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ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY A GENERALIZED DIFFERENTIAL OPERATOR AND MULTIPLIER TRANSFORMATION

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Abstract. In this paper we define a new operator using the a generalized differential operator and multiplier transformation. Denote by $ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)$, the operator given by $ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell) : \mathcal{A}_n \to \mathcal{A}_n$, $ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z) = (1-\alpha)I(m, \lambda, \ell)f(z)+\alpha D_{m,\lambda,\mu}(\alpha_1, \beta_1)f(z)$, for $z \in U$, where $I(m, \lambda, \ell)f(z)$ denote the multiplier transformation, $D_{m,\lambda,\mu}(\alpha_1, \beta_1)f(z)$ is the generalized differential operator and $\mathcal{A}_n = \left\{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + ..., z \in U\right\}$ is the class of normalized analytic functions. A certain subclass denote by $\mathcal{ID}_m(\delta, \lambda, \mu, \ell, \alpha, \alpha_1, \beta_1)$ of analytic functions in the open unit disc is introduced by means of the new operator. By making use of the concept of differential subordination we will derive various properties and characteristics of the class $\mathcal{ID}_m(\delta, \lambda, \mu, \ell, \alpha, \alpha_1, \beta_1)$. Also several differential subordinations are established regarding the operator $ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)$.

1. Introduction

Denote by U the unit disk of the complex plane, $U = \{z \in C : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorpic functionsin U.

Let

$$\mathcal{A}(p, n) = \left\{ f \in \mathcal{H}(U) : f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, z \in U \right\},$$

with $\mathcal{A}(1, n) = \mathcal{A}_n$ and

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + ..., z \in U\},$$

where $p, n \in \mathbb{N}, a \in \mathbb{C}$.

Denote by $K = \left\{ f \in \mathcal{A}_n : \Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0, z \in U \right\}$, the class of normalized convex functions in U.

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If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$ for all $z \in U$ such that $f(z) = g(w(z))$ for all $z \in U$.

If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$. let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h an univalent function in U. If p is analytic in U. and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad \text{for} \quad z \in U, \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a diominant, if $p \prec q$ for all p satisfying (1).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of U.

Definition 1 [3]. For $f \in \mathcal{A}(p, n)$, $p, n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, \ell \geq 0$, the operator $I_p(m, \lambda, \ell)f(z)$ is defined by the following infinite series

$$I_p(m, \lambda, \ell)f(z) := z^p + \sum_{j=p+n}^{\infty} \left(\frac{p + \lambda(j - p) + \ell}{p + \ell} \right)^m a_j z^j$$

Remark 1. It follows from the above definition that

$$I_p(0, \lambda, \ell)f(z) = f(z),$$

$$(p + \ell)I_p(m + 1, \lambda, \ell)f(z) = [p(1 - \lambda) + \ell]\, I_p(m, \lambda, \ell)f(z) + \lambda z(I_p(m, \lambda, \ell)f(z))',$$

for $z \in U$.

Remark 2. If $p = 1$, we have $I_1(m, \lambda, \ell)f(z) = I(m, \lambda, \ell)$ and

$(\ell+1)I(m+1, \lambda, \ell)f(z) = [\ell+1-\lambda]I(m, \lambda, \ell)f(z)+\lambda z(I(m, \lambda, \ell)f(z))'$, for $z \in U$.

Remark 3. If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then

$$I(m, \lambda, \ell)f(z) = z + \sum_{j=n+1}^{\infty} \left(\frac{1 + \lambda(j - 1) + \ell}{\ell + 1} \right)^m a_j z^j,$$

for $z \in U$.

Remark 4. For $\ell = 0, \lambda \geq 0$, the operator $D_\lambda^m = I(m, \lambda, 0)$ was introduced and studied by Al-Oboudi [2], which reduced to the Sălăgean differential operator $S^m = I(m, 1, 0)$ [8] for $\lambda = 1$. The operator $I_\ell^m = I(m, 1, \ell)$ was studied recently by Cho and Srivastava [5] and Cho and Kim [4]. The operator $I_m = I(m, 1, 1)$ was studied by Uralegaddi and Somanatha [10],the operator $D_\lambda^\delta = I(\delta, \lambda, 0)$, with $\delta \in R, \delta \geq 0$, was introduced by Acu and Owa [1].

