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FIXED POINTS CLASSIFICATION OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS AS A DYNAMICAL SYSTEM

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Abstract. In this article, we have tried to classify fixed points stability of nonlinear fractional differential equations (NFDE) as a dynamical system. In this attempt, various types of the fixed points have been analyzed and the conditions on which a NFDE provides an asymptotic stable fixed point have been proved. Our analytical discussions have been combined with some examples to observe the accuracy of our claim. Finally, the semi stability of a limit cycle which exists in a system of NFDE has been analyzed.

1. Introduction

Although the theory of fractional calculus is a 300-year-old topic which can be traced back to Leibniz, Riemann, Liouville, Grünwald and Letnikov, the applications of fractional calculus to physics and engineering are recent focus of interest. It is well known that fractional differential equations (FDE) are a generalization of ordinary differential equations to an arbitrary order. These equations have attracted considerable attention because of their ability to model complex phenomena. Indeed, FDE capture nonlocal relation in space and time. Due to the extensive applications of FDE in engineering and science, research in this area has grown significantly worldwide. On the other hand, the considerable applications of FDE in modeling dynamical phenomena provide the motivation to generalize the concepts of dynamical systems to FDE. There are a good number of articles published in this regard and some of them noteworthy such as that of Tavazoei and Haeri in [13] in which they proved no existence of periodic solution in time invariant fractional order systems. Wang and Li in [14] studied the existence of limit cycles in the fractional order systems. Ahmad and his coworkers suggested some conditions on existence of Hopf bifurcation in fractional order dynamical systems [6]. They also discussed stability and numerical solutions of fractional-order predatorprey [1]. Matouk studied chaos, feedback control and synchronization in FDE [8, 9]. Matignon in his appealing work [7] analyzed the stability of finite-dimensional linear FDE and

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Taghvafard and Erjaee in [12] discussed the stability of synchronization in chaotic FDE systems. In [3] Deng and his coworkers studied the stability of n-dimensional linear FDE with time delays. For such linear systems, a characteristic equation was introduced using Laplace transform. They found out that if all roots of the characteristic equation have negative parts, then the equilibrium of linear system with fractional order is Lyapunov globally asymptotical stable. That is almost the same as that of classical differential equations. In [5] Diethelma and Ford discussed existence, uniqueness, and structural stability of solutions of NFDE. Zhang in[15] chose to study stability and Lyapunov functions for FDE.
In this article, an attempt was made to classify the fixed points of dynamical systems presented by FDE. For this purpose, in section 2 some basic definitions and results in fractional calculus have been stated. In section 3, we have analyzed the stability types of the fixed points in NFDE systems. Using our main theorem discussed in section 3, the fixed points of FDE systems have been classified in section 4. Finally, in section 5 we have considered a system satisfying Lienard theorem in the form of FDE. Then, similar to the study of Wang and Li in [14], we have shown that this system has a limit cycle. We have also shown that this limit cycle is at least semi stable.

2. PRELIMINARIES

In this section, we recall some basic definitions and results in fractional calculus.
Definition 1. Let $\alpha > 0$. Then function E_α is defined by

$$E_\alpha(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\alpha+1)}.$$

Whenever this series converges, it is called Mittag_ Leffler function of orderα.
Theorem 1. Let $\alpha > 0$. Then Mittag Leffler function E_α behaves as follows:
(a) $E_\alpha(re^{i\phi}) \to 0$ for $r \to \infty$ if $|\phi| > \frac{\alpha\pi}{2}$.
(b) $E_\alpha(re^{i\phi})$ remain bounded for $r \to \infty$ if $|\phi| = \frac{\alpha\pi}{2}$.
(c) $|E_\alpha(re^{i\phi})| \to \infty$ if $|\phi| < \frac{\alpha\pi}{2}$.
Proof. Refer to [4].
Definition 2. Let $\alpha \in \mathbf{R}^+$ then operator, J_a^α defined on $L_1[a,b]$

$$J_a^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt$$

for $a \leq x \leq b$ is called Riemann-Liouville fractional integral operator of order α. Here $L_p[a,b] := \{f : [a,b] \to \mathbf{R}; f$ is measurable on$[a,b]$ and $\int_a^b |f(x)|^p dx < \infty\}$.
Definition 3. Let $n \in \mathbf{N}$, $f \in A^n[a,b]$ and $t \in [a,b]$, then for $n-1 < \alpha \leq n$ the Cuputo's definition of the fractional-order derivative is defined as

$${}^cD_a^\alpha = \frac{1}{\Gamma(n-\alpha)} \int_a^x (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)dt.$$

Definition 4. By A^n or $A^n[a,b]$ we denote the set of functions with an absolutely continuous $(n-1)$st derivative, i.e. the functions f for which there exists (almost everywhere) a function $g \in L_1[a,b]$ such that

$$f^{(n-1)} = f^{(n-1)}(a) - \int_a^x g(t)dt.$$

Remark 1. The subscript a in definition 3 denote the left limit related to the operator D^α. Whenever it's clear what is the left limit of the operator or it doesn't have any difference whatever it is, for simplicity we eliminate it from notation. Also for simplicity in notification we eliminate the left superscript c in definition 3 so whenever we use the notation D^α we mean Caputo derivative else, it will be mentioned.

