathbb{R})$.
 (f_2) There is a constant $\mu > 2$ such that

$$0 < \mu F(t, u) \leq uf(t, u) \text{ for every } t \in [0, T] \text{ and } u \in \mathbb{R} \setminus \{0\}.$$

We recall the notion of solution for (11).

Definition 1 $u \in E^\alpha$ be a weak solution of (11) if

$$\int_0^T {}_0D_t^\alpha u(t) {}_0D_t^\alpha v(t) dt = \int_0^T f(t, u(t))v(t) dt, \text{ for any } v \in E^\alpha. \quad (12)$$

Associated to the boundary problem (11) we have the functional $I : E^\alpha \rightarrow \mathbb{R}$ defined by

$$I(u) = \frac{1}{2} \int_0^T |{}_0D_t^\alpha u(t)|^2 dt - \int_0^T F(t, u(t)) dt, \quad (13)$$

where $F(t, s) = \int_0^s f(t, \xi) d\xi$. Following [28], we can show $I \in C^1(E^\alpha, \mathbb{R})$ and we have

$$I'(u)v = \int_0^T {}_0D_t^\alpha u(t) {}_0D_t^\alpha v(t) dt - \int_0^T f(t, u(t))v(t) dt, \quad \forall v \in E^\alpha.$$

Therefore critical points of I are weak solutions of (11).

We are going to check that I satisfies the conditions of mountain pass theorem.

First we consider two technically lemmas

Lemma 1 If f satisfies (f_2), then for every $t \in [0, T]$ the following inequalities hold

$$F(t, u) \leq F(t, \frac{u}{|u|})|u|^\mu, \text{ if } 0 < |u| \leq 1; \quad (14)$$

and

$$F(t, u) \geq F(t, \frac{u}{|u|})|u|^\mu, \text{ if } |u| \geq 1. \quad (15)$$

We note, according to Lemma 1, f is superquadratic at infinity and subquadratic at the origin.

Proof. By (f_2) we note that

$$\mu F(t, \sigma u) \leq \sigma u f(t, \sigma u).$$

Let $h(\sigma) = F(t, \sigma u)$, then

$$\frac{d}{d\sigma} (f(\sigma)\sigma^{-\mu}) \geq 0. \quad (16)$$

We conclude integrating (16) from 1 until $\frac{1}{|u|}$ and from $\frac{1}{|u|}$ until 1 .

Lemma 2 Let $m = \inf\{F(t, u) / t \in [0, T], |u| = 1\}$. Then for any $\xi \in \mathbb{R} \setminus \{0\}$ and $u \in E^\alpha \setminus \{0\}$, we have

$$\int_0^T F(t, \xi u(t)) dt \geq m|\xi|^\mu \int_0^T |u(t)|^\mu dt - Tm. \quad (17)$$

Proof. Fix $\xi \in \mathbb{R} \setminus \{0\}$ and $u \in E^\alpha \setminus \{0\}$, Let

$$\begin{aligned} A &= \{t \in [0, T] / |\xi u(t)| \leq 1\}, \text{ and} \\ B &= \{t \in [0, T] / |\xi u(t)| \geq 1\}. \end{aligned}$$

Now by (15), we obtain

$$\begin{aligned} \int_0^T F(t, \xi u(t)) dt &\geq \int_B F(t, \xi u(t)) dt \geq \int_B F(t, \frac{\xi u(t)}{|\xi u(t)|}) |\xi u(t)|^\mu dt \\ &\geq m \int_B |\xi u(t)|^\mu dt = m \int_0^T |\xi u(t)|^\mu dt - m \int_A |\xi u(t)|^\mu dt \\ &\geq m |\xi|^\mu \int_0^T |u(t)|^\mu dt - mT. \end{aligned}$$

Lemma 3 Under the condition (f_1) , the functional defined by (13) is well defined, $I \in C^1(E^\alpha, \mathbb{R})$ and

$$I'(u)v = \int_0^T {}_0D_t^\alpha u(t) {}_0D_t^\alpha v(t) dt - \int_0^T f(t, u(t))v(t) dt, \quad \forall v \in E^\alpha. \quad (18)$$

Moreover, it satisfies the Palais -Smale condition.

Proof. Using the continuity of f , we obtain the continuity and differentiability of I and we get (18).