Definition 2 [9]. For $f \in \mathcal{A}(1, n)$, $n \in \mathbb{N}$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda, \mu \geq 0$ where $0 \leq \mu \leq \lambda \leq 1$, the operator $D_{m,\lambda,\mu}(\alpha_1, \beta_1)f(z)$ is defined by the following infinite series

$$D_{m,\lambda,\mu}(\alpha_1, \beta_1)f(z) = z + \sum_{j=n+1}^{\infty} \vartheta_j^m \sigma_j a_j z^j, \tag{2}$$

where

$$\vartheta_j = 1 + (\lambda\mu j + \lambda - \mu)(j - 1) \tag{3}$$

and

$$\sigma_j = \frac{(\alpha_1)_{j-1}(\alpha_2)_{j-1}...(\alpha_q)_{j-1}}{(\beta_1)_{j-1}(\beta_2)_{j-1}...(\beta_s)_{j-1(j-1)!}}, \tag{4}$$

where $\alpha_1,\alpha_2,...,\alpha_q$ and $\beta_1,\beta_2,...,\beta_s$ ($q,s\in\mathbb{N}\cup\{0\},\ q\leq s+1$) be a complex numbers such that $\beta_k\neq 0,-1,-2,...$ for $k\in\{1,2,...,s\}$.

Remark 5. It follows from the above definition if $q=2,\ s=1,\ \alpha_1=\beta_1$ and $\alpha_2=1$ then $D_{0,\lambda,\mu}(\alpha_1,\beta_1)f(z)=f(z)$,

$$D_{m+1,\lambda,\mu}(\alpha_1,\beta_1)f(z)=\lambda\mu z^2[D_{m,\lambda,\mu}(\alpha_1,\beta_1)f(z)]''+(\lambda-\mu)z[D_{m,\lambda,\mu}(\alpha_1,\beta_1)f(z)]'+(1-\lambda+\mu)[D_{m,\lambda,\mu}(\alpha_1,\beta_1)f(z)] \quad (5)$$

Remark 6. For $m=0$ the operator $D_{m,\lambda,\mu}(\alpha_1,\beta_1)f(z)$ reduces to the well-known Dziok-Srivastava operator [6] and for $\mu=0,\ q=2,\ s=1,\ \alpha_1=\beta_1$ and $\alpha_2=1$, it reduces to the operator introduced by F. M. Al-Oboudi [2]. Further we remark that, when $\lambda=1,\ \mu=0,\ q=2,\ s=1,\ \alpha_1=\beta_1$ and $\alpha_2=1$ the operator $D_{m,\lambda,\mu}(\alpha_1,\beta_1)f(z)$ reduces to the operator introduced by G. S. Sălăgean [8].

Lemma 1 (Miller and Mocanu [[7], Th. 3.1.6, p. 71]). Let h be a convex function with $h(0)=a$, and let $\gamma\in\mathbb{C}\backslash\{0\}$ be a complex number with $\Re\gamma\geq 0$. If $p\in\mathcal{H}[a,n]$ and

$$p(z)+\frac{1}{\gamma}zp'(z)\prec h(z),\ \text{for}\ \ z\in U,$$

then

$$p(z)\prec g(z)\prec h(z),\ \text{for}\ \ z\in U,$$

where

$$g(z)=\frac{\gamma}{nz^{\gamma/n}}\int_0^z h(t)t^{(\gamma/n)-1}dt,\ \ \text{for}\ \ z\in U,$$

Lemma 2 (Miller and Mocanu [7]). Let g be a convex functioin in U and let $h(z)=g(z)+n\alpha zg'(z)$, for $z\in U$, where $\alpha>0$ and n is a positive integer. If $p(z)=g(0)+p_nz^n+p_{n+1}z^{n+1}+...$, for $z\in U$ is holomorphic in U and

$$p(z)+\alpha zp'(z)\prec h(z),\ \text{for}\ z\in U,$$

then

$$p(z)\prec g(z),\ \text{for}\ z\in U,$$

and this result is sharp.

2. Main results

Definition 3. Let $\alpha,\lambda,\mu,\ell\geq 0,\ 0\leq\mu\leq\lambda\leq 1$ and $n,m\in\mathbb{N}$. Denote by $ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)$ the operator given by $ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell):\mathcal{A}_n\to\mathcal{A}_n$,

$$ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)=(1-\alpha)I(m,\lambda,\ell)f(z)+\alpha D_{m,\lambda,\mu}(\alpha_1,\beta_1)f(z),$$

for $z\in U$.

