

## POSITIVE SOLUTIONS OF INTEGRAL BOUNDARY VALUE PROBLEM INVOLVING RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE

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ABSTRACT. In this paper, by using the lower and upper solutions method and fixed point theorem on cone, we consider the existence and uniqueness of positive solution of the integral boundary value problem for nonlinear differential equation involving Riemann-Liouville fractional derivative. An example demonstrates the application of our results.

### 1. INTRODUCTION

Fractional-order models are found to be more accurate than integer-order models, that is, there are more degrees of freedom in the fractional-order models. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order([1]-[5]). Fractional-order differential equations also serve better for the description of hereditary properties of various materials and processes than integer-order differential equations. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details, see ([6]-[23]) and the references therein.

In present, there are some papers which deal with the existence and multiplicity of solutions for nonlinear fractional differential equations' boundary value problems by means of some fixed point theorems([24]-[35]). By the use of some fixed point theorems on cones, Bai and Lü[28] and Zhang[36], investigated the existence of positive solutions for the equation

$$D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad 1 < \alpha \leq 2,$$

with one of the boundary conditions

$$\begin{aligned} u(0) &= u(1) = 0, \\ u(0) + u'(0) &= u(1) + u'(0) = 0, \end{aligned}$$

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respectively. The adomian decomposition method is used by Jafari[37] for solving the problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + \mu f(t, u(t)) = 0, & 0 < t < 1, & 1 < \alpha \leq 2, \\ u(0) = 0, & u(1) = c. \end{cases}$$

In 2010, by the use of fixed point index theory, Bai[38] obtained existence results of positive solution for the following nonlinear fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, & 0 < t < 1, & 1 < \alpha \leq 2, \\ u(0) = 0, & \beta u(\eta) = u(1). \end{cases}$$

From the above works, we can see a fact, although the fractional boundary value problems have been studied by some authors, to the best of our knowledge, there have been a few works using the lower and upper solution method [39, 40]. However, the Schauder fixed-point theorem cannot ensure the solutions to be positive. Since only positive solutions are useful for many applications, motivated by the above works, in this paper, we study the existence and uniqueness of positive solutions of the following integral boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, & 0 < t < 1, & 1 < \alpha \leq 2, \\ u(0) = 0, & u(1) = \int_0^1 u(s)ds, \end{cases} \quad (1)$$

where  $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$  is a continuous function and  $D_{0+}^{\alpha}$  is the standard Riemann-Liouville fractional derivative.

Integral boundary-value problems constitute a very interesting and important class of problem. They include two, three, multi-point and nonlocal boundary-value problems as special cases. The theory of integral boundary-value problems arises in different areas of applied mathematics and physics. For instance, heat conduction, chemical engineering, underground water flow, and plasma physics can all be reduced to nonlocal problems with integral boundary conditions. For some recent work on integral boundary value problems for nonlinear differential equations of fractional order, see [41]-[52] and the references therein.

To our knowledge, no paper has considered the integral boundary value problem (1) by using the lower and upper solutions method. Our aim is to study the existence and uniqueness of positive solution of integral boundary value problem (1). However, with integral boundary value condition, it will become more complicated. Therefore, we shall use the lower and upper solutions method and fixed point theorem to overcome this difficulty. Some ideas of this paper are from [39, 40]. Our results generalize and complement some previous findings of [39, 40] and some other known results.

We organize the rest of this paper as follows: in Section 2, we derive the corresponding Green' function named by fractional Green' function. Here, we give some properties of the Green' function. Consequently problem (1) is reduced to a equivalent Fredholm integral equation. Then in Section 3, using some fixed-point theorems, the existence and uniqueness of positive solutions are obtained. An example demonstrates the application of our Theorem.

## 2. PRELIMINARIES

We need the following lemmas that will be used to prove our main results.

**Lemma 2.1** [1] Let  $\alpha > 0$  and  $u \in C(0, 1) \cap L(0, 1)$ . Then fractional differential equation

$$D_{0+}^{\alpha} u(t) = 0$$

has

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_N t^{\alpha-N}, \quad C_i \in R, \quad i = 1, 2, \dots, N, \quad N = [\alpha] + 1$$

as unique solution.

