Journal of Fractional Calculus and Applications
Vol. 14(1) Jan. 2023, pp. 182-190.
ISSN: 2090-5858.
http://math-frac.org/Journals/JFCA/

PROPERTIES FOR A CLASS RELATED TO A NEW FRACTIONAL DIFFERENTIAL OPERATOR

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Abstract. The object of this paper is to introduce and study some properties such as coefficient estimate, distortion theorems, radii of starlikness, convexity and close-to convexity of a class defined by Al-Oboudi - Al-Amoudi operator.

1. Introduction

Let $\mathcal{A}$ denote the class of functions:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in U=$\{z \in \mathbb{C} : |z| < 1\}$ and $\Im \subset \mathcal{A}$ for which

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0). \tag{1.2}$$

The fractional derivative of order α for analytic function f defined in a simply connected domain that contains zero is defined by [16]:

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \int_0^z \frac{f(t)}{(z-t)^{\alpha}} dt\ ,\ 0 \leq \alpha < 1, \tag{1.3}$$

$$\Omega^{\alpha} f(z) = \Gamma(2-\alpha)\, z^{\alpha} D_z^{\alpha} f(z)$$

$$= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} a_k z^k, \tag{1.4}$$

where multiplicity of $(z-t)^{-\alpha}$ is removed by requiring $\log(z-t)$, to be real when $z - t > 0$ (see also $[16], [17]$). Define the new operator

1991 *Mathematics Subject Classification.* Mathematics Subject Classification: 30C45.

Key words and phrases. Coefficient estimate, extreme points, closure theorem, growth and distortion.

Submitted Oct. 27, 2022. Revised Nov. 6, 2022.

$$\begin{aligned} D_\lambda^{0,0} f(z) &= f(z), \\ D_\lambda^{1,\alpha} f(z) &= (1-\lambda)\,\Omega^\alpha f(z) + \lambda z\left(\Omega^\alpha f(z)\right)' = D_\lambda^\alpha f(z),\ \lambda \geq 0,\ 0 \leq \alpha < 1, \\ D_\lambda^{2,\alpha} f(z) &= D_\lambda^\alpha \left(D_\lambda^\alpha f(z)\right), \\ D_\lambda^{n,\alpha} f(z) &= D_\lambda^\alpha \left(D_\lambda^{n-1,\alpha} f(z)\right),\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \end{aligned}$$

If f is given by (1.1) then

$$D_\lambda^{n,\alpha} f(z) = z + \sum_{k=2}^{\infty} \Psi_{k,n}(\alpha,\lambda) a_k z^k \tag{1.5}$$

where

$$\Psi_{k,n}(\alpha,\lambda) = \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left[1+\lambda(k-1)\right]^n. \tag{1.6}$$

Observe that this operator modified the operator of $[2,3]$, and we note that:
(*i*) $D_\lambda^{0,\alpha} = D_z^\alpha$ (see [16], [17]),
(*ii*) $D_1^{n,0} = D^n$ [19],
(*iii*) $D_\lambda^{n,0} = D_\lambda^n$ [1].

Definition 1. *For $\lambda, \mu \geq 0$, $\gamma \geq 1$, $0 \leq \alpha, \beta < 1$, $0 \leq \delta \leq 1$, $n \in \mathbb{N}_0$, a function $f \in \mathcal{A}$ is in the class $S_\lambda^{n,\alpha}(\delta,\gamma,\mu,\beta)$, if it satisfy.*

$$Re\left\{\frac{\gamma z G'(z)}{G(z)} - (\gamma-1)\right\} > \mu\left|\frac{\gamma z G'(z)}{G(z)} - \gamma\right| + \beta, \tag{1.7}$$

where

$$G(z) = (1-\delta) D_\lambda^{n,\alpha} f(z) + \delta z\left(D_\lambda^{n,\alpha} f(z)\right)'. \tag{1.8}$$

Let

$$TS_\lambda^{n,\alpha}(\delta,\gamma,\mu,\beta) := S_\lambda^{n,\alpha}(\delta,\gamma,\mu,\beta) \cap \Im. \tag{1.9}$$