Remark 7. If

$$f\in\mathcal{A}_n,\ f(z)=z+\sum_{j=n+1}^{\infty}a_jz^j,$$

then

$$ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)=z+\sum_{j=n+1}^{\infty}\left\{\alpha\left[1+(\lambda\mu j+\lambda-\mu)(j-1)\right]^m\sigma_j+(1-\alpha)\left[\frac{1+\lambda(j-1)+\ell}{\ell+1}\right]^m\right\}a_jz^j,\ \text{for}\ z\in U.$$

Remark 8. For $\alpha = 0$, $ID^0_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z) = I(m, \lambda, \ell)f(z)$, where $z \in U$ and for $\alpha = 1$, $ID^1_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z) = D_{m,\lambda,\mu}(\alpha_1, \beta_1)f(z)$.

Definition 4. Let $\delta \in [0, 1)$, $\alpha, \lambda, \mu, \ell \geq 0$ where $0 \leq \mu \leq \lambda \leq 1$, $n, m \in \mathbb{N}$, where $\alpha_1, \alpha_2, ..., \alpha_q$ and $\beta_1, \beta_2, ..., \beta_s$ $(\ q, s \in \mathbb{N} \cup \{0\},\ q \leq s + 1)$ be complex numbers such that $\beta_k \neq 0, -1, -2, ...,$ for $k \in \{1, 2, ..., s\}$.

A function $f \in \mathcal{A}_n$ is said to be in the class $\mathcal{ID}_m(\delta, \lambda, \mu, \ell, \alpha, \alpha_1, \beta_1)$ if it satisfies the inequality

$$\Re \left(ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)\right)' > \delta, \quad \text{for} \quad z \in U \tag{6}$$

Theorem 1. The class $\mathcal{ID}_m(\delta, \lambda, \mu, \ell, \alpha, \alpha_1, \beta_1)$ is convex.

Proof. Let the functions

$$f_k(z) = z + \sum_{j=n+1}^{\infty} a_{jk} z^j, \quad \text{for } k = 1, 2, \quad z \in U,$$

be in the class $\mathcal{ID}_m(\delta, \lambda, \mu, \ell, \alpha, \alpha_1, \beta_1)$. It is sufficient to show that the function

$$h(z) = \eta_1 f_1(z) + \eta_2 f_2(z)$$

is in the class $\mathcal{ID}_m(\delta, \lambda, \mu, \ell, \alpha, \alpha_1, \beta_1)$ with η_1 and η_2 nonnegative such that $\eta_1 + \eta_2 = 1$, since $h(z) = z + \sum\limits_{j=n+1}^{\infty} (\eta_1 a_{j1} + \eta_2 a_{j2}) z^j$, for $z \in U$, then

$$ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)h(z) = z + \sum_{j=n+1}^{\infty} \Big\{\alpha \left[1 + (\lambda\mu j + \lambda - \mu)(j - 1)\right]^m \sigma_j + (1 - \alpha)\left[\frac{1 + \lambda(j - 1) + \ell}{\ell + 1}\right]^m\Big\} (\eta_1 a_{j1} + \eta_2 a_{j2}) z^j, \quad \text{for} \quad z \in U. \tag{7}$$

Differentiating (7) we obtain

$$\left(ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)h(z)\right)' = 1 + \sum_{j=n+1}^{\infty} \Big\{\alpha \left[1 + (\lambda\mu j + \lambda - \mu)(j - 1)\right]^m \sigma_j + (1 - \alpha)\left[\frac{1 + \lambda(j - 1) + \ell}{\ell + 1}\right]^m\Big\} (\eta_1 a_{j1} + \eta_2 a_{j2}) j z^{j-1}, \quad \text{for} \quad z \in U.$$

Hence

$$\Re \left(ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)h(z)\right)' = 1 +$$

$$\Re\left(\eta_1 \sum_{j=n+1}^{\infty} j \left\{\alpha \left[1 + (\lambda\mu j + \lambda - \mu)(j - 1)\right]^m \sigma_j + (1 - \alpha)\left[\tfrac{1+\lambda(j-1)+\ell}{\ell+1}\right]^m\right\} a_{j1} z^{j-1}\right)$$
$$+\Re\left(\eta_2 \sum_{J=n+1}^{\infty} j \left\{\alpha \left[1 + (\lambda\mu j + \lambda - \mu)(j - 1)\right]^m \sigma_j + (1 - \alpha)\left[\tfrac{1+\lambda(j-1)+\ell}{\ell+1}\right]^m\right\} a_{j2} z^{j-1}\right). \tag{8}$$