Theorem 2. Let $\alpha > 0, m = \lceil\alpha\rceil$ and $\lambda \in \mathbf{R}$. Then the solution of the initial value problem

$$D^\alpha y(x) = \lambda y(x) + q(x), \qquad y^{(k)}(0) = y_0^k \qquad (k = 0, ...m-1),$$

where $q \in C[0,h]$ is a given function, and can be expressed in the form of

$$y(x) = \sum_{k=0}^{m-1} y_0^{(k)} u_k(x) + \tilde{y}(x)$$

with

$$\tilde{y}(x) = \begin{cases} J_0^\alpha q(x) & \text{if} \quad \lambda = 0 \\ \frac{1}{\lambda}\int_0^x (q(x-t)\acute{u}_0(t)dt & \text{if} \quad \lambda \neq 0 \end{cases}$$

$u_k(x) := J_0^k e_\alpha(x), k = 0, 1, ...m-1$, and $e_\alpha(x) := E_\alpha(\lambda x^\alpha)$.

Proof. Refer to [4].

Remark 2. In the case $0 < \alpha < 1$, it is easy to see that the solution of the above initial value problem is

$$y(x) = y_0^{(0)} E_\alpha(\lambda x^\alpha) + \alpha \int_0^x q(x-t)t^{\alpha-1}E_\alpha^t(\lambda t^\alpha)dt.$$

Definition 5. The autonomous system $D^\alpha \mathbf{x} = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$ is said to be:
- stable iff $\forall \mathbf{x}_0, \exists M, \forall t \geq 0, ||\mathbf{x}(t)|| \leq M$,
- asymptotically stable iff $\lim_{t\to\infty} ||\mathbf{x}(t)|| = 0$, with some norm.

Theorem 3. The autonomous system $D^\alpha \mathbf{x} = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$ is:
- asymptotically stable iff $|arg(eig\mathbf{A})| > \frac{\alpha\pi}{2}$. In this case, the components of the state decay towards zero as $t^{-\alpha}$.
- stable iff either it is asymptotically stable, or those critical eigenvalues which satisfy $|arg(eig\mathbf{A})| = \frac{\alpha\pi}{2}$ have geometric multiplicity one.

Proof. Refer to [7].

Definition 6. Let $m, n \in \mathbf{N}$ and $\alpha \in \mathbf{R}^+$ and $f(x)$ having $\lceil m\alpha\rceil$ continuous derivtives on a neighborhood of $x_0 \in [a,b]$. Then the fractional order Taylor expansion of $f(x)$ around the $x = x_0$, is defined as follows:

$$f(x) = \sum_{n=0}^{m-1} \frac{D_a^{n\alpha} f(x)|_{x=x_0}}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} + J_a^{m\alpha} D_a^{m\alpha} f(x)|_{x=x_0}.$$

Definition 7. Suppose that $\mathbf{f} = (f_1, f_2)$, Then the fractional Jacobian matrix of $\mathbf{f}(\mathbf{x})$, $\mathbf{D}^\alpha\mathbf{f}(\mathbf{x})$, is defined as follow

$$\mathbf{D}^\alpha\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{\partial^\alpha f_1}{\partial x_1^\alpha}(x) & \frac{\partial^\alpha f_1}{\partial x_2^\alpha}(x) \\ \frac{\partial^\alpha f_2}{\partial x_1^\alpha}(x) & \frac{\partial^\alpha f_2}{\partial x_2^\alpha}(x) \end{pmatrix}.$$

It should be noted that the functions f_1 and f_2 must satisfy the conditions introduce in Definition 3, of course with respect to x_1 and x_2 or in the other words, the elements of the Jacobian matrix should be defined.

Grünwald-Letnikov approximation. Here, to evaluate the components of the matrix $\mathbf{D}^\alpha \mathbf{f}(\mathbf{x})$, the following approximation will be used, arising from the Grünwald-Letnikov [11]definition; i.e.,

$$ {}_aD_t^\alpha = \frac{1}{h^\alpha} \sum_{j=0}^{[\frac{(t-a)}{h}]} (-1)^j \binom{\alpha}{j} f(t - jh). $$