Note that:
$S_\lambda^{n,\alpha}(0,1,\mu,\beta) = SP_{\alpha,\lambda}^n(\mu,\beta)$, $S_\lambda^{n,\alpha}(1,1,\mu,\beta) = UCV_{\alpha,\lambda}^n(\mu,\beta)$,
$[2,3$, with $\Psi_{k,n}(\alpha,\lambda)$ of the form (1.6)$]$. For different values of $n, \alpha, \lambda, \delta, \gamma, \mu,$ and β, we get the classes defined by $[3-15]$, [18] and [20].

2. Coefficient estimate

In the rest of the paper, let $0 \leq \alpha, \beta < 1$, $\lambda, \mu \geq 0$, $\gamma \geq 1$, $0 \leq \delta \leq 1$, $n \in \mathbb{N}_0$, $\Psi_{k,n}(\alpha,\lambda)$ as (1.6) and f given by (1.2).

Theorem 1. *For $f \in \Im$, $f \in TS_\lambda^{n,\alpha}(\delta,\gamma,\mu,\beta)$, if and only if*

$$\sum_{k=2}^{\infty}\left[1-\beta+\gamma\left(k-1\right)\left(1+\mu\right)\right]\left[1+\left(k-1\right)\delta\right]\Psi_{k,n}(\alpha,\lambda)a_k\leq 1-\beta. \tag{2.1}$$

Proof . Assume that (2.1) hold. And using the fact that for real β and complex number w,

$$\text{Re}(w)\geq\beta\Leftrightarrow|w+(1-\beta)|-|w-(1+\beta)|\ \geq 0, \tag{2.2}$$

it is sufficient to show that

$$\left|\frac{\gamma zG^{'}(z)}{G(z)}-(\gamma-1)-\mu\left|\frac{\gamma zG^{'}(z)}{G(z)}-\gamma\right|-(1+\beta)\right|\leq$$
$$\left|\frac{\gamma zG^{'}(z)}{G(z)}-(\gamma-1)-\mu\left|\frac{\gamma zG^{'}(z)}{G(z)}-\gamma\right|+(1-\beta)\right|. \tag{2.3}$$

For the right-hand side of (2.3)

$$\begin{aligned} R\quad:&=\left|\frac{\gamma zG^{'}(z)}{G(z)}-(\gamma-1)-\mu\left|\frac{\gamma zG^{'}(z)}{G(z)}-\gamma\right|+(1-\beta)\right| \\ &=\frac{1}{|G(z)|}\left|\gamma zG^{'}(z)+\left(2-\beta-\gamma\right)G(z)-\mu e^{i\theta}\left|\gamma zG^{'}(z)-\gamma G(z)\right|\right| \\ &>\frac{|z|}{|G(z)|}\{2-\beta-\sum_{k=2}^{\infty}\left[2-\beta+\gamma\left(k-1\right)\left(1+\mu\right)\right]\left[1+\left(k-1\right)\delta\right]\Psi_{k,n}(\alpha,\lambda)\left|a_k\right|\}. \end{aligned}$$

Similarly, for the left-hand side of (2.3) we have

$$\begin{aligned} L\quad:&=\left|\frac{\gamma zG^{'}(z)}{G(z)}-(\gamma-1)-\mu\left|\frac{\gamma zG^{'}(z)}{G(z)}-\gamma\right|-(1+\beta)\right| \\ &=\frac{1}{|G(z)|}\left|\gamma zG^{'}(z)-(\gamma-1)G(z)-\mu e^{i\theta}\left|\gamma zG^{'}(z)-\gamma G(z)\right|-(1+\beta)G(z)\right| \\ &<\frac{|z|}{|G(z)|}\{\beta+\sum_{k=2}^{\infty}\left[\gamma\left(k-1\right)\left(1+\mu\right)-\beta\right]\left[1+\left(k-1\right)\delta\right]\Psi_{k,n}(\alpha,\lambda)\left|a_k\right|\}. \end{aligned}$$

Since

$$R-L>$$

$$\frac{|z|}{|G(z)|}\{2\left(1-\beta\right)-2\sum_{k=2}^{\infty}\left[1-\beta+\gamma\left(k-1\right)\left(1+\mu\right)\right]\left[1+\left(k-1\right)\delta\right]\Psi_{k,n}(\alpha,\lambda)\left|a_k\right|\}\geq 0,$$

then the required condition (2.3) is satisfied, so $f\in TS_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$.