Taking into account that $f_1, f_2 \in \mathcal{ID}_m(\delta, \lambda, \mu, \ell, \alpha, \alpha_1, \beta_1)$ we deduce

$$\Re\left(\eta_{k}\sum_{j=n+1}^{\infty} j\left\{\alpha\left[1+(\lambda\mu j+\lambda-\mu)(j-1)\right]^{m}\sigma_{j}+(1-\alpha)\left[\frac{1+\lambda(j-1)+\ell}{\ell+1}\right]^{m}\right)\right\} a_{jk}z^{j-1}\right)$$
$$>\eta_k(\delta-1),\ \text{for } k=1,2. \qquad (9)$$

Using (9) we get from (8)

$$\Re\left(ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)h(z)\right)'>1+\eta_1(\delta-1)+\eta_2(\delta-1)=\delta,\ \text{for } z\in U,$$

which is equivalent that $\mathcal{ID}_m(\delta,\lambda,\mu,\ell,\alpha,\alpha_1,\beta_1)$ is convex.

Theorem 2. Let g be a convex function in U and let $h(z)=g(z)+\frac{1}{c+2}zg'(z)$, where $z\in U,\ c>0,\ f\in\mathcal{A}_n$ and

$$F(z)=I_c(f)(z)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t)dt,\ \text{for } z\in U,$$

then

$$\left[ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)\right]'\prec h(z),\qquad \text{for}\ \ z\in U, \qquad (10)$$

implies

$$\left[ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)F(z)\right]'\prec g(z),\qquad \text{for}\ \ z\in U,$$

and this result is sharp.

Proof. We obtain that

$$z^{c+1}F(z)=(c+2)\int_0^z t^c f(t)dt. \qquad (11)$$

Differentiating (11), with respect to z, we have $(c+1)F(z)+zF'(z)=(c+2)f(z)$ and

$$(c+1)ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)F(z)+z\left[ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)F(z)\right]'=$$
$$(c+2)ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z),\ \text{for } z\in U. \qquad (12)$$

Differentiating (12) we have

$$\left[ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)F(z)\right]'+\frac{1}{c+2}z\left[ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)F(z)\right]''=\left[ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)\right]' \qquad (13)$$

Using (13), the differential subordination (10) becomes

$$\left[ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)F(z)\right]'+\frac{1}{c+2}z\left[ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)F(z)\right]''$$
$$\prec g(z)+\frac{1}{(c+2)}zg'(z). \qquad (14)$$

If we denote

$$p(z)=\left[ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)F(z)\right]'\ \text{for}\ \ z\in U, \qquad (15)$$

then $p\in\mathcal{H}[1,n]$.

Replacing (15) in (14) we obtain

$$p(z)+\frac{1}{c+2}zp'(z)\prec g(z)+\frac{1}{(c+2)}zg'(z),\ \text{for } z\in U.$$

Using lemma 2 we have

$$p(z) \prec g(z), \text{ for } z \in U,$$

$$i.e. \;\; (ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)F(z))' \prec g(z), \text{for } z \in U,$$

and g is the best dominant.
Theorem 3. Let $h(z) = \frac{1+(2\delta-1)z}{1+z}$, where $\delta \in [0, 1)$ and $c > 0$. If $f \in \mathcal{A}_n, \; \alpha, \lambda, \mu, \ell \geq 0, \; n, m \in \mathbb{N}$ and $I_c(f)(z) = \frac{c+2}{z^{c+1}} \int\limits_0^z t^c f(t)dt$, for $z \in U$, then

$$\mathcal{ID}_m(\delta, \lambda, \mu, \ell, \alpha, \alpha_1, \beta_1) \subset I_c\left[\mathcal{ID}_m(\delta^*, \lambda, \mu, \ell, \alpha, \alpha_1, \beta_1)\right] \tag{16}$$

where $\delta^* = 2\delta - 1 + \frac{(c+2)(2-2\delta)}{n}\beta(\frac{c+2}{n} - 2)$ and $\beta(x) = \int\limits_0^1 \frac{t^{x+1}}{t+1}dt$.
Proof. The function h is convex and using the same steps as in the proof of Theorem 2, we get from the hypothesis of Theorem 3 that

$$p(z) + \frac{1}{c+2}zp'(z) \prec h(z),$$

where $p(z)$ is defined in (15). Using lemma 1 we deduce that

$$p(z) \prec g(z) \prec h(z), \quad i.e. \;\; (ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)F(z))' \prec g(z) \prec h(z),$$

where

$$g(z) = \frac{c+2}{nz^{\frac{c+2}{n}}} \int\limits_0^z t^{\frac{c+2}{n}-1} \frac{1+(2\delta-1)t}{1+t}dt = (2\delta - 1) + \frac{(c+2)(2-2\delta)}{nz^{\frac{c+2}{n}}} \int\limits_0^z \frac{t^{\frac{c+2}{n}-1}}{1+t}dt.$$