Diethelm method to solve FDE. Diethelm method [4], which is a PECE (predict, evaluate, correct, evaluate) numerical type method with Caputo derivative will be used. This method is based on fractional PC (predictor-corrector) algorithm whereby the following FDE

$$ {}_aD_t^\alpha = f(t, y(t)), \qquad 0 \leq t \leq T, $$
$$ y^{(k)}(0) = y_0^k, \qquad k = 0, 1, ...m - 1, \qquad (m = \lceil \alpha \rceil) $$

is equivalent to the Volterra integral equation

$$ y(t) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, y(s)) ds. $$

Now, set $h = \frac{T}{N}, t_n = nh, n = 0, 1...N$, and let $y_h(t_n)$ be an approximation of $y(t_n)$. Then the approximation of $y_h(t_{n+1})$ is given by

$$ y_h(t_{n+1}) = \sum_{k=0}^{m-1} c_k \frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, y_h^p(t_{n+1})) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_h(t_j)), $$

where

$$ a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n - \alpha)(n + 1)^\alpha & \text{if} \quad j = 0 \\ (n - j - 2)^{\alpha+1} + (n - j)^{\alpha+1} - 2(n - j - 1)^{\alpha+1} & \text{if} \quad 1 \leq j \leq n - 1 \\ 1 & \text{if} \quad j = n, \end{cases} $$

and

$$ y_h^p(t_{n+1}) = \sum_{k=0}^{m-1} c_k \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j)), $$

in which $b_{j,n+1} = \frac{h^\alpha}{\alpha}((n + 1 - j)^\alpha - (n - j)^\alpha))$. Therefore, the estimation error of the approximation is $max_{j=0,1,...n}|y(t_j) - y_h(t_j) = O(h^p)$ where $p = min(2, 1 + \alpha)$,[12, 14].

3. STABILITY ANALYSES

In this section, we state and prove the following theorem which can be considered as a corollary of the above stated Theorem 3.

Theorem 4. Let $\alpha > 0$, $\mathbf{f}$ satisfy conditions of Definition 3 and $\bar{\mathbf{x}}$ be the equilibrium point of the autonomous system of NFDE $D^\alpha \mathbf{x} = \mathbf{f}(\mathbf{x})$. Then $\bar{\mathbf{x}}$ is
- asymptotically stable iff $|arg(eig\mathbf{D}^\alpha f(\bar{\mathbf{x}}))| > \frac{\alpha\pi}{2}$.
- stable iff either it is asymptotically stable, or those critical eiganvalues which satisfy $|arg(eig\mathbf{D}^\alpha f(\bar{\mathbf{x}}))| = \frac{\alpha\pi}{2}$ have geometric multiplicity one.

proof. To study the stability properties of $\bar{\mathbf{x}}$, it is convenient to introduce a new variable $\mathbf{y}(t) = \mathbf{x}(t) - \bar{\mathbf{x}}$, and so the equilibrium point $\bar{\mathbf{x}}$ of $D^\alpha \mathbf{x} = \mathbf{f}(\mathbf{x})$ corresponds to the equilibrium point $\mathbf{y} = 0$ of the FDE

$$D^\alpha(\mathbf{y}) = \mathbf{f}(\mathbf{y} + \bar{\mathbf{x}}). \tag{1}$$

Using series expansion in fractional form, we can expand $f(\mathbf{y}+\bar{\mathbf{x}})$ about $\bar{\mathbf{x}}$ to obtain

$$\mathbf{f}(\mathbf{y} + \bar{\mathbf{x}}) = \mathbf{f}(\bar{\mathbf{x}}) + \frac{1}{\Gamma(1+\alpha)}\mathbf{D}^\alpha \mathbf{f}(\mathbf{x})\mathbf{y} + \mathbf{g}(\mathbf{y}),$$

where the remainder function $\mathbf{g}(\mathbf{y})$ satisfies

$$\mathbf{g}(\mathbf{0}) = 0 \qquad \text{and} \qquad \mathbf{D}^\alpha \mathbf{g}(\mathbf{0}) = 0. \tag{2}$$

Since $\mathbf{f}(\bar{\mathbf{x}}) = 0$, FDE $D^\alpha(\mathbf{y}) = \mathbf{f}(\mathbf{y} + \bar{\mathbf{x}})$ can be written in the form of

$$D^\alpha \mathbf{y} = \frac{1}{\Gamma(1+\alpha)}\mathbf{D}^\alpha \mathbf{f}(\bar{\mathbf{x}})\mathbf{y} + \mathbf{g}(\mathbf{y}) \tag{3}$$

The properties in (2) show that near the origin $\mathbf{g}(\mathbf{y})$ is small compared to $\mathbf{y}$. Therefore, (3) yields

$$D^\alpha \mathbf{y} = \frac{1}{\Gamma(1+\alpha)}\mathbf{D}^\alpha \mathbf{f}(\bar{\mathbf{x}})\mathbf{y}.$$

Now, according to Theorem 3 this system is asymptotically stable iff $|arg(eig\mathbf{D}^\alpha f(\bar{\mathbf{x}}))| > \frac{\alpha\pi}{2}$. This means $\mathbf{y}(t)$ approaches to $(0,0)$ as t goes toward infinity. So $(0,0)$ is an asymptotically stable fixed point of $D^\alpha \mathbf{y} = \mathbf{f}(\mathbf{y} + \bar{\mathbf{x}})$ or equivalently $\bar{\mathbf{x}}$ is asymptotically stable fixed point of system $D^\alpha \mathbf{x} = \mathbf{f}(\mathbf{x})$. Similarly $\bar{\mathbf{x}}$ is a stable fixed point of the system $D^\alpha \mathbf{x} = \mathbf{f}(\mathbf{x})$ if and only if the second condition of the theorem is satisfied.