**Lemma 2.2** [1] Assume that  $u \in C(0, 1) \cap L(0, 1)$  with a fractional derivative of order  $\alpha > 0$  that belongs to  $C(0, 1) \cap L(0, 1)$ . Then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) - C_1 t^{\alpha-1} - C_2 t^{\alpha-2} - \dots - C_N t^{\alpha-N},$$

for some  $C_i \in R$ ,  $i = 1, 2, \dots, n$ ,  $n = [\alpha] + 1$ .

In the following, we present the Green function of fractional differential equation with integral boundary value condition.

**Theorem 2.1** Let  $1 < \alpha < 2$ , Assume  $y(t) \in C[0, 1]$ , then the following equation

$$D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \quad (2)$$

$$u(0) = 0, \quad u(1) = \int_0^1 u(s) ds, \quad (3)$$

has a unique solution

$$u(t) = \int_0^1 G(t, s) y(s) ds$$

where

$$G(t, s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1}(\alpha-1+s) - [t-s]^{\alpha-1}(\alpha-1)}{(\alpha-1)\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{[t(1-s)]^{\alpha-1}(\alpha-1+s)}{(\alpha-1)\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

*Proof.* We may apply Lemma 2.2 to reduce equation (2) to an equivalent integral equation

$$u(t) = -I_{0+}^{\alpha} y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2},$$

for some  $C_1, C_2 \in R$ . Therefore, the general solution of (2) is

$$u(t) = - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}. \quad (4)$$

By  $u(0) = 0$ , we can get  $C_2 = 0$ .

In addition,  $u(1) = - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + C_1$ , it follows

$$C_1 = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \int_0^1 u(s) ds. \quad (5)$$

Take (5) into (4), we have

$$u(t) = - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + t^{\alpha-1} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + t^{\alpha-1} \int_0^1 u(s) ds. \quad (6)$$

Let  $\int_0^1 u(s)ds = A$ , by (6), we can get

$$\begin{aligned} \int_0^1 u(t)dt &= -\int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds dt + \int_0^1 t^{\alpha-1} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds dt + A \int_0^1 t^{\alpha-1} dt \\ &= -\int_0^1 \frac{(1-s)^\alpha}{\alpha\Gamma(\alpha)} y(s) ds + \int_0^1 \frac{(1-s)^{\alpha-1}}{\alpha\Gamma(\alpha)} y(s) ds + \frac{A}{\alpha} \\ &= \int_0^1 \frac{s(1-s)^{\alpha-1}}{\alpha\Gamma(\alpha)} y(s) ds + \frac{A}{\alpha}. \end{aligned}$$

So,

$$A = \frac{\alpha}{\alpha-1} \int_0^1 \frac{s(1-s)^{\alpha-1}}{\alpha\Gamma(\alpha)} y(s) ds = \int_0^1 \frac{s(1-s)^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)} y(s) ds.$$

Combine with (6), we have

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + t^{\alpha-1} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + t^{\alpha-1} \int_0^1 \frac{s(1-s)^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)} y(s) ds \\ &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \int_0^1 \frac{[t(1-s)]^{\alpha-1}(\alpha-1+s)}{(\alpha-1)\Gamma(\alpha)} y(s) ds \\ &= \int_0^t \frac{[t(1-s)]^{\alpha-1}(\alpha-1+s) - (t-s)^{\alpha-1}(\alpha-1)}{(\alpha-1)\Gamma(\alpha)} y(s) ds \\ &\quad + \int_t^1 \frac{[t(1-s)]^{\alpha-1}(\alpha-1+s)}{(\alpha-1)\Gamma(\alpha)} y(s) ds \\ &= \int_0^1 G(t,s)y(s) ds. \end{aligned}$$

This complete the proof. □

**Remark 2.1** Obviously, the Green function  $G(t, s)$  satisfies the following properties:

- (i)  $G(t, s) > 0, t, s \in (0, 1)$ ;
- (ii)  $G(t, s) \leq \frac{2}{(\alpha-1)\Gamma(\alpha)}, 0 \leq t, s \leq 1$ .

