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QUASI CONVOLUTION RESULTS FOR CLASSES RELATED TO $q-p-$ SĂLĂGEAN OPERATOR

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Abstract. By using $q-$difference operator and $p-$Sălăgean operator, we defined $q-p-$ Sălăgean operator. We defined a new class of close-to convex function associated with $q-p-$ Sălăgean operator. Also, we obtained coefficient estimate theorem for this class. Several modified-Hadamard product for this class are introduced.

1. Introduction

Let $T(p,j)$ be the class of functions

$$f(z)=z^p-\sum_{k=p+j}^{\infty}a_kz^k(a_k\geq 0,p,j\in\mathbb{N}:=\{1,2,3,...\}), \tag{1}$$

that are analytic in $\mathbf{U}=\{z\in\mathbb{C}:|z|<1\}$.The $q-$difference operator, which was introduced by Jackson ([11]), and see also (([1]-[3]),[5],[6],[9],[16],[17] and [10]) is defined by

$$\partial_q f(z)=\begin{cases}\frac{f(z)-f(qz)}{(1-q)z} & \text{if} \quad z\neq 0\\ f'(0) & \text{if} \quad z=0\end{cases}.$$

For $f(z)$ given by (1)

$$\partial_q f(z)=[p]_qz^{p-1}-\sum_{k=p+j}^{\infty}[k]_qa_kz^{k-1},$$

where,

$$[k]_q=\frac{1-q^k}{1-q},\ \text{and, as } q\to 1^-\Rightarrow [k]_q\to k. \tag{2}$$

Now we define $q-p-$ Sălăgean operator by

$$D^0_{p,q}f(z)=f(z)$$

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$$D^1_{p,q}f(z) = \frac{z}{[p]_q}\partial_q(f(z)) = z^p - \sum_{k=p+j}^{\infty} \frac{[k]_q}{[p]_q} a_k z^k,$$

$$D^2_{p,q}f(z) = \frac{z}{[p]_q}\partial_q(D_{p,q}f(z)),$$

and

$$\begin{aligned} D^n_{p,q}f(z) &= \frac{z}{[p]_q}\partial_q(D^{n-1}_{p,q}f(z)) \\ &= z^p - \sum_{k=p+j}^{\infty} \left(\frac{[k]_q}{[p]_q}\right)^n a_k z^k, \quad n \in \mathbb{N} = \mathbb{N} \cup \{0\}. \end{aligned} \tag{3}$$

Note that:

(i) Putting $q \to 1^-$in (3) we have the operator D^n_p which was introduced and studied by Shenen et al [15], Kamali and Orhan [12], Aouf and Mostafa [7] and Aouf et al [8];

(ii) Putting $p = 1$,$q \to 1^-$in (3) we have the Sălăgean operator D^n ([14]).

Definition 1.1. *For some α $(0 \le \alpha < [p]_q)$ and β $(0 < \beta \le 1)$, a function $f(z) \in T(p,j)$ is in the class $T_{p,q}(n,j,\alpha,\beta)$ if it satisfies*

$$\left| \frac{\frac{\partial q(D^n_{p,q}f(z))}{z^{p-1}} - [p]_q}{\frac{\partial q(D^n_{p,q}f(z))}{z^{p-1}} + [p]_q - 2\alpha} \right| < \beta. \tag{4}$$

The object of this paper is to introduce a new class $T_{p,q}(n,j,\alpha,\beta)$ by using the definition $q-$difference operator and $q-p-$Sălăgean operator. As well as we calculate the coefficient estimates for functions belong to this class. Also we get modified Hadamard product for functions in this class.

2. Coefficients estimates for the functions belongs to $T_{p,q}(n,j,\alpha,\beta)$

Unless otherwise mentioned, let $0 \le \alpha < [p]_q, p, j \in \mathbb{N}, 0 \le q < 1, 0 < \beta \le 1$ and $n \in \mathbb{N}$.

In this section, we will calculate the coefficients estimate theorem for the functions belongs to the class $T_{p,q}(n,j,\alpha,\beta)$.

Theorem 2.1. *Let $f(z)$ be given by* (1)*, then $f(z) \in T_{p,q}(n,j,\alpha,\beta)$ if and only if*

$$\sum_{k=p+j}^{\infty} (1+\beta)[k]_q \left(\frac{[k]_q}{[p]_q}\right)^n a_k \le 2\beta([p]_q - \alpha). \tag{5}$$

