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CAUCHY-TYPE PROBLEMS OF A FUNCTIONAL DIFFERINTEGRAL EQUATIONS WITH ADVANCED ARGUMENTS

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Abstract. In this article, weighted Cauchy type problems of fractional order with advanced arguments are considered. An existence result is obtained by the help of the well-known Schauder fixed point theorem, also a uniqueness result is given. An example is also given to illustrate the efficiency of the main theorem.

1. Introduction

Differential equations with advanced and deviated arguments are found to be important mathematical tools for the better understanding of several real world problems in physics, mechanics, engineering, economics, etc. see ([1], [5]).
One of the basic problems considered in the theory of differential equations with advanced arguments is to establish convenient conditions guaranteeing the existence of solutions of those equations. For the general theory and applications of differential equations with advanced and deviated arguments, we refer the reader to the references ([2], [8]-[10], [16]-[19]). For the fractional differential equations with advanced and deviated arguments see ([3], [12]) and the references therein.
Let $C[0,T]$ be the space of continuous functions defined on $[0,T]$.
Consider the Cauchy type fractional problems with advanced argument

$$(I)\ \begin{cases} D^{\alpha}u(t) &= f(t,u(\phi(t)))\ a.e.\ t\in(0,T],\ T<\infty \\ t^{1-\alpha}\,u(t)|_{t=0} &= b \end{cases}$$

$$(II)\ \begin{cases} D^{\alpha}u(t) &= f(t,u(\phi(t)))\ a.e.\ t\in(0,T],\ T<\infty \\ I^{1-\alpha}\,u(t)|_{t=0} &= b\,\Gamma(\alpha) \end{cases}$$

where D^{α} denote the Riemann-Liouville derivative of order $\alpha\in(0,1]$. Here we prove that the two problems (I) and (II) are equivalent and that there exists of at least one solution of the two problems when the function $\phi(t)$ is continuous advanced function.

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In [6] El-Sayed and Abd El-Salam has been studied the existence of at least one L_1-solution of the problem (I) where the function f satisfies the Carathèodory conditions and $\phi(t)$ is nondecreasing differentiable function such that $\phi' \geq M > 0$.

Nonlinear fractional differential equation with weighted initial data has been carried out by various researchers. In present, there are some papers which deal with the existence and multiplicity of solutions for weighted nonlinear fractional differential equations see ([4], [6], [7]) and the references therein.
The interest in the study of fractional-order differential equations lies in the fact that fractional-order models are more accurate than integer-order models, that is, there are more degrees of freedom in the fractional-order models. Fractional-order differential equations are also better for the description of hereditary properties of various materials and processes than integer-order differential equations. As a consequence, the subject of fractional differential equations is gaining much importance and attention, see the monographs of Kilbas et al. [11], Podlubny [13].

2. Preliminaries

In this section, we present some definitions, lemmas and notation which will be used in our theorems.
Definition 2.1. (see [13]-[15]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a Lebesgue-measurable function $f : R^+ \to R$ is defined by

$$I_a^\alpha \; f(t) \;=\; \frac{1}{\Gamma(\alpha)} \int_a^t \; (t-s)^{\alpha-1} \; f(s) \; ds,$$

when $a = 0$ we write $I_a^\alpha \; f(t) = I^\alpha \; f(t)$.
And we have, for $\alpha,\ \beta \in R^+$:
(r_1) $I_a^\alpha :\ L_1 \longrightarrow L_1$.
(r_2) $I_a^\alpha \, I_a^\beta \, f(t) \;=\; I_a^{\alpha\,+\,\beta} \, f(t)$.

Definition 2.2. (see [13]-[15]) The Riemann-Liouville fractional derivative of order $\alpha \in (0,1]$ of a Lebesgue-measurable function $f : R^+ \to R$ is defined by

$$D^\alpha \; f(t) \;= \frac{d}{dt} \; I^{1-\alpha} \; f(t) \;=\; \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \; \int_0^t \; (t-s)^{-\alpha} \; f(s) \; ds.$$

Theorem 2.1. (Schauder fixed point Theorem)
Let S be a non-empty, closed , convex and bounded subset of the Banach space X and let $Q : S \to S$ be a continuous and compact operator. Then the operator equation $Qu \;=\; u$ has at least one fixed-point in S.