Conversely, let $f\in TS_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$, choosing the values of z, on the positive real axis, the inequality (1.7) reduces to:

$$\frac{1-\sum_{k=2}^{\infty}\left[1+(k-1)\gamma\right]\left[1+(k-1)\delta\right]\Psi_{k,n}(\alpha,\lambda)a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\left[1+(k-1)\delta\right]\Psi_{k,n}(\alpha,\lambda)a_k z^{k-1}}-\beta>$$

$$\mu\left|\frac{\sum_{k=2}^{\infty}\left[1+(k-1)\gamma\right]\left[1+(k-1)\delta\right]\Psi_{k,n}(\alpha,\lambda)a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\left[1+(k-1)\delta\right]\Psi_{k,n}(\alpha,\lambda)a_k z^{k-1}}\right|.$$

Letting z$\longrightarrow 1^-$, we obtain the desired inequality.

Corollary 2. *If* $f\in TS_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta), then$

$$a_k\leq\frac{1-\beta}{\left[1-\beta+\gamma\left(k-1\right)\left(1+\mu\right)\right]\left[1+\left(k-1\right)\delta\right]\Psi_{k,n}(\alpha,\lambda)}\quad\left(k\geq2\right). \tag{2.4}$$

Equality holds for

$$f(z)=z-\frac{1-\beta}{\left[1-\beta+\gamma\left(k-1\right)\left(1+\mu\right)\right]\left[1+\left(k-1\right)\delta\right]\Psi_{k,n}(\alpha,\lambda)}z^k\quad\left(k\geq2\right). \tag{2.5}$$

3. Growth and distortion theorem

Theorem 3. *Let* $f\in TS_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$. *Then, for* $|z|=r<1$

$$r-\frac{1-\beta}{B_2\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)}r^2\leq\left|f\left(z\right)\right|\leq r+\frac{1-\beta}{B_2\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)}r^2 \tag{3.1}$$

and

$$1-\frac{2(1-\beta)}{B_2\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)}r\leq\left|f'\left(z\right)\right|\leq 1+\frac{2(1-\beta)}{B_2\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)}r, \tag{3.2}$$

where

$$B_k\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)=\left[1-\beta+\gamma\left(k-1\right)\left(1+\mu\right)\right]\left[1+\left(k-1\right)\delta\right]\Psi_{k,n}(\alpha,\lambda)\quad\left(k\geq2\right). \tag{3.3}$$

The function gives the sharpness is

$$f\left(z\right)=z-\frac{1-\beta}{B_2\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)}z^2. \tag{3.4}$$

Proof. Since $f\in \mathrm{TS}_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$, then from Theorem 2.1 it follows that

$$\sum_{k=2}^{\infty}B_k\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)a_k\leq1-\beta.$$

We have

$$B_2(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)\sum_{k=2}^{\infty}a_k = \sum_{k=2}^{\infty}B_2(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)a_k \leq \sum_{k=2}^{\infty}B_k(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)a_k \leq 1-\beta$$

and therefore

$$\sum_{k=2}^{\infty}a_k \leq \frac{1-\beta}{B_2(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)}. \tag{3.5}$$

From (1.2) and (3.5) we have

$$|f(z)| \leq |z| + |z|^2\sum_{k=2}^{\infty}a_k|z|^{k-2} \leq r + r^2\sum_{k=2}^{\infty}a_k \leq r + \frac{1-\beta}{B_2(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)}r^2$$

and

$$|f(z)| \geq |z| - |z|^2\sum_{k=2}^{\infty}a_k|z|^{k-2} \geq r - r^2\sum_{k=2}^{\infty}a_k \geq r - \frac{1-\beta}{B_2(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)}r^2.$$