Since g is convex and $g(U)$ is symmetric with respect to the real axis, we deduce

$$\Re\left[ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)F(z)\right]' \geq \min_{|z|=1} \Re g(z) = \Re g(1) = \delta^*$$

$$= 2\delta - 1 + \frac{(c+2)(2-2\delta)}{n}\beta\left(\frac{c+2}{n} - 2\right). \tag{17}$$

From (17) we deduce inclusion (16).
Theorem 4. Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $\alpha, \lambda, \mu, \ell \geq 0, \; n, m \in \mathbb{N}, \; f \in \mathcal{A}_n$ and satisfies the differential subordination

$$(ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z))' \prec h(z), \quad \text{for } \; z \in U \tag{18}$$

then

$$\frac{ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)}{z} \prec g(z), \quad \text{for } \; z \in U,$$

and this result is sharp.
Proof. By using the properties of operator $ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)$, we have

$$ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z) = z + \sum_{j=n+1}^{\infty} \left\{\alpha\left[1 + (\lambda\mu j + \lambda_{-}\mu)(j-1)\right]^m \sigma_j + \right.$$

$$\left.(1-\alpha)\left[\frac{1+\lambda(j-1)+\ell}{\ell+1}\right]^m\right\} a_j z^j, \quad \text{for} \quad z \in U.$$

Consider

$$\begin{aligned} p(z) &= \frac{ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)}{z} = \\ &= \frac{z+\sum\limits_{j=n+1}^{\infty}\left\{\alpha\left[1+(\lambda\mu j+\lambda-\mu)(j-1)\right]^m\sigma_j+(1-\alpha)\left[\frac{1+\lambda(j-1)+\ell}{\ell+1}\right]^m\right\}a_jz^j}{z} \\ &= 1+p_nz^n+p_{n+1}z^{n+1}+..., \quad \text{for} \quad z\in U. \end{aligned}$$

We deduce that $p \in \mathcal{H}[1,n]$. Let $ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z) = zp(z)$, for $z \in U$. Differentiating we obtain $\left(ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)\right)' = p(z)+zp'(z)$, for $z \in U$. Then (18) becomes

$$p(z)+zp'(z) \prec h(z) = g(z)+zg'(z), \quad \text{for} \quad z\in U.$$

By using lemma 2, we have

$$p(z) \prec g(z), \quad \text{for} \quad z\in U, \quad i.e. \quad \frac{ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)}{z} \prec g(z), \quad \text{for} \quad z\in U.$$

Theorem 5. Let h be holomorhic function which satisfies the inequality $\Re\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, for $z\in U$, and $h(0)=1$. If $\alpha,\lambda,\mu,\ell \geq 0$, $n,m\in\mathbb{N}$, $f\in\mathcal{A}_n$ and satisfies the differential subordination

$$\left(ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)\right)' \prec h(z) \text{ for } z\in U, \tag{19}$$

then

$$\frac{ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)}{z} \prec q(z), \quad \text{for} \quad z\in U,$$

where $q(z) = \frac{1}{nz^{1/n}}\int\limits_0^z h(t)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best dominant.

Proof. Let

$$\begin{aligned} p(z) &= \frac{ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)}{z} = \\ &= \frac{z+\sum\limits_{j=n+1}^{\infty}\left\{\alpha\left[1+(\lambda\mu j+\lambda-\mu)(j-1)\right]^m\sigma_j+(1-\alpha)\left[\frac{1+\lambda(j-1)+\ell}{\ell+1}\right]^m\right\}a_jz^j}{z} \\ &= 1+\sum_{j=n+1}^{\infty}\left\{\alpha\left[1+(\lambda\mu j+\lambda-\mu)(j-1)\right]^m\sigma_j+(1-\alpha)\left[\frac{1+\lambda(j-1)+\ell}{\ell+1}\right]^m\right\}a_jz^{j-1} \\ &= 1+\sum_{j=n+1}^{\infty}p_jz^{j-1}, \quad \text{for } z\in U, \ p\in\mathcal{H}[1.n]. \end{aligned}$$