**Lemma 2.3**[53] Let  $D$  be a subset of the cone  $P$  of semi-order Banach space  $E$ ,  $T : D \rightarrow E$  be nondecreasing. If there exist  $x_0, y_0 \in D$  such that  $x_0 \leq y_0, \langle x_0, y_0 \rangle \subset D$  and  $x_0, y_0$  are the lower and upper solutions of equation  $x - T(x) = 0$ , then the equation  $x - T(x) = 0$  has maximum solution and minimum solutions  $x^*, y^*$  in  $\langle x_0, y_0 \rangle$ , such that  $x^* \leq y^*$ , when one of the following conditions holds

- (1)  $P$  is normal and  $T$  is compact continuous;
- (2)  $P$  is regular and  $T$  is continuous;
- (3)  $E$  is reflexive,  $P$  is normal and  $T$  is continuous or weak continuous.

### 3. MAIN RESULT

Let  $E = C[0, 1]$  be the Banach space endowed with the sup norm and define the cone  $P \subset E$  by

$P = \{u \in E \mid u(t) \geq 0, 0 \leq t \leq 1\}$ . Define the operator  $T : P \rightarrow P$  as follows,

$$Tu(t) := \int_0^1 G(t,s)f(s, u(s))ds,$$

then the equation (1) has a solution if and only if the operator  $T$  has a fixed point.

We firstly give the definition of lower and upper solutions of the operator  $T$ .

**Definition 3.1** Let  $v(t), w(t) \in E$ , we say that  $v(t)$  is called a lower solution of operator  $T$  if

$$v(t) \leq Tv(t),$$

and  $w(t)$  is called an upper solution of operator  $T$  if

$$w(t) \geq Tw(t).$$

**Theorem 3.1** Assume that

(H<sub>1</sub>)  $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$  is continuous,  $f(t, \cdot)$  is nondecreasing for each  $t \in [0, 1]$ , and there exists a positive constant  $a$ , such that  $f(t, \cdot)$  is strictly increasing on  $[0, a]$  for each  $t \in [0, 1]$ .

(H<sub>2</sub>)  $0 < \lim_{u \rightarrow +\infty} f(t, u(t)) < +\infty$  for each  $t \in [0, 1]$ .

Then the equation (1) has a positive solution.

*Proof.* We will prove the theorem through four steps.

Step 1:  $T : P \rightarrow P$  is completely continuous.

The operator  $T : P \rightarrow P$  is continuous in view of nonnegativeness and continuity of  $G(t, s)$  and  $f(t, u)$ .

Let  $\Omega \subset P$  be bounded, which is to say there exists a positive constant  $M > 0$  such that  $\|u\| \leq M, \forall u \in \Omega$ . Let  $L = \max_{0 \leq t \leq 1, 0 \leq u \leq M} |f(t, u)| + 1$ . Then  $\forall u \in \Omega$ , we have

$$|Tu(t)| \leq \int_0^1 G(t, s) f(s, u(s)) ds \leq L \int_0^1 G(t, s) ds.$$

Hence  $T(\Omega)$  is bounded.

For each  $u \in \Omega, \forall t_1, t_2 \in [0, 1]$  satisfy  $t_1 < t_2$ , we have