3. Existence of continuous solutions

Here we study the existence of at least one continuous solution of the nonlocal Cauchy problems (I) and (II), assuming that:
$(\mathbf{h}_1)$ The function $f : [0,T] \times R \to R$ is measurable in t for all $u : [0,T] \to R$ and continuous in u for all $t \in [0,T]$ and there exist a bounded measurable function b and a function $a \in L_1[0,T]$ such that

$$|f(t,u)| \;\leq\; a(t) \;+\; b(t)\; |u| \text{ for all } (t,u) \in [0,T] \times R,$$

$(\mathbf{h}_2)$ $\phi : [0,T] \;\to\; [0,T]$ is continuous advanced function for all $t \in [0,T]$ and $t \;< \phi(t)$.

Definition 3.1. By a solution of the nonlocal Cauchy problems (I) and (II) we mean a functions $\{u : t^{1-\alpha}\ u(t)$ is continuous on the interval $[0,T]\}$ and this function satisfies (I) and (II).
Let $C[0,T] = \big\{u : u(t)$ is continuous on $[0,T] : ||u||_C = \max_{t\in[0,T]} |u(t)| \big\}$,

$$C_{1-\alpha}[0,T] = \big\{u : t^{1-\alpha}\, u(t) \text{ is continuous on } [0,T] \text{ with the weighted norm } ||u||_{C_{1-\alpha}} = ||t^{1-\alpha}\, u(t)||_C\big\}.$$

Lemma 3.1. (see [6]) The solution of the nonlocal problem (I) can be expressed by the fractional-order integral equation

$$u(t) = b\ t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f(s, u(\phi(s)))\, ds. \tag{1}$$

Lemma 3.2. The two problems (I) and (II) are equivalent.
Proof. From problem (II),

$$D^\alpha u(t) = f(t, u(\phi(t)))$$

i.e.

$$D\, I^{1-\alpha}\, u(t) = f(t, u(\phi(t))).$$

Integrating both sides from 0 to t, we get

$$I^{1-\alpha}\, u(t) - I^{1-\alpha}\, u(t)|_{t=0} = \int_0^t f(t, u(\phi(s)))\, ds$$

$$I^{1-\alpha}\, u(t) = b\,\Gamma(\alpha) + \int_0^t f(t, u(\phi(s)))\, ds,$$

operating by I^α on both sides, we get

$$I\, u(t) = \frac{b\, t^\alpha}{\alpha} + I^{1+\alpha}\, f(t, u(\phi(t))),$$

differentiating both sides, then we get (1).
Conversely, let $u(t)$ be a solution of (1), operating by $I^{1-\alpha}$ on both sides of it, then

$$I^{1-\alpha}\, u(t) = I^{1-\alpha}\, b\, t^{\alpha-1} + I^{1-\alpha}\, I^\alpha\, f(t, u(\phi(t))) = b\,\Gamma(\alpha) + \int_0^t f(t, u(\phi(s)))\, ds$$

and

$$I^{1-\alpha}\, u(t)|_{t=0} = b\,\Gamma(\alpha),$$

then problem (II) and equation (1) are equivalent.

Theorem 3.1. Assume that the hypothesis (h_1) and (h_2) hold. If $\frac{B\,\Gamma(\alpha)\, T^{1+\alpha}}{\Gamma(2\alpha)} < 1$, then the nonlocal problem (I) has at least one solution $u \in C_{1-\alpha}[0,T]$.
Proof. Define the subset $Q_r \subset C_{1-\alpha}[0,T]$ by

$$Q_r = \{u(t) \in C_{1-\alpha}[0,T] : ||u(t)||_{C_{1-\alpha}[0,T]} \le r\}$$

and

$$r \le \frac{b + \frac{M\, T^{1-\beta}}{\Gamma(\alpha-\beta+1)}}{1 - \frac{B\,\Gamma(\alpha)\, T^{1+\alpha}}{\Gamma(2\alpha)}},$$

where $M = \max_{[0,T]} I^\beta\, a(t),\ 0 < \beta < \alpha$ and $\sup |b(t)| = B$.
The set Q_r is nonempty, closed and convex.