In virtue of Theorem 2.1, we also have

$$\frac{B_2(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{2}\sum_{k=2}^{\infty}ka_k = \sum_{k=2}^{\infty}\frac{B_2(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{2}ka_k \leq \sum_{k=2}^{\infty}B_k(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)a_k \leq 1-\beta,$$

which yields

$$\sum_{k=2}^{\infty}ka_k \leq \frac{2(1-\beta)}{B_2(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)}.$$

Thus,

$$|f'(z)| \leq 1 + \sum_{k=2}^{\infty}ka_k|z|^{k-1} \leq 1 + r\sum_{k=2}^{\infty}ka_k \leq 1 + \frac{2(1-\beta)}{B_2(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)}r$$

and

$$|f'(z)| \geq 1 - \sum_{k=2}^{\infty}ka_k|z|^{k-1} \geq 1 - r\sum_{k=2}^{\infty}ka_k \geq 1 - \frac{2(1-\beta)}{B_2(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)}r.$$

4. Extreme points

Next, we examine the extreme points for $TS_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$.

Theorem 4. *Let,*

$$f_1(z) = z \quad \textit{and} \quad f_k(z) = z - \frac{1-\beta}{B_k(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)}z^k \quad (k \geq 2). \tag{4.1}$$

Then $f \in TS_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$ if and only if

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z), \tag{4.2}$$

where $\lambda_k \geq 0$ $(k \geq 1)$, $\sum_{k=1}^{\infty}\lambda_k = 1$ and $B_k(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)$ is given by (3.3).

Proof. Assume that f as in (4.2), then

$$\begin{aligned} f(z) &= \lambda_1 z+\sum_{k=2}^{\infty}\lambda_k\left[z-\frac{1-\beta}{B_k(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)}z^k\right] \\ &= z - \sum_{k=2}^{\infty}\lambda_k\frac{1-\beta}{B_k(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)}z^k. \end{aligned}$$

Since

$$\sum_{k=2}^{\infty}B_k(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)\lambda_k\frac{1-\beta}{B_k(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)}=(1-\beta)\sum_{k=2}^{\infty}\lambda_k=(1-\beta)(1-\lambda_1)\leq 1-\beta,$$

it follows that $f \in \mathrm{TS}_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$. Conversely, suppose $f \in \mathrm{TS}_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$ and

consider

$$\lambda_k=\frac{B_k(n,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{1-\beta}a_k,\ k\geq 2 \qquad \textit{and} \qquad \lambda_1 := 1-\sum_{k=2}^{\infty}\lambda_k.$$

Then

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z).$$

Corollary 5. *The extreme points of the class* $TS_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$ *are given by* (4.1).

5. Closure theorem

Let the functions $f_j \in \Im$ $(j=1,2,...,p)$ defined by,

$$f_j(z)=z-\sum_{k=2}^{\infty}a_{k,j}z^k \quad (a_{k,j}\geq 0). \tag{5.1}$$

Theorem 6. *Let* $f_j \in TS_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$ $(j=1,2,...,p)$ *and let* $c_j \geq 0$ *such that* $\sum_{j=1}^{p}c_j=1$. *Then the function* $h \in TS_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$,

$$h(z)=\sum_{j=1}^{p}c_j f_j(z).$$

Proof. In virtue of the definition of h,

$$h(z)=\sum_{j=1}^{p}c_j\left[z-\sum_{k=2}^{\infty}a_{k,j}z^k\right]=z-\sum_{k=2}^{\infty}\left(\sum_{j=1}^{p}c_j a_{k,j}\right)z^k.$$

Since $f_j \in \mathrm{TS}_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$, for every $j=1,2,...,p$, then

$$\sum_{k=2}^{\infty} B_k\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)a_{k,j}\leq 1-\beta.$$