Differentiating, we obtain $\left(ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)\right)' = p(z)+zp'(z)$, for $z\in U$ and (19) becomes

$$p(z)+zp'(z) \prec h(z), \quad \text{for} \quad z\in U.$$

Using lemma 1, we have

$$p(z) \prec q(z), \text{ for } z \in U, \text{ i.e. } \frac{ID_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)}{z} \prec q(z)$$
$$= \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)t^{\frac{1}{n}-1}dt, \text{ for } z \in U,$$

and q is the best dominant

Theorem 6. Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$ for $z \in U$. If $\alpha, \lambda, \mu, \ell \geq 0$ where $0 \leq \mu \leq \lambda \leq 1$, $n, m \in N$, $f \in \mathcal{A}_n$ and the differential subordination

$$\left(\frac{zID^{\alpha}_{m+1,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)}{ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)}\right)' \prec h(z), \text{ for } z \in U \tag{20}$$

holds then

$$\frac{ID^{\alpha}_{m+1,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)}{ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)} \prec g(z), \text{ for } z \in U$$

and this result is sharp.

Proof. For $f \in \mathcal{A}_n$, $f(z) = z + \sum\limits_{j=n+1}^{\infty} a_j z^j$ we have

$$ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z) = z+$$
$$\sum_{j=n+1}^{\infty} \left\{\alpha \left[1 + (\lambda\mu j + \lambda - \mu)(J - 1)\right]^m \sigma_j + (1 - \alpha) \left[\tfrac{1+\lambda(j-1)+\ell}{\ell+1}\right]^m\right\} a_j z^j, \text{ for } z \in U.$$

Consider

$$p(z) = \frac{ID^{\alpha}_{m+1,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)}{ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)}$$
$$= \frac{z + \sum\limits_{j=n+1}^{\infty} \left\{\alpha \left[1 + (\lambda\mu j + \lambda - \mu)(j - 1)\right]^{m+1} \sigma_j + (1 - \alpha) \left[\tfrac{1+\lambda(j-1)+\ell}{\ell+1}\right]^{m+1}\right\} a_j z^j}{z + \sum\limits_{j=n+1}^{\infty} \left\{\alpha \left[1 + (\lambda\mu j + \lambda - \mu)(j - 1)\right]^{m} \sigma_j + (1 - \alpha) \left[\tfrac{1+\lambda(j-1)+\ell}{\ell+1}\right]^{m}\right\} a_j z^j}.$$

We have

$$p'(z) = \frac{\left(ID^{\alpha}_{m+1,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)\right)'}{ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)} - p(z)\frac{\left(ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)\right)'}{ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)}$$

and we obtain

$$p(z) + zp'(z) = \left(\frac{zID^{\alpha}_{m+1,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)}{ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)}\right)'.$$

Relation (20) becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z), \text{ for } z \in U.$$

By using lemma 2, we have

$$p(z) \prec g(z) \text{ for } z \in U, \text{ i.e.} = \frac{ID^{\alpha}_{m+1,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)}{ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)} \prec g(z), \text{ for } z \in U.$$

Theorem 7. Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $\alpha, \lambda, \mu, \ell \geq 0$ where $0 \leq \mu \leq \lambda \leq 1$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and the differential subordination

$$\begin{aligned}
&\frac{(\ell+1)^2}{\lambda^2 z} ID^{\alpha}_{m+2,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z) - \frac{2(\ell+1)(\ell+1-\lambda)}{\lambda^2 z} ID^{\alpha}_{m+1,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z) \\
&+\frac{(\ell+1-\lambda)^2}{\lambda^2 z} ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z) - \frac{\alpha(\ell+1)^2}{\lambda^2 z} D_{m+2,\lambda,\mu}(\alpha_1,\beta_1)f(z) \\
&+\frac{\alpha}{\lambda z}\left[\frac{1}{\mu} + \frac{2(\ell+1)(\ell+1-\lambda)}{\lambda}\right] D_{m+1,\lambda,\mu}(\alpha_1,\beta_1)f(z) \\
&-\frac{\alpha}{\lambda z}\left[\frac{(\ell+1-\lambda)^2}{\lambda} + \frac{(1-\lambda+\mu)}{\mu}\right] D_{m,\lambda,\mu}(\alpha_1,\beta_1)f(z) \\
&+\left[\alpha - \frac{\alpha(\lambda-\mu)}{\lambda\mu}\right]\left[D_{m,\lambda,\mu}(\alpha_1,\beta_1)f(z)\right]' \prec h(z), \text{ for } z \in U \qquad (21)
\end{aligned}$$

holds, then

$$\left[ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)\right]' \prec g(z), \text{ for } z \in U.$$

This result is sharp.