4. FIXED POINTS CLASSIFICATION OF NFDE SYSTEMS

Using Theorem (3), fixed points of a NFDE system, $D^\alpha \mathbf{x} = \mathbf{f}(\mathbf{x})$ can be classified as in the following theorem

Theorem 5. Let $\alpha > 0$, $\mathbf{f}$ satisfy conditions of Definition 3 and $\bar{\mathbf{x}}$ be the equilibrium point of the autonomous NFDE system $D^\alpha \mathbf{x} = \mathbf{f}(\mathbf{x})$. Then $\bar{\mathbf{x}}$ is

i) asymptotically stable or stable node, respectively, if all the eigenvalues of $\mathbf{D}^\alpha \mathbf{f}(\bar{\mathbf{x}})$ are real and satisfy $|arg(eig\mathbf{D}^\alpha \mathbf{f}(\bar{\mathbf{x}}))| > \frac{\alpha\pi}{2}$ or those critical eigenvalues which satisfy $|arg(eig\mathbf{D}^\alpha \mathbf{f}(\bar{\mathbf{x}}))| = \frac{\alpha\pi}{2}$ have geometric multiplicity one.

ii) unstable node if all the eigenvalues of $\mathbf{D}^\alpha \mathbf{f}(\bar{\mathbf{x}})$ satisfy $|arg(eig\mathbf{D}^\alpha \mathbf{f}(\bar{\mathbf{x}}))| < \frac{\alpha\pi}{2}$.

iii) saddle point if some eigenvalues satisfy $|arg(eig\mathbf{D}^\alpha \mathbf{f}(\bar{\mathbf{x}}))| > \frac{\alpha\pi}{2}$ and some others satisfy$|arg(eig\mathbf{D}^\alpha \mathbf{f}(\bar{\mathbf{x}}))| < \frac{\alpha\pi}{2}$.

iv) stable improper node if all eigenvalues are the same and satisfy $|arg(eig\mathbf{D}^\alpha \mathbf{f}(\bar{\mathbf{x}}))| > \frac{\alpha\pi}{2}$.

v) unstable improper node if all eigenvalues are the same and satisfy $|arg(eig\mathbf{D}^\alpha \mathbf{f}(\bar{\mathbf{x}}))| < \frac{\alpha\pi}{2}$.

vi) stable focus if eigenvalues are complex and satisfy $|arg(eig\mathbf{D}^\alpha \mathbf{f}(\bar{\mathbf{x}}))| > \frac{\alpha\pi}{2}$.

vii) unstable focus if eigenvalues are complex and satisfy $|arg(eig\mathbf{D}^\alpha \mathbf{f}(\bar{\mathbf{x}}))| < \frac{\alpha\pi}{2}$.

Proof. Here, we consider the case in $\mathbf{R}^2$, the case for higher dimension will be the same. To start with, we consider NFDE system $\mathbf{D}^\alpha(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ and its linearization as

$$D^\alpha \mathbf{x} = \frac{1}{\Gamma(\alpha+1)}\mathbf{D}^\alpha \mathbf{f}(\bar{\mathbf{x}})\mathbf{x}. \tag{4}$$

Now, we suppose matrix $\mathbf{D}^{\alpha}\mathbf{f}(\bar{\mathbf{x}})$ has two real eigenvalues. Then system (4), without loss of generality, can be written in the normal form:

$$\begin{cases} D^{\alpha}x_1 = \lambda_1 x_1 \\ D^{\alpha}x_2 = \lambda_2 x_2. \end{cases} \tag{5}$$

According to Remark 1 system (5) has the following solution

$$\begin{cases} x_1(t) = x_1(0)E_{\alpha}(\lambda_1 t^{\alpha}) \\ x_2(t) = x_2(0)E_{\alpha}(\lambda_2 t^{\alpha}). \end{cases} \tag{6}$$