$$\begin{aligned} |Tu(t_2) - Tu(t_1)| &= \left| \int_0^1 G(t_2, s) f(s, u(s)) ds - \int_0^1 G(t_1, s) f(s, u(s)) ds \right| \\ &= \int_0^{t_1} |G(t_2, s) - G(t_1, s)| f(s, u(s)) ds + \int_{t_1}^{t_2} |G(t_2, s) - G(t_1, s)| f(s, u(s)) ds \\ &\quad + \int_{t_2}^1 |G(t_2, s) - G(t_1, s)| f(s, u(s)) ds \\ &\leq \int_0^{t_1} \left| \frac{[t_2(1-s)]^{\alpha-1}(\alpha-1+s) - (t_2-s)^{\alpha-1}(\alpha-1)}{(\alpha-1)\Gamma(\alpha)} \right. \\ &\quad \left. - \frac{[t_1(1-s)]^{\alpha-1}(\alpha-1+s) - (t_1-s)^{\alpha-1}(\alpha-1)}{(\alpha-1)\Gamma(\alpha)} \right| f(s, u(s)) ds \\ &\quad + \int_{t_1}^{t_2} \left| \frac{[t_2(1-s)]^{\alpha-1}(\alpha-1+s) - (t_2-s)^{\alpha-1}(\alpha-1)}{(\alpha-1)\Gamma(\alpha)} \right. \\ &\quad \left. - \frac{[t_1(1-s)]^{\alpha-1}(\alpha-1+s)}{(\alpha-1)\Gamma(\alpha)} \right| f(s, u(s)) ds \\ &\quad + \int_{t_2}^1 \left| \frac{[t_2(1-s)]^{\alpha-1}(\alpha-1+s)}{(\alpha-1)\Gamma(\alpha)} \right. \\ &\quad \left. - \frac{[t_1(1-s)]^{\alpha-1}(\alpha-1+s)}{(\alpha-1)\Gamma(\alpha)} \right| f(s, u(s)) ds \\ &\leq \frac{L}{(\alpha-1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}(\alpha-1+s) ds (t_2^{\alpha-1} - t_1^{\alpha-1}) \\ &\leq \frac{2L}{(\alpha-1)\Gamma(\alpha)} (t_2^{\alpha-1} - t_1^{\alpha-1}). \end{aligned}$$

Since  $t^{\alpha-1}$  is uniformly continuous when  $t \in [0, 1]$  and  $1 < \alpha \leq 2$ , it's easy to prove that  $T(\Omega)$  is equicontinuous. The Arzela-Ascoli Theorem implies that  $\overline{T(\Omega)}$  is compact. That is,  $T : P \rightarrow P$  is completely continuous.

Step 2:  $T$  is an increasing operator.

In fact, by  $(H_1)$ , let  $u_1 \leq u_2$ , we have

$$Tu_1(t) = \int_0^1 G(t, s)f(s, u_1(s))ds \leq \int_0^1 G(t, s)f(s, u_2(s))ds \leq Tu_2(t).$$

Step 3: By  $(H_2)$ ,  $\exists M_1 > 0, N > 0$  such that  $u \geq N$ , it holds  $f(t, u(t)) \leq M_1$ . On the other hand, by  $(H_1)$ ,  $f : [0, 1] \times [0, N]$  is continuous,  $\exists M_2 > 0$  such that  $u \leq N$ , it holds  $f(t, u(t)) \leq M_2$ . Let  $M = \max\{M_1, M_2\}$ , then we have  $f(t, u(t)) \leq M, \forall u \geq 0$ .

Now we consider the following equation

$$\begin{cases} D_{0+}^\alpha w(t) + M = 0, & 0 < t < 1, & 1 < \alpha \leq 2, \\ w(0) = 0, & w(1) = \int_0^1 w(s)ds, \end{cases} \tag{7}$$

From Theorem 2.1, we have the solution of (7) is

$$w(t) = \int_0^1 G(t, s)Mds \geq \int_0^1 G(t, s)f(s, w(s))ds = Tw(t).$$

which implies that  $w(t)$  is an upper solution of the operator  $T$ .

On the other hand, it's obvious that  $v(t) \equiv 0$  is a lower solution of the operator  $T$ , and we have

$$v(t) \leq w(t).$$

Step 4: Since  $P$  is a normal cone, Lemma 2.3 implies that  $T$  has a fixed point  $u \in \langle 0, w(t) \rangle$ . Therefore, the equation (1) has a positive solution.

This complete the proof. □

**Example 3.1** Consider the fractional differential equation

$$\begin{cases} D_{0+}^{\frac{3}{2}} u(t) + (b + t) \arctan(1 + u(t)) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = \int_0^1 u(s)ds, \end{cases} \tag{8}$$

where  $b > 0$  is a constant.

Note that  $\arctan(1 + u(t)) < \frac{\pi}{2}$  and  $\lim_{u \rightarrow +\infty} f(t, u(t)) = \lim_{u \rightarrow +\infty} (b + t) \arctan(1 + u(t)) = \frac{(b + t)\pi}{2} < +\infty$  for each  $t \in [0, 1]$ . Then the conditions  $(H_1)$  and  $(H_2)$  hold.