Proof. Let

$$\begin{aligned}
p(z) &= \left(ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)\right)' \\
&= (1-\alpha)(I(m,\lambda,\ell f(z))' + \alpha(D^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1)f(z))' \qquad (22) \\
&= 1 + \sum_{j=n+1}^{\infty}\Big\{\alpha\left[1+(\lambda\mu j+\lambda-\mu)(j-1)\right]^m \sigma_j \\
&+(1-\alpha)\left[\frac{1+\lambda(j-1)+\ell}{\ell+1}\right]^m\Big\} j a_j z^{j-1} = 1 + p_n z^n + p_{n+1}z^{n+1} + \ldots.
\end{aligned}$$

We deduce that $p \in \mathcal{H}[1,n]$. By using the properties of operators $ID^{\alpha}_{m,\lambda,\mu}$ and $I(m,\lambda,\ell)$, after a short Calculation, we obtain

$$\begin{aligned}
p(z)+zp'(z) &= \frac{(\ell+1)^2}{\lambda^2 z} ID^{\alpha}_{m+2,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z) - 2\tfrac{(\ell+1)(\ell+1-\lambda)}{\lambda^2 z} ID^{\alpha}_{m+1,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z) \\
&+\frac{(\ell+1-\lambda)^2}{\lambda^2 z} ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z) - \frac{\alpha(\ell+1)^2}{\lambda^2 z} D_{m+2,\lambda,\mu}(\alpha_1,\beta_1)f(z) \\
&+\frac{\alpha}{\lambda z}\left[\frac{1}{\mu} + \frac{2(\ell+1)(\ell+1-\lambda)}{\lambda}\right] D_{m+1,\lambda,\mu}(\alpha_1,\beta_1)f(z) \\
&-\frac{\alpha}{\lambda z}\left[\frac{(\ell+1-\lambda)^2}{\lambda} + \frac{(1-\lambda+\mu)}{\mu}\right] D_{m,\lambda,\mu}(\alpha_1,\beta_1)f(z) \\
&+\left[\alpha - \frac{\alpha(\lambda-\mu)}{\lambda\mu}\right]\left[D_{m,\lambda,\mu}(\alpha_1,\beta_1)f(z)\right]'.
\end{aligned}$$

Using the notation in (21), the differential subordination becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z).$$

By using Lemma 2, we have

$$p(z) \prec g(z), \quad \text{for } z \in U, \text{ i.e. } \left(ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)\right)' \prec g(z),$$

for $z \in U$, and this result is sharp.

Theorem 8. Let $h(z) = \dfrac{1+(2\beta-1)z}{1+z}$ be a convex function in U, where $0 \leq$

$\beta < 1$. If $\alpha, \lambda, \mu, \ell \geq 0$ where $0 \leq \mu \leq \lambda \leq 1,\ n, m \in \mathbb{N},\ f \in \mathcal{A}_n$ and satisfies the differential subordination

$$\begin{aligned} &\frac{(\ell+1)^2}{\lambda^2 z} ID^{\alpha}_{m+2,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z) - 2\tfrac{(\ell+1)(\ell+1-\lambda)}{\lambda^2 z} ID^{\alpha}_{m+1,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z) \\ &+\frac{(\ell+1-\lambda)^2}{\lambda^2 z} ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z) - \frac{\alpha(\ell+1)^2}{\lambda^2 z} D_{m+2,\lambda,\mu}(\alpha_1, \beta_1)f(z) \\ &+\frac{\alpha}{\lambda z}\left[\frac{1}{\mu} + \frac{2(\ell+1)(\ell+1-\lambda)}{\lambda}\right] D_{m+1,\lambda,\mu}(\alpha_1, \beta_1)f(z) \\ &-\frac{\alpha}{\lambda z}\left[\frac{(\ell+1-\lambda)^2}{\lambda} + \frac{(1-\lambda+\mu)}{\mu}\right] D_{m,\lambda,\mu}(\alpha_1, \beta_1)f(z) \\ &+\left[\alpha - \frac{\alpha(\lambda-\mu)}{\lambda\mu}\right] [D_{m,\lambda,\mu}(\alpha_1, \beta_1)f(z)]' \prec h(z), \quad \text{for} \quad z \in U, \end{aligned} \tag{23}$$

then

$$\left(ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)\right)' \prec q(z), \text{ for } \quad z \in U,$$

where q is given by

$$q(z) = 2\beta - 1 + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt, \quad \text{for} \quad z \in U.$$

The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 7 and considering $p(z) = \left(ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z)\right)'$, the differential subordination 23 becomes

$$p(z) + zp'(z) \prec h(z) = \frac{1+(2\beta-1)z}{1+z}, \text{ for } \quad z \in U.$$