Considering Theorem 1, here $t \in \mathbf{R}$ and arguments of λ_1 and λ_2 correspond to ϕ . Therefore, if $|arg(\lambda_1)| > \frac{\alpha\pi}{2}$ and $|arg(\lambda_2)| > \frac{\alpha\pi}{2}$, then $lim_{t\to\infty}x_1(t) = 0$ and $lim_{t\to\infty}x_2(t) = 0$, which proves the asymptotic stability of the fixed point in part (i). Note that if $|arg(eig\mathbf{D}^{\alpha}\mathbf{f}(\bar{\mathbf{x}}))| = \frac{\alpha\pi}{2}$, then, from Theorem 1(b), $\mathbf{x}(t)$ remains bounded and consequently $\bar{\mathbf{x}}$ is stable and this complete the proof of part (i). Obviously, if $|arg(\lambda_1)| < \frac{\alpha\pi}{2}$ and $|arg(\lambda_2)| < \frac{\alpha\pi}{2}$, then $lim_{t\to\infty}x_1(t) = \infty$ and $lim_{t\to\infty}x_2(t) = \infty$, which proves part (ii). On the other hand, if $|arg(\lambda_1)| < \frac{\alpha\pi}{2}$ and $|arg(\lambda_2)| > \frac{\alpha\pi}{2}$, then $lim_{t\to\infty}x_1(t) = \infty$ and $lim_{t\to\infty}x_2(t) = 0$, which means that the fixed point is a saddle. Similar proofs can be stated for the cases (iv) and (v) for which we have the same eigenvalues.

Note that for cases (i) to (iii), to obtain the direction field near the equilibrium point, from (6) we have:

$$\frac{dx_2}{dx_1} = \frac{d(x_2(0)E_{\alpha}(\lambda_2 t^{\alpha})}{d(x_1(0)E_{\alpha}(\lambda_1 t^{\alpha})} = \frac{x_2(0)\lambda_2 E_{\alpha}(\lambda_2 t^{\alpha})}{x_1(0)\lambda_1 E_{\alpha}(\lambda_1 t^{\alpha})} = \frac{x_2(0)\lambda_2 x_2}{x_1(0)\lambda_1 x_1}. \tag{7}$$

In cases (iv) and (v), since $\lambda_1 = \lambda_2$ the slope is obtained as follows:

$$\frac{dx_2}{dx_1} = \frac{x_2(0)}{x_1(0)}$$

To prove parts (vi) and (vii) it is assumed that $\mathbf{D}^{\alpha}\mathbf{f}(\bar{\mathbf{x}})$ has complex eigenvalues. Therefore, the normal form of the linearized system $D^{\alpha}\mathbf{x} = \mathbf{f}(\mathbf{x})$ is as follows (assuming that $\lambda_{1,2} = a \pm ib$)

$$\begin{aligned} D^{\alpha}x_1 &= ax_1 + bx_2 \\ D^{\alpha}x_2 &= -bx_1 + ax_2. \end{aligned} \tag{8}$$

To show that the trajectories are in spiral shape near the equilibrium point, we consider system (8) in the matrix form $D^{\alpha}\mathbf{x} = \mathbf{B}\mathbf{x}$, where

$$\mathbf{B} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Noting that Theorem 2 holds for matrix form too ([2]), we need to calculate $E_{\alpha}(\mathbf{B}t^{\alpha})$, i.e.

$$E_{\alpha}(\mathbf{B}t^{\alpha}) = \sum_{k=0}^{\infty} \frac{\mathbf{B}^k t^{\alpha k}}{\Gamma(\alpha k + 1)}$$

However, it is easy to see that ([10])

$$\mathbf{B}^k = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^k = \begin{pmatrix} Re(\lambda^k) & -Im(\lambda^k) \\ Im(\lambda^k) & Re(\lambda^k) \end{pmatrix}.$$

Therefore,

$$E_\alpha(\mathbf{B}t^\alpha) = \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \begin{pmatrix} Re(\lambda^k) & -Im(\lambda^k) \\ Im(\lambda^k) & Re(\lambda^k) \end{pmatrix}$$

On the other hand, if we write $\lambda = re^{i\theta}$, where $r = \sqrt{a^2+b^2}$ and $\theta = arctg(\frac{b}{a})$, then $\lambda^k = r^k e^{ik\theta} = r^k(cos(k\theta) + i\ sin(k\theta))$ and therefore

$$E_\alpha(\mathbf{B}t^\alpha) = \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \begin{pmatrix} cos(k\theta) & -sin(k\theta) \\ sin(k\theta) & cos(k\theta) \end{pmatrix}.$$

As it can be seen, the rotation matrix is apparent in the solution of the system which shows that the trajectories of (8) are in spiral shape near the equilibrium point. Note that according to Theorem 5, $\bar{\mathbf{x}}$ is asymptotically stable if and only if $|arg\lambda_{1,2}| > \frac{\alpha\pi}{2}$, stable if $|arg\lambda_{1,2}| = \frac{\alpha\pi}{2}$ and unstable if $|arg\lambda_{1,2}| < \frac{\alpha\pi}{2}$. This completes the Theorem. Now, we consider some examples.

Example 1. The following NFDE system with $\alpha = 0.98$

$$\begin{cases} D^\alpha x_1 = 4 - x_1^2 \\ D^\alpha x_2 = -x_2 + x_1 x_2^2, \end{cases} \tag{9}$$

has a fixed point at (2, 0) and at this point we have

$$\mathbf{D}^{0.98}\mathbf{f}|_{(2,0)} = \begin{pmatrix} -3.8218 & 0 \\ 0 & -0.9647 \end{pmatrix}.$$

Obviously, $\lambda_1 = -3.8218, \lambda_2 = -0.9647$ and the absolute values of arguments λ_1, λ_2 are equal to π which are more than $\frac{\alpha\pi}{2}$. Therefore, (2, 0) is a stable node for NFDE system (9). See Figure(1) for numerical results.