Proof. Assume that (5) holds, we find from (1), (4) that

$$\begin{aligned} &\left|\partial q(D^n_{p,q}f(z)) - [p]_q z^{p-1}\right| - \beta \left|\partial q(D^n_{p,q}f(z)) + [p]_q z^{p-1} - 2\alpha z^{p-1}\right| \\ &= \left|\sum_{k=p+j}^{\infty} [k]_q \left(\frac{[k]_q}{[p]_q}\right)^n a_k z^{k-1}\right| - \beta \left|2([p]_q - \alpha)z^{p-1} + \sum_{k=p+j}^{\infty} [k]_q \left(\frac{[k]_q}{[p]_q}\right)^n a_k z^{k-1}\right| \\ &\le -2\beta([p]_q - \alpha)\left|z\right|^{p-1} + (1+\beta)\sum_{k=p+j}^{\infty} [k]_q \left(\frac{[k]_q}{[p]_q}\right)^n a_k \left|z\right|^{k-1} \\ &< -2\beta([p]_q - \alpha) + (1+\beta)\sum_{k=p+j}^{\infty} [k]_q \left(\frac{[k]_q}{[p]_q}\right)^n a_k \le 0. \end{aligned}$$

Hence, by the maximum modulus theorem, we have

$$\left| \frac{\frac{\partial q(D^n_{p,q}f(z))}{z^{p-1}} - [p]_q}{\frac{\partial q(D^n_{p,q}f(z))}{z^{p-1}} + [p]_q - 2\alpha} \right| < \beta,$$

thus, $f(z) \in T_{p,q}(n,j,\alpha,\beta)$.

Conversely, let $f(z) \in T_{p,q}(n,j,\alpha,\beta)$, then from (4), we find that

$$\left| \frac{\frac{\partial q(D^n_{p,q}f(z))}{z^{p-1}} - [p]_q}{\beta \frac{\partial q(D^n_{p,q}f(z))}{z^{p-1}} + \beta([p]_q - 2\alpha)} \right| =$$

$$\left| \frac{\sum_{k=p+j}^{\infty} -[k]_q \left(\frac{[k]_q}{[p]_q}\right)^n a_k z^{k-1}}{2\beta([p]_q - \alpha) z^{p-1} - \beta \sum_{k=p+j}^{\infty} [k]_q \left(\frac{[k]_q}{[p]_q}\right)^n a_k z^{k-1}} \right| < 1.$$

Now, since $Re(z) \leq |z|$ for all z, we have

$$Re \left\{ \frac{\sum_{k=p+j}^{\infty} [k]_q \left(\frac{[k]_q}{[p]_q}\right)^n a_k z^{k-1}}{2\beta([p]_q - \alpha) z^{p-1} - \beta \sum_{k=p+j}^{\infty} [k]_q \left(\frac{[k]_q}{[p]_q}\right)^n a_k z^{k-1}} \right\} < 1. \tag{6}$$

Now, choose values of z on the real axis so that $\frac{\partial q(D^n_{p,q}f(z))}{z^{p-1}}$ is real and letting $z \to 1^-$ through real values, we have

$$\sum_{k=p+j}^{\infty} [k]_q \left(\frac{[k]_q}{[p]_q}\right)^n a_k \leq 2\beta([p]_q - \alpha) - \beta \sum_{k=p+j}^{\infty} [k]_q \left(\frac{[k]_q}{[p]_q}\right)^n a_k.$$

This gives the required condition. □

3. Modified-Hadamard product theorems for $T_{p,q}(n,j,\alpha,\beta)$

In this section, we will introduce new theorems for modified Hadamard product for the functions belongs to the class $T_{p,q}(n,j,\alpha,\beta)$. As will as, we will get the sharp function.

Let $f_v(v = 1,2,...,s)$ be defined by

$$f_v(z) = z^p - \sum_{k=p+j}^{\infty} a_{k,v} z^k \qquad (a_{k,v} \geq 0). \tag{7}$$

The modified Hadamard product of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=p+j}^{\infty} a_{k,1} a_{k,2} z^k. \tag{8}$$

Theorem 3.2. *Let $f_v(z) \in T_{p,q}(n,j,\alpha_v,\beta)(v = 1,2,...,s)$ defined by (7), then $(f_1 * f_2 * ... * f_s)(z) \in T_{p,q}(n,j,\omega,\beta)$, where*

$$\omega = [p]_q - \frac{\prod_{v=1}^{s}([p]_q - \alpha_v)}{\left[(1+\beta)[p+j]_q \left(\frac{[p+j]_q}{[p]_q}\right)^n\right]^{s-1}}. \tag{9}$$

The result is sharp for the functions

$$f_v(z) = z^p - \frac{2\beta([p]_q - \alpha_v)}{(1+\beta)[p+j]_q \left(\frac{[p+j]_q}{[p]_q}\right)^n} z^{p+j} \quad . \tag{10}$$

Proof. To prove the Theorem we will use the mathematical induction. For $s = 1$. we see that $\gamma = [p]_q - \alpha_1$. For $s = 2$, Theorem 1 gives

$$\sum_{k=p+j}^{\infty} \frac{[k]_q(1+\beta)\left(\frac{[k]_q}{[p]_q}\right)^n}{2\beta([p]_q - \alpha_v)} a_{k,v} \leq 1 \quad (v = 1,2). \tag{11}$$