Let $F: Q_r \;\rightarrow\; Q_r$ be an operator defined by

$$F\,u(t) \;=\; b\;t^{\alpha-1} \;+\; \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\; f(s,u(\phi(s)))\,ds. \tag{2}$$

For $u \in Q_r$, then F is a continuous operator, since, if $\{u_n(t)\}$ is a sequence in Q_r converges to $u(t), \forall t \in [0,T]$, then

$$\lim_{n\to\infty} Fu_n(t) \;\;=\;\; bt^{\alpha-1} + \lim_{n\to\infty}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f(s,u_n(\phi(s)))\,ds,$$

by assumptions (h_1) and (h_2) and the Lebesgue dominated convergence theorem we deduce that

$$\lim_{n\to\infty} Fu_n(t) \;=\; Fu(t).$$

Now from equation (2), let $u \in Q_r$, then

$$\begin{aligned}
|\,t^{1-\alpha}\,(Fu)(t)\,| \;&\leq\; b \;+\; t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\;|\,f(s,u(\phi(s)))\,|\,ds\\
&\leq\; b \;+\; t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\;(|a(s)| \,+\, |b(s)|\;|u(\phi(s))|)\,ds\\
&\leq\; b + t^{1-\alpha}\,I^{\alpha}\;|a(t)| + t^{1-\alpha}\;I^{\alpha}\;|b(t)|\,|u(\phi(t))|\\
&\leq\; b + t^{1-\alpha}\,I^{\alpha-\beta}\;I^{\beta}\;|a(t)| + B\,t^{1-\alpha}\;I^{\alpha-\beta}\;I^{\beta}\;|u(\phi(t))|\\
&\leq\; b + T^{1-\alpha}\,M\;\int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\;ds\\
&+\;\; B\;T^{1-\alpha}\;I^{\alpha-\beta}\;\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\;s^{\alpha-1}\,(\phi(s))^{1-\alpha}\,|u(\phi(s))|\,ds\\
&\leq\; b + T^{1-\alpha}\,M\,\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + B\;T^{1-\alpha}\,||u||_{C_{1-\alpha}}\,I^{\alpha-\beta}\;\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\;s^{\alpha-1}\,ds\\
&\leq\; b \;+\; \frac{M\,T^{1-\beta}}{\Gamma(\alpha-\beta+1)} \,+\, B\,T^{1-\alpha}\,||u||_{C_{1-\alpha}}\;I^{\alpha-\beta}\;\frac{t^{\alpha+\beta-1}\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\\
&\leq\; (b + \frac{M\,T^{1-\beta}}{\Gamma(\alpha-\beta+1)}) \,+\, \frac{B\,\Gamma(\alpha)\,T^{1+\alpha}}{\Gamma(2\alpha)}\;||u||_{C_{1-\alpha}}.
\end{aligned}$$

Then $\{Fu(t)\}$ is uniformly bounded in Q_r.
In what follows we show that F is a completely continuous operator. For $t_1, t_2 \in [0,T],\; t_1 < t_2$ such that $|t_2 - t_1| < \delta$, from (2) we have

$$\begin{aligned}
&\left|t_2^{1-\alpha}(Fu)(t_2) - t_1^{1-\alpha}(Fu)(t_1)\right|\\
&\leq\;\; \left|t_2^{1-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(\phi(s)))ds \;-\; t_1^{1-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(\phi(s)))ds\right|\\
&\leq\;\; \left|t_2^{1-\alpha}\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(\phi(s)))ds \;+\; t_2^{1-\alpha}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(\phi(s)))ds\right.\\
&\qquad \left. -\; t_1^{1-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(\phi(s)))ds\right|\\
&\leq\;\; \left|\left(t_2^{1-\alpha} - t_1^{1-\alpha}\right)\;\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(\phi(s)))ds + t_2^{1-\alpha}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(\phi(s)))ds\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left|t_2^{1-\alpha}-t_1^{1-\alpha}\right| \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} |\, f(s,u(\phi(s)))\,|\, ds + t_2^{1-\alpha} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} |\, f(s,u(\phi(s)))\,|\, ds \\
&\leq \left|t_2^{1-\alpha}-t_1^{1-\alpha}\right| \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \left(|a(s)| + |b(s)|\, |u(\phi(s))|\right) ds \\
&\quad + t_2^{1-\alpha} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \left(|a(s)| + |b(s)|\, |u(\phi(s))|\right) ds \\
&\leq \left|t_2^{1-\alpha}-t_1^{1-\alpha}\right| \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} |a(s)|\, ds + \left|t_2^{1-\alpha}-t_1^{1-\alpha}\right| \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} |b(s)|\, |u(\phi(s))| ds \\
&\quad + t_2^{1-\alpha} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} |a(s)|\, ds + t_2^{1-\alpha} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} |b(s)|\, |u(\phi(s))| ds
\end{aligned}
$$