To show that I satisfies the Palais - Smale condition, let $\{u_k\} \in E^\alpha$ such that

$$|I(u_k)| \leq M, \quad \lim_{k \rightarrow \infty} I'(u_k) = 0. \quad (19)$$

First we prove that $\{u_k\}$ is bounded. We have

$$\begin{aligned} I(u_k) &= \frac{1}{2} \|u_k\|_\alpha^2 - \int_0^T F(t, u_k(t)) dt. \\ I'(u_k)u_k &= \|u_k\|_\alpha^2 - \int_0^T f(t, u_k(t))u_k(t) dt. \end{aligned}$$

Then by (19)

$$\left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_k\|_\alpha^2 \leq I(u_k) - \frac{1}{\mu} I'(u_k)u_k \leq M + M \|u_k\|_\alpha.$$

Since $\mu > 2$ it follows that $\{u_k\}$ is bounded in E^α . Since E^α is reflexive space, going to a subsequence if necessary, we may assume that $u_k \rightharpoonup u$ in E^α , thus we have

$$\begin{aligned} \langle I'(u_k) - I'(u), u_k - u \rangle &= \langle I'(u_k), u_k - u \rangle - \langle I'(u), u_k - u \rangle \\ &\leq \|I'(u_k)\| \|u_k - u\|_\alpha - \langle I'(u), u_k - u \rangle \rightarrow 0. \end{aligned} \quad (20)$$

as $k \rightarrow \infty$. Moreover according (9) and Proposition 4, section 3, we get that u_k is bounded in $C[0, T]$ and

$$\lim_{k \rightarrow \infty} \|u_k - u\|_\infty = 0.$$

Hence we have

$$\int_0^T [f(t, u_k(t)) - f(t, u(t))](u_k(t) - u(t)) dt \rightarrow 0, \quad k \rightarrow \infty.$$

Moreover, an easy computation show that

$$\langle I'(u_k) - I'(u), u_k - u \rangle = \|u_k - u\|_\alpha^2 - \int_0^T (f(t, u_k(t)) - f(t, u(t)))(u_k(t) - u(t))dt.$$

So $\|u_k - u\|_\alpha \rightarrow 0$ as $k \rightarrow \infty$. That is $\{u_k\}$ converges strongly to u in E^α .

Now we can prove our main theorem.

Theorem 1 Let $\alpha \in (1/2, 1]$ and suppose that f satisfies (f_1) and (f_2) . The equation (11) possesses a nontrivial weak solution $u \in E^\alpha$.

Proof. In our case is clear that $I(0) = 0$. Now we show that I satisfies the geometry conditions of mountain pass theorem. By (9) we have

$$\max_{t \in [0, T]} |u(t)| \leq C \|u\|_\alpha, \quad \forall u \in E^\alpha,$$

where $C = \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(\frac{\alpha+1}{2})^{1/2}}$. Now let $C_1 = \frac{1}{C}$, it follow by the inequality from above and (14), if $\|u\|_\alpha \leq C_1$

$$\begin{aligned} \int_0^T F(t, u(t))dt &\leq \int_0^T F(t, \frac{u(t)}{|u(t)|})|u(t)|^\mu dt \\ &\leq M \|u\|_{L^\mu}^\mu \leq MTC^\mu \|u\|_\alpha^\mu. \end{aligned}$$

Then

$$\begin{aligned} I(u) &= \frac{1}{2} \|u\|_\alpha^2 - \int_0^T F(t, u(t))dt \\ &\geq \frac{1}{2} \|u\|_\alpha^2 - MTC^\mu \|u\|_\alpha^\mu, \quad \text{if } \|u\|_\alpha \leq C_1, \end{aligned}$$

and consequently

$$I(u) \geq \frac{1}{2} C_1^2 - MTC^\mu C_1^\mu, \quad \text{if } \|u\|_\alpha = C_1.$$

Now let $\rho < \min\{C_1, (\frac{1}{2MTC^\mu})^{\frac{1}{\mu-2}}\}$ and $\beta = \frac{\rho^2}{2} - MTC^\mu \rho^\mu$,

$$I(u) \geq \beta \quad \text{if } \|u\|_\alpha = \rho.$$

Hence I satisfies the first geometry condition of mountain pass theorem.

Now by Lemma 1, we have that for every $\xi \in \mathbb{R} \setminus \{0\}$ and $u \in E^\alpha \setminus \{0\}$

$$\begin{aligned} I(\xi u) &= \frac{\xi^2}{2} \|u\|_\alpha^2 - \int_0^T F(t, \xi u(t))dt \\ &\leq \frac{\xi^2}{2} \|u\|_\alpha^2 - m|\xi|^\mu \int_0^T |u(t)|^\mu dt + Tm. \end{aligned}$$

Taking ξ large enough and let $e = \xi u$ then $I(e) \leq 0$. Therefore I satisfies the mountain pass condition, so by mountain pass theorem we get a nontrivial weak solution of (11).

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