Example 2. As a saddle point example, we consider

$$\begin{cases} D^\alpha x_1 = -x_1 \\ D^\alpha x_2 = -x_2 + x_1^2. \end{cases} \tag{10}$$

The only fixed point of this system is the origin. Evaluating $\mathbf{D}^\alpha\mathbf{f}|_{(0,0)}$ and then $|arg(eig\mathbf{D}^\alpha\mathbf{f}|_{(0,0)})|$ with $\alpha = 0.98$, we can see that one of them is zero which is less than $\frac{\alpha\pi}{2}$ and the other one is π which is greater than $\frac{\alpha\pi}{2}$. Therefore, as we can see in Figure (2) we have a saddle point in (0, 0).

Example 3. Consider the following system

$$\begin{cases} D^\alpha x_1 = x_1 + x_2 \\ D^\alpha x_2 = x_1^2 x_2 + x_2. \end{cases} \tag{11}$$

This system has a fixed point at (0,0) for which we have

$$\mathbf{D}^{0.98}\mathbf{f}|_{(2,0)} = \begin{pmatrix} 0.9134 & 0.9134 \\ 0 & 0.9134 \end{pmatrix}$$

and $\lambda_1 = \lambda_2 = 0.9134$. Hence, (0, 0) is an improper node. Again note that since the absolute value of the arguments of the eigenvalues are zero which are less than $\frac{\alpha\pi}{2}$, with $\alpha = 0.98$, therefore, (0, 0) is an unstable improper node. See the numerical

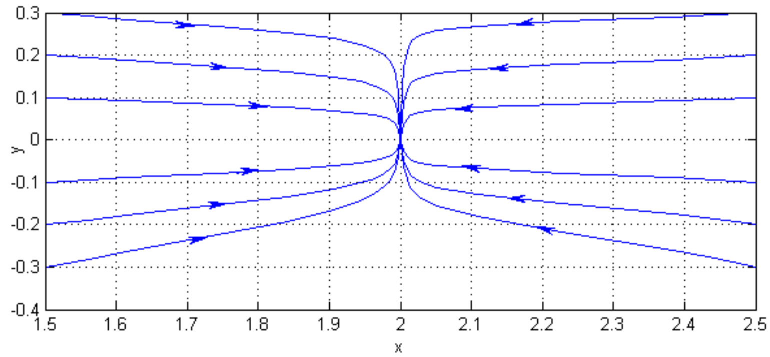

Figure 1. Stable node for NFDE system (9) with $\alpha = 0.98$.

results in Figure (3).

Example 4. As an example of stable focus, we can consider the following system

$$\begin{cases} D^\alpha x_1 = x_1(1 - \frac{x_1}{2} - x_2) \\ D^\alpha x_2 = x_2(x_1 - 1 - \frac{x_2}{2}. \end{cases} \tag{12}$$

This system has a fixed point at (6/5, 2/5) with

$$\mathbf{D}^{0.98}\mathbf{f}|_{(2,0)} = \begin{pmatrix} -0.5718 & -1.1478 \\ 0.3826 & -0.1892 \end{pmatrix},$$

$\lambda_{1,2} = -0.3805 \pm i0.6354$, and the absolute value of the argument is equal to 2.1110 which is more than $\frac{\alpha\pi}{2}$ with $\alpha = 0.98$. Therefore (6/5, 2/5) is a stable focus. We can see the numerical results in Figure 4.

Example 5. As an example of unstable focus consider

$$\begin{cases} D^\alpha x_1 = x_2 \\ D^\alpha x_2 = x_1(1 - x_1^2) + x_2, \end{cases} \tag{13}$$

This system has a fixed point at (1,0) with

$$\mathbf{D}^{0.98}\mathbf{f}|_{(2,0)} = \begin{pmatrix} 0 & 0.9134 \\ -1.8257 & 0.9134 \end{pmatrix},$$

$\lambda_{1,2} = -0.4566 \pm i1.208$ and absolute value of the argument λ_1, λ_2 is equal to 1.2094 which is less than $\frac{\alpha\pi}{2}$ with $\alpha = 0.98$. Therefore (1,0) is an unstable focus. See Figure (5) for numerical results.