Since $0<\beta<\alpha$, then the last equation can be written as

$$\left|t_2^{1-\alpha}(Fu)(t_2) - t_1^{1-\alpha}(Fu)(t_1)\right|$$

$$
\begin{aligned}
&\leq \left|t_2^{1-\alpha}-t_1^{1-\alpha}\right| I^{\alpha-\beta} I^{\beta} a(t_1) + B \left|t_2^{1-\alpha}-t_1^{1-\alpha}\right| I^{\alpha-\beta} I^{\beta} |u(\phi(t_1))| \\
&\quad + t_2^{1-\alpha} I_{t_1}^{\alpha-\beta} I_{t_1}^{\beta} a(t_2) + B\, t_2^{1-\alpha} I_{t_1}^{\alpha-\beta} I_{t_1}^{\beta} |u(\phi(t_2))| \\
&\leq M \left|t_2^{1-\alpha}-t_1^{1-\alpha}\right| \int_0^{t_1} \frac{(t_1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\, ds + M\, t_2^{1-\alpha} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\, ds \\
&\quad + B \left|t_2^{1-\alpha}-t_1^{1-\alpha}\right| I^{\alpha-\beta} \int_0^{t_1} \frac{(t_1-y)^{\beta-1}}{\Gamma(\beta)} y^{\alpha-1} (\phi(y))^{1-\alpha} |u(\phi(y))|\, dy \\
&\quad + B\, t_2^{1-\alpha} I_{t_1}^{\alpha-\beta} \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} s^{\alpha-1} (\phi(s))^{1-\alpha} |u(\phi(s))|\, ds \\
&\leq M \left|t_2^{1-\alpha}-t_1^{1-\alpha}\right| \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + M\, t_2^{1-\alpha} \frac{(t_2-t_1)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
&\quad + B\, ||u||_{C_{1-\alpha}} \left|t_2^{1-\alpha}-t_1^{1-\alpha}\right| I^{\alpha-\beta} \int_0^{t_1} \frac{(t_1-y)^{\beta-1}}{\Gamma(\beta)} y^{\alpha-1}\, dy \\
&\quad + B\, ||u||_{C_{1-\alpha}}\, t_2^{1-\alpha} I_{t_1}^{\alpha-\beta} \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} s^{\alpha-1}\, ds.
\end{aligned}
$$

Since $s>0$ for s in (t_1,t_2), then $s^{\alpha-1}$ is bounded by $t_1^{\alpha-1}$

$$
\begin{aligned}
&\leq M \left|t_2^{1-\alpha}-t_1^{1-\alpha}\right| \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + M\, T^{1-\alpha} \frac{(t_2-t_1)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
&\quad + B\, ||u||_{C_{1-\alpha}} \left|t_2^{1-\alpha}-t_1^{1-\alpha}\right| I^{\alpha-\beta} \frac{t_1^{\alpha+\beta-1}\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \\
&\quad + B\, ||u||_{C_{1-\alpha}}\, t_2^{1-\alpha}\, t_1^{\alpha-1} I_{t_1}^{\alpha-\beta} \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\, ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq M \left|t_2^{1-\alpha}-t_1^{1-\alpha}\right| \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + M\, T^{1-\alpha} \frac{(t_2-t_1)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
&\quad + \frac{B\, t_1^{2\alpha-1}\Gamma(\alpha)\, \Gamma(\alpha-\beta)}{\Gamma(2\alpha)} ||u||_{C_{1-\alpha}} \left|t_2^{1-\alpha}-t_1^{1-\alpha}\right| + B\, ||u||_{C_{1-\alpha}} \frac{(t_2-t_1)^{\beta}}{\Gamma(\beta+1)}.
\end{aligned}
$$