This gives that

$$\sum_{k=p+j}^{\infty} \frac{[k]_q(1+\beta)\left(\frac{[k]_q}{[p]_q}\right)^n}{\sqrt{\prod_{v=1}^{2} 2\beta([p]_q-\alpha_v)}} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{12}$$

To prove the case where $s=2$, we have to find the largest ω such that

$$\sum_{k=p+j}^{\infty} \frac{[k]_q(1+\beta)\left(\frac{[k]_q}{[p]_q}\right)^n}{2\beta([p]_q-\omega)} a_{k,1}a_{k,2} \leq 1, \tag{13}$$

such that

$$\frac{\sqrt{a_{k,1}a_{k,2}}}{2\beta([p]_q-\omega)} \leq \frac{1}{\sqrt{\prod_{v=1}^{2} 2\beta([p]_q-\alpha_v)}} \quad . \tag{14}$$

Then, using (12), we need to find the largest ω such that

$$\frac{1}{2\beta([p]_q-\omega)} \leq \frac{[k]_q(1+\beta)\left(\frac{[k]_q}{[p]_q}\right)^n}{\prod_{v=1}^{2} 2\beta([p]_q-\alpha_v)}, \tag{15}$$

that is

$$\omega \leq [p]_q - \frac{\prod_{v=1}^{2} 2\beta([p]_q-\alpha_v)}{2\beta[k]_q(1+\beta)\left(\frac{[k]_q}{[p]_q}\right)^n} \quad . \tag{16}$$

Defining the function $\vartheta(k)$ by

$$\vartheta(k) = [p]_q - \frac{\prod_{v=1}^{2} 2\beta([p]_q-\alpha_v)}{2\beta[k]_q(1+\beta)\left(\frac{[k]_q}{[p]_q}\right)^n}, \tag{17}$$

we see that $\vartheta'(k) \geq 0$ for $k \geq p+j$. This implies that

$$\omega \leq \vartheta(p+j) = [p]_q - \frac{\prod_{v=1}^{2} 2\beta([p]_q-\alpha_v)}{2\beta[p+j]_q(1+\beta)\left(\frac{[p+j]_q}{[p]_q}\right)^n}. \tag{18}$$

Therefore, the result is true for $s=2$.

Suppose that the result is true for any positive integer s. Then we have $(f_1 * f_2 * ... * f_s * f_{s+1})(z) \in T_{p,q}(n, j, \bigtriangledown, \beta)$, where

$$\bigtriangledown = [p]_q - \frac{([p]_q-\omega)2\beta([p]_q-\alpha_{s+1})}{[p+j]_q(1+\beta)\left(\frac{[p+j]_q}{[p]_q}\right)^n}. \tag{19}$$

After simple calculations, we have

$$\bigtriangledown = [p]_q - \frac{\prod_{v=1}^{s+1} 2\beta([p]_q-\alpha_v)}{2\beta\left[[p+j]_q(1+\beta)\left(\frac{[p+j]_q}{[p]_q}\right)^n\right]^s}. \tag{20}$$

Thus the result is true for $s+1$, the results is true for any positive integer s.

Putting $\alpha_v = \alpha$ $(v = 1, 2, ..., s)$ in Theorem 2, we have :

Corollary 3.0. *If $f_v(z) \in T_{p,q}(n, j, \alpha, \beta)(v = 1, 2, ..., s)$, then $(f_1 * f_2 * ... * f_s)(z) \in T_{p,q}(n, j, \omega_1, \beta)$, where*

$$\omega_1 = [p]_q - \frac{[2\beta]^{s-1}([p]_q-\alpha)]^s}{\left[[p+j]_q(1+\beta)\left(\frac{[p+j]_q}{[p]_q}\right)^n\right]^{s-1}} \tag{21}$$

The result is sharp for the functions

$$f_v(z) = z^p - \frac{2\beta([p]_q-\alpha)}{(1+\beta)[p+j]_q\left(\frac{[p+j]_q}{[p]_q}\right)^n} z^{p+j} \tag{22}$$

Putting $\beta = 1$ in Theorem 2, we have:

Corollary 3.0. *If $f_v(z) \in T_{p,q}(n,j,\alpha_v,1) = T_{p,q}(n,j,\alpha_v)(v=1,2,...,s)$, then $(f_1 * f_2 * ... * f_s)(z) \in T_{p,q}(n,j,\gamma)$, where*

$$\gamma = [p]_q - \frac{\prod_{v=1}^{s}([p]_q - \alpha_v)}{\left[[p+j]_q\left(\frac{[p+j]_q}{[p]_q}\right)^n\right]^{s-1}}. \tag{23}$$