By using Lemma 1, for $\gamma = 1$, we have $p(z) \prec q(z)$, i.e.,

$$\begin{aligned} (ID_{m,\lambda,\mu}(\alpha_1, \beta_1)f(z))' \prec q(z) &= \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt \\ &= \frac{1}{nz^{\frac{1}{n}}} \int_0^z t^{\frac{1}{n}-1} \frac{1+(2\beta-1)t}{1+t} dt \\ &= 2\beta - 1 + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt, \text{for} \quad z \in U. \end{aligned}$$

Theorem 9. Let h be holomorphic function which satisfies the inequality $\Re\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}$, for $z \in U$, and $h(0) = 1$. If $\alpha, \lambda, \mu, \ell \geq 0$ where $0 \leq \mu \leq \lambda \leq 1,\ n, m \in \mathbb{N},\ f \in \mathcal{A}_n$ and satisfies the differential subordination

$$\begin{aligned} &\frac{(\ell+1)^2}{\lambda^2 z} ID^{\alpha}_{m+2,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z) - 2\tfrac{(\ell+1)(\ell+1-\lambda)}{\lambda^2 z} ID^{\alpha}_{m+1,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z) \\ &+\frac{(\ell+1-\lambda)^2}{\lambda^2 z} ID^{\alpha}_{m,\lambda,\mu}(\alpha_1, \beta_1, \ell)f(z) - \frac{\alpha(\ell+1)^2}{\lambda^2 z} D_{m+2,\lambda,\mu}(\alpha_1, \beta_1)f(z) \\ &+\frac{\alpha}{\lambda z}\left[\frac{1}{\mu} + \frac{2(\ell+1)(\ell+1-\lambda)}{\lambda}\right] D_{m+1,\lambda,\mu}(\alpha_1, \beta_1)f(z) \end{aligned}$$

$$-\frac{\alpha}{\lambda z}\left[\frac{(\ell+1-\lambda)^2}{\lambda}+\frac{(1-\lambda+\mu)}{\mu}\right]D_{m,\lambda,\mu}(\alpha_1,\beta_1)f(z)$$
$$+\left[\alpha-\frac{\alpha(\lambda-\mu)}{\lambda\mu}\right][D_{m,\lambda,\mu}(\alpha_1,\beta_1)f(z)]' \prec h(z), \quad \text{for} \quad z\in U. \tag{24}$$

then

$$[ID_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)]' \prec q(z), \text{ for } z\in U,$$

where q is given by $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best dominant .

proof. Using the properties of operator $ID^{\alpha}_{m,\lambda,\mu}$ and considering

$$p(z)=\left[ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)\right]',$$

we obtain

$$p(z)+zp'(z)=\frac{(\ell+1)^2}{\lambda^2 z}ID^{\alpha}_{m+2,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)-2\tfrac{(\ell+1)(\ell+1-\lambda)}{\lambda^2 z}ID^{\alpha}_{m+1,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)$$
$$+\frac{(\ell+1-\lambda)^2}{\lambda^2 z}ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)-\frac{\alpha(\ell+1)^2}{\lambda^2 z}D_{m+2,\lambda,\mu}(\alpha_1,\beta_1)f(z)$$
$$+\frac{\alpha}{\lambda z}\left[\frac{1}{\mu}+\frac{2(\ell+1)(\ell+1-\lambda)}{\lambda}\right]D_{m+1,\lambda,\mu}(\alpha_1,\beta_1)f(z)$$
$$-\frac{\alpha}{\lambda z}\left[\frac{(\ell+1-\lambda)^2}{\lambda}+\frac{(1-\lambda+\mu)}{\mu}\right]D_{m,\lambda,\mu}(\alpha_1,\beta_1)f(z)$$
$$+\left[\alpha-\frac{\alpha(\lambda-\mu)}{\lambda\mu}\right][D_{m,\lambda,\mu}(\alpha_1,\beta_1)f(z)]' \prec h(z), \quad \text{for} \quad z\in U.$$

Then (24) becomes

$$p(z)+zp'(z)\prec h(z), \text{ for } \quad z\in U.$$

Since $p\in\mathcal{H}[1,n]$, using Lemma 1, we deduce $p(z)\prec q(z)$, for $\quad z\in U$ i.e.

$$\left(ID^{\alpha}_{m,\lambda,\mu}(\alpha_1,\beta_1,\ell)f(z)\right)'\prec q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt, \text{ for } \ z\in U,$$

and q is the best dominant.

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