Hence, we get

$$\sum_{j=1}^{p} c_j\left(\sum_{k=2}^{\infty} B_k\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)a_{k,j}\right)\leq \sum_{j=1}^{p} c_j\left(1-\beta\right)=1-\beta,$$

therefore

$$h\left(z\right)\in TS_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta).$$

Corollary 7. *The class $TS_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$ is closed under convex linear combination.*

Proof. Assume that $f_j\in TS_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$, $(j=1,2)$ given by (5.1).
It is sufficient to show that the function $h\in TS_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$,

$$h\left(z\right)=cf_1\left(z\right)+\left(1-c\right)f_2\left(z\right)\quad\left(0\leq c\leq 1\right).$$

By taking $p=2, c_1=c$ and $c_2=1-c$ in Theorem 5.1 we obtain the corollary.

6. Radii of starlikeness, convexity and close-to-convexity

Theorem 8. *Let $f\in TS_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$. Then f is starlike of order ρ $(0\leq\rho<1)$ in $|z|<r_1\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)$, where*

$$r_1\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)=\inf_{k\geq 2}\left[\frac{\left(1-\rho\right)B_k\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)}{\left(k-\rho\right)\left(1-\beta\right)}\right]^{\frac{1}{k-1}}.$$

Proof. To prove the theorem we must show that

$$\left|\frac{zf'\left(z\right)}{f\left(z\right)}-1\right|\leq 1-\rho\quad\left(0\leq\rho<1\right).$$

For $z\in U$ with $|z|<r_1\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)$. We have

$$\left|\frac{zf'\left(z\right)}{f\left(z\right)}-1\right|=\left|\frac{-\sum_{k=2}^{\infty}\left(k-1\right)a_k z^{k-1}}{1-\sum_{k=2}^{\infty}a_k z^{k-1}}\right|\leq\frac{\sum_{k=2}^{\infty}\left(k-1\right)a_k\left|z\right|^{k-1}}{1-\sum_{k=2}^{\infty}a_k\left|z\right|^{k-1}}.$$

Thus

$$\left|\frac{zf'\left(z\right)}{f\left(z\right)}-1\right|\leq 1-\rho,$$

if

$$\sum_{k=2}^{\infty}\left(\frac{k-\rho}{1-\rho}\right)a_k\left|z\right|^{k-1}\leq 1, \tag{6.1}$$

in virtue of Theorem 2.1 we have

$$\sum_{k=2}^{\infty}\frac{B_k\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)a_k}{1-\beta}\leq 1,$$

hence, the inequality (6.1) will be true if

$$\left(\frac{k-\rho}{1-\rho}\right)|z|^{k-1} \leq \frac{B_k\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)}{1-\beta} \ \left(k \geq 2\right),$$

or if

$$|z| \leq \left[\frac{\left(1-\rho\right)B_k\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)}{\left(k-\rho\right)\left(1-\beta\right)}\right]^{\frac{1}{k-1}} \left(k \geq 2\right).$$

Corollary 9. *Let* $f \in TS_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$, *then* f *is convex of order* ρ $(0 \leq \rho < 1)$ *in* $|z| < r_2\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)$, *where*

$$r_2\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right) = \inf_{k\geq 2}\left[\frac{\left(1-\rho\right)B_k\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)}{k\left(k-\rho\right)\left(1-\beta\right)}\right]^{\frac{1}{k-1}}.$$

Corollary 10. *Let* $f \in TS_{\lambda}^{n,\alpha}(\delta,\gamma,\mu,\beta)$, *then* f *is close-to- convex of order* ρ $(0 \leq \rho < 1)$ *in* $|z| < r_3\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)$, *where*

$$r_3\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right) = \inf_{k\geq 2}\left[\frac{\left(1-\rho\right)B_k\left(n,\alpha,\lambda,\delta,\gamma,\mu,\beta\right)}{k\left(1-\beta\right)}\right]^{\frac{1}{k-1}}.$$

7. Acknowledgements

The authors would like to thanks the referees of the paper for their helpful suggestions.

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