The result is sharp for the functions

$$f_v(z) = z^p - \frac{([p]_q - \alpha_v)}{[p+j]_q\left(\frac{[p+j]_q}{[p]_q}\right)^n} z^{p+j} \quad (v=1,2,...,s). \tag{24}$$

□

Theorem 3.3. *If $f_v(z) \in T_{p,q}(n,j,\alpha_v,\beta)(v=1,2,...,s)$, then*

$$h(z) = z^p - \sum_{k=p+j}^{\infty}(\sum_{v=1}^{s} a_{k,v}^2)z^k \tag{25}$$

belongs to $T_{p,q}(n,j,\omega_2,\beta)$ where

$$\omega_2 = [p]_q - \frac{s[2\beta([p]_q - \alpha_0)]^2}{2\beta(1+\beta)([p+j]_q\left(\frac{[p+j]_q}{[p]_q}\right)^n} \quad (\alpha_0 = \min\{\alpha_1,\alpha_2,...,\alpha_s\}). \tag{26}$$

The result is sharp for the functions $f_v(z)$ given by (10).

Proof. Since Theorem 2 gives

$$\sum_{k=p+j}^{\infty}\left\{\frac{(1+\beta)[k]_q\left(\frac{[k]_q}{[p]_q}\right)^n}{2\beta([p]_q - \alpha_v)}\right\}^2 a_{k,v}^2 \le \left\{\sum_{k=p+j}^{\infty}\frac{(1+\beta)[k]_q\left(\frac{[k]_q}{[p]_q}\right)^n}{2\beta([p]_q - \alpha_v)}\right\}^2 \le 1, \tag{27}$$

then, for $(v=1,2,...,s)$ we have:

$$\sum_{k=p+j}^{\infty}\frac{1}{s}\left\{\frac{(1+\beta)[k]_q\left(\frac{[k]_q}{[p]_q}\right)^n}{2\beta([p]_q - \alpha_v)}\right\}^2 (\sum_{v=1}^{s} a_{k,v}^2) \le 1. \tag{28}$$

So, we may find the largest ω_2 such that

$$\sum_{k=p+j}^{\infty}\left\{\frac{(1+\beta)[k]_q\left(\frac{[k]_q}{[p]_q}\right)^n}{2\beta([p]_q - \omega_2)}\right\}(\sum_{v=1}^{s} a_{k,v}^2) \le 1. \tag{29}$$

Inequalities (28) and (29) leads to

$$\omega_2 \le [p]_q - \frac{s[2\beta([p]_q - \alpha_v)]^2}{2\beta(1+\beta)[k]_q\left(\frac{[k]_q}{[p]_q}\right)^n},$$

that is

$$\omega_2 \le [p]_q - \frac{s[2\beta([p]_q - \alpha_v)]^2}{2\beta(1+\beta)[p+j]_q\left(\frac{[p+j]_q}{[p]_q}\right)^n}. \tag{30}$$

This completes the proof □

Putting $\alpha_v = \alpha$ $(v=1,2,...,s)$ in Theorem 3, we have:

Corollary 3.0. *If $f_v(z) \in T_{p,q}(n,j,\alpha_v,\beta)(v=1,2,...,s)$, and $h(z)$ is defined by (25), then $h(z) \in T_{p,q}(n,j,\omega_3,\beta)$, where*

$$\omega_3 = [p]_q - \frac{s[2\beta([p]_q - \alpha)]^2}{2\beta(1+\beta)([p+j]_q\left(\frac{[p+j]_q}{[p]_q}\right)^n}. \tag{31}$$

The result is sharp for the functions defined by (22).

Putting $\beta = 1$ in Theorem 3, we have:

Corollary 3.0. *If $f_v(z) \in T_{p,q}(n, j, \alpha_v)(v = 1, 2, ..., s)$, and $h(z)$ is defined by (25), then $h(z) \in T_{p,q}(n, j, \omega_4)$, where*

$$\omega_4 = [p]_q - \frac{s[([p]_q - \alpha_0)]^2}{([p+j]_q \left(\frac{[p+j]_q}{[p]_q}\right)^n} \quad (\alpha_0 = \min\{\alpha_1, \alpha_2, ..., \alpha_s\}) \tag{32}$$

The result is sharp for the functions defined by (23).

Remark 1. *Putting $n = 0, q \to 1$ in the above results we get the result of ([4]).*

4. **Conclusions**

Throughout the paper, we used the definition of $q-$difference operator and $p-$Sălăgean operator to introduce the operator $q - p-$Sălăgean operator. Also, we use this operator to introduce the class $T_{p,q}(n, j, \alpha, \beta)$. After that, we calculate the coefficient estimates for functions belong to this class. As will as, we get modified Hadamard product for functions in this class. Finally, we obtained the sharpness function for our results.

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