5. Semi Stability of Limit Cycles in NFDE Systems

Suppose NFDE system $D^\alpha \mathbf{x} = \mathbf{f}(\mathbf{x})$ has a limit cycle surrounding just a fixed point. Then it is clear that this limit cycle is at least semi stable whenever the fixed point is unstable. On the other hand, limit cycle is at least semi unstable whenever the fixed point is stable. In this case, it is enough to analyze the stability of the fixed point inside the limit cycle. However, fixed point stability of a NFDE system can be analyzed by Theorem 5 in the above section. Therefore, using Theorem 5

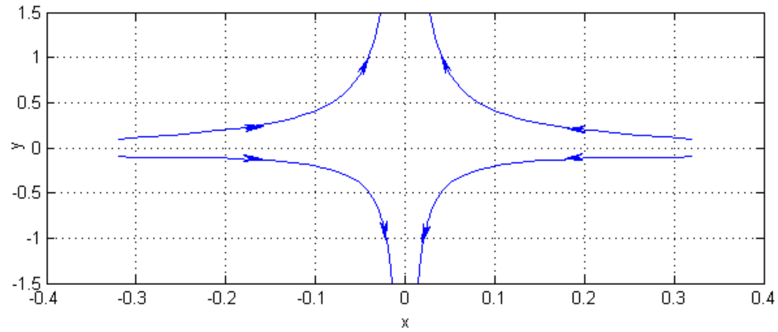

FIGURE 2. Saddle point for NFDE system (10) with $\alpha = 0.98$.

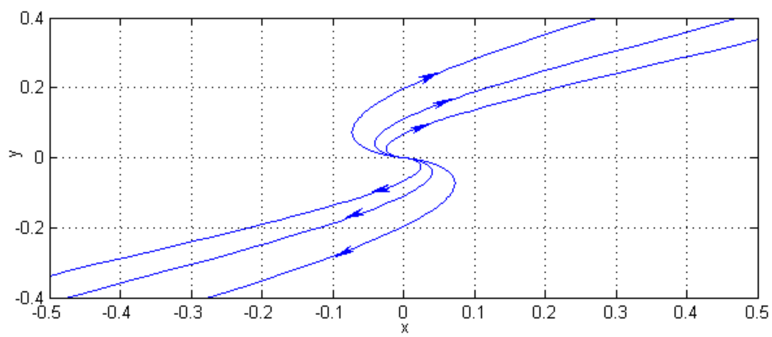

FIGURE 3. Unstable improper node for NFDE system (11) with $\alpha = 0.98$.

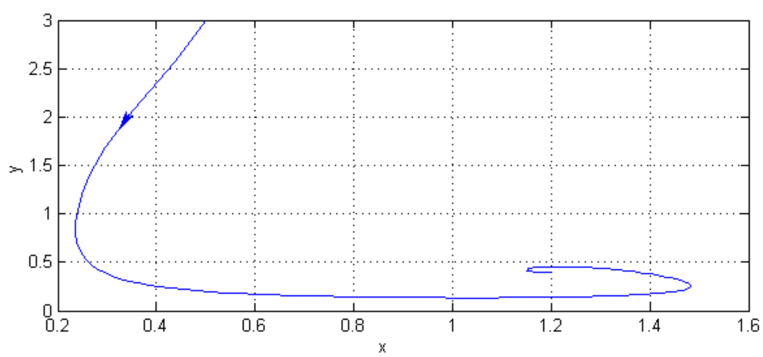

FIGURE 4. Stable focus point for NFDE system (12) with $\alpha = 0.98$.

we can analyze (at least) semi stability of existing limit cycle in a NFDE system. For this purpose, we consider fractional order which satisfies Lienard theorem. As is well known, in the classical case such a system has a limit cycle containing a fixed point[10]. Here, for the fractional order, first the stability of the fixed point inside the limit cycle is analyzed, then using Diethelm method the numerical results are

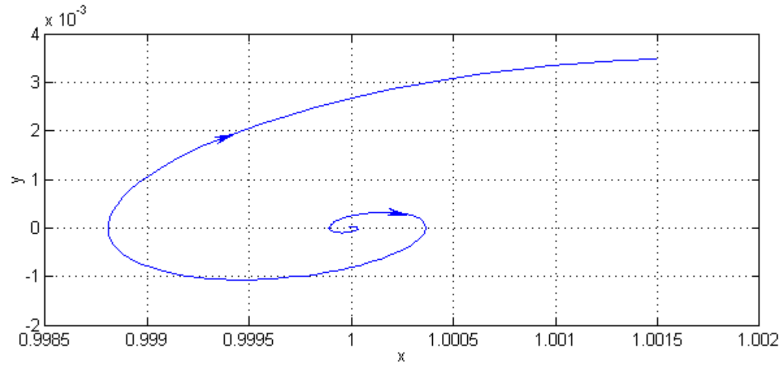

FIGURE 5. Unstable focus point for NFDE system (13) with $\alpha = 0.98$.

illustrated. Note that as long as the limit cycle exists in the classical ODE system this limit cycle also exists for its NFDE counterpart, provided that stability of fixed point inside the limit cycle does not change its stability as α varies.