Hence the class $\{Fu(t)\}$ is equi-continuous, by Arzelá-Ascoli Theorem then $\{Fu(t)\}$ is relatively compact. Since all conditions of Schauder fixed point Theorem are hold, then F has a fixed point in Q_r. Therefor the the nonlocal problem (I) and (II) has at least one solution $u \in C_{1-\alpha}[0,T]$.

Theorem 3.2. Let $f : [0,T] \times R \to R$ be continuous and satisfy the Lipschitz condition

$$|f(t,u_1) \ - \ f(t,u_2)| \ \leq \ L \ |u_1 \ - \ u_2|, \ L > 0 \text{ for all } u_1, \, u_2 \in R.$$

If the condition (h_2) is satisfied and

$$\frac{2^{1-2\alpha}\sqrt{\pi}\, L\, T^\alpha}{\Gamma(\alpha+\frac{1}{2})} \ < \ 1,$$

then the problems (I) and (II) have a unique solution $u \in C_{1-\alpha}[0,T]$.
Proof. Let F be an operator defined by (2), then $F: \ C_{1-\alpha}[0,T] \ \to \ C_{1-\alpha}[0,T]$ and

$$\begin{aligned}
\left|t^{1-\alpha}(Fu)(t) - t^{1-\alpha}(Fv)(t)\right| \ &\leq \ t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \ |f(s,u(\phi(s))) - f(s,v(\phi(s)))|\, ds \\
&\leq \ Lt^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |u(\phi(s)) - v(\phi(s))|\, ds \\
&\leq \ L\, t^{1-\alpha}\int_0^t \frac{s^{\alpha-1}(t-s)^{\alpha-1}}{\Gamma(\alpha)} \ (\phi(s))^{1-\alpha}\, |u(\phi(s)) - v(\phi(s))|\, ds \\
&\leq \ L\, t^{1-\alpha} \ ||u-v||_{C_{1-\alpha}} \ \int_0^t \frac{s^{\alpha-1}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, ds \\
&\leq \ L\, t^{1-\alpha} \ ||u-v||_{C_{1-\alpha}} \frac{t^{2\alpha-1}\Gamma(\alpha)}{\Gamma(2\alpha)} \\
&\leq \ \frac{2^{1-2\alpha}\sqrt{\pi}\, L\, T^\alpha}{\Gamma(\alpha+\frac{1}{2})} \ ||u-v||_{C_{1-\alpha}}.
\end{aligned}$$

This means that

$$\left|\left|t^{1-\alpha}(Fu)(t) - t^{1-\alpha}(Fv)(t)\right|\right|_C \leq \frac{2^{1-2\alpha}\sqrt{\pi}\, L\, T^\alpha}{\Gamma(\alpha+\frac{1}{2})} \ ||u-v||_{C_{1-\alpha}}.$$

Then by using Banach fixed point Theorem, the operator F has a unique fixed point $u(t) \in C_{1-\alpha}$.

4. Example

In this section we provide an example illustrating the main results.

Example 4.1 Consider the nonlinear fractional differential problem

$$\begin{aligned}
D^{\frac{1}{2}} u(t) \ &= \ \cos^2 t \ + \ \frac{u(2t) + \sin u(2t)}{3+t^2} \ a.e. \ t \in (0,1], \\
t^{\frac{1}{2}}\, u(t)|_{t=0} \ &= \ b
\end{aligned}$$

Observe, the above problem is a special case of problem (I). Indeed if we put $\phi(t) = 2t$,

$$f(t,u) \;=\; \cos^2 t \;+\; \frac{u+\sin u}{3+t^2} \qquad \text{and} \qquad \alpha = \frac{1}{2}.$$

Then we can easy check that the assumptions of Theorem 3.1 are satisfied.
Then the problem has at least one solution $u \in C_{1-\alpha}[0,1]$.

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