If fact, the solution of problem (8) is equivalent to a fixed point of the operator  $T$ , here  $Tu(t) = \int_0^1 (b + s)G(t, s) \arctan(1 + u(s))ds$ . Take  $w(t) = \frac{\pi}{2} \int_0^1 G(t, s)(b + s)ds$  and  $v(t) \equiv 0$ , then

$$w(t) \geq \int_0^1 G(t, s)f(s, w(s))ds = Tw(t),$$

which implies  $w(t)$  is an upper solution of the operator  $T$ . It is obvious that  $v(t) \equiv 0$  is a lower solution of the operator  $T$ .

Thus, by Theorem 3.1, we can get that the problem (8) has a positive solution.

**Theorem 3.2** Assume that function  $f$  satisfies

$$|f(t, u) - f(t, v)| \leq a(t)|u - v|, \tag{9}$$

where  $t \in [0, 1], u, v \in [0, \infty), a : [0, 1] \rightarrow [0, \infty)$  is a continuous function. If

$$\int_0^1 s^{\alpha-1}(1 - s)^{\alpha-1}(\alpha - 1 + s)a(s)ds < (\alpha - 1)\Gamma(\alpha), \tag{10}$$

then the equation (1) has a unique positive solution.

*Proof.* If  $T^n$  is a contraction operator for  $n$  sufficiently large, then the equation (1) has a unique positive solution.

In fact, by the definition of Green function  $G(t, s)$ , for  $u, v \in P$ , we have the estimate

$$\begin{aligned} |Tu(t) - Tv(t)| &= \int_0^1 G(t, s) |f(s, u(s)) - f(s, v(s))| ds \\ &\leq \int_0^1 G(t, s) a(s) |u(s) - v(s)| ds \\ &\leq \int_0^1 \frac{[t(1-s)]^{\alpha-1} (\alpha-1+s)}{(\alpha-1)\Gamma(\alpha)} a(s) \|u-v\| ds \\ &\leq \frac{\|u-v\| t^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (\alpha-1+s) a(s) ds \end{aligned}$$

Denote  $K = \int_0^1 (1-s)^{\alpha-1} (\alpha-1+s) a(s) ds$ , then

$$|Tu(t) - Tv(t)| \leq \frac{K t^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)} \|u-v\|.$$

Similarly,

$$\begin{aligned} |T^2u(t) - T^2v(t)| &= \int_0^1 G(t, s) |f(s, Tu(s)) - f(s, Tv(s))| ds \\ &\leq \int_0^1 G(t, s) a(s) |Tu(s) - Tv(s)| ds \\ &\leq \int_0^1 G(t, s) a(s) \frac{K s^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)} \|u-v\| ds \\ &\leq \int_0^1 \frac{K [t(1-s)]^{\alpha-1} (\alpha-1+s)}{(\alpha-1)^2 \Gamma^2(\alpha)} a(s) s^{\alpha-1} \|u-v\| ds \\ &\leq \frac{K \|u-v\| t^{\alpha-1}}{(\alpha-1)^2 \Gamma^2(\alpha)} \int_0^1 s^{\alpha-1} (1-s)^{\alpha-1} (\alpha-1+s) a(s) ds, \\ &= \frac{KH t^{\alpha-1}}{(\alpha-1)^2 \Gamma^2(\alpha)} \|u-v\| \end{aligned}$$

where  $H = \int_0^1 s^{\alpha-1} (1-s)^{\alpha-1} (\alpha-1+s) a(s) ds$ . By mathematical induction, it follows

$$|T^n u(t) - T^n v(t)| \leq \frac{KH^{n-1} t^{\alpha-1}}{(\alpha-1)^n \Gamma^n(\alpha)} \|u-v\|,$$

by (10), for  $n$  large enough, we have

$$\frac{KH^{n-1}}{(\alpha-1)^n \Gamma^n(\alpha)} = \frac{K}{(\alpha-1)\Gamma(\alpha)} \left( \frac{H}{(\alpha-1)\Gamma(\alpha)} \right)^{n-1} < 1.$$

Hence, it holds

$$\|T^n u - T^n v\| < \|u-v\|,$$

which implies  $T^n$  is a contraction operator for  $n$  sufficiently large, then the equation (1) has a unique positive solution.

This complete the proof.  $\square$

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