Remark 2. From above discussion, we emphasis that the condition on which a limit cycle exists in a NFDE systems is the same as in ODE counterpart systems. Provided that as we change the value of derivative order α, the stability type of equilibrium point inside limit cycle does not change. Of course, this stability can be checked by above Theorem 5. In another word, no need to prove the existence of limit cycles in NFDE systems as long as we know the existence of limit cycle in ODE counterpart systems.

Example 6. Consider the following system which satisfies the Lienard theorem conditions

$$\begin{cases} \dot{x} = y - \dfrac{(x^3 - x)}{x^2 + 1} \\ \dot{y} = -x, \end{cases} \tag{14}$$

This system has one stable limit cycle containing just one unstable fixed point (0,0) [10]. Now, according to Theorem 5, $(0,0)$ is an unstable fixed point. Since, matrix $\mathbf{D}^{\alpha}\mathbf{f}$ with $\alpha = 1$ is in the following form

$$\mathbf{Df} = \begin{pmatrix} -\dfrac{3x^2 - 1}{x^2 + 1} + \dfrac{2(x^3 - x)x}{(x^2 + 1)^2} & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{Df}(0,0) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

The eigenvalues of this matrix are $\dfrac{1}{2} \pm i\dfrac{1}{2}\sqrt{3}$ with positive real parts which confirms instability of (0,0). Note that $|arg(eig\mathbf{Df}(0,0)| = \dfrac{\pi}{3} < \dfrac{\alpha\pi}{2} = \dfrac{\pi}{2}$ (See Figure 6 (a)). Converting this Lienard system to a fractional one with various fractional orders i.e., 0.97 and 0.7, we have the following results. For $\alpha = 0.97$ we obtain

$$\mathbf{D}^{0.97}\mathbf{f}|_{(0,0)} = \begin{pmatrix} 0.9352 & 0.9630 \\ -0.9630 & 0 \end{pmatrix},$$

with eigenvalues $0.4676 \pm i0.8419$ and

$$|arg(eig\mathbf{D}^{0.97}\mathbf{f}|_{(0,0)}| = 1.0638 < \frac{0.97\pi}{2} = 1.5237$$

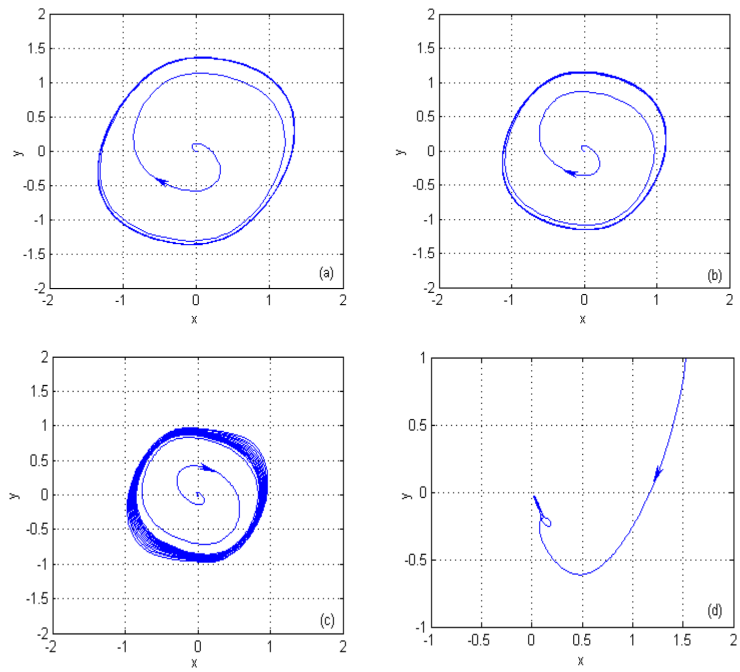

FIGURE 6. (a) Semi stable limit cycle in Lienard system with derivative order and initial value $(x_0, y_0) = (0.001, 0.002)$. (b) and (c) Semi stable limit cycle in Lienard system with order $\alpha = 0.97$, initial value $(x_0, y_0) = (0.001, 0.002)$, order$\alpha = 0.9$ and initial value $(x_0, y_0) = (0.001, 0.002)$, respectively. (d) Limit cycle no longer exists in Lienard system with $\alpha = 0.7$ and initial value $(x_0, y_0) = (1, 2)$.

This shows that (0, 0) is an unstable fixed point and the system has a limit cycle which is at least semi stable. See Figure 6 (b) for numerical results. Now, for $\alpha = 0.7$ we have

$$|arg(eig\mathbf{D}^{0.7}\mathbf{f}|_{(0,0)}| = 1.1714 > \frac{0.7\pi}{2} = 1.0996,$$

which shows that (0, 0) is asymptotically stable in fact it is a stable focus. Therefore, as we claimed above, limit cycle no longer exists in this NFDE system, since the stability type of the fixed point (0, 0) inside limit cycle has been changed from unstable case (for $\alpha = 0.97$) to stable case (for $\alpha = 0.7$) (Figure 6(d)).

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