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POSITIVE SOLUTIONS AND MONOTONE ITERATIVE SEQUENCES FOR A CLASS OF HIGHER ORDER BOUNDARY VALUE PROBLEMS OF FRACTIONAL ORDER

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Abstract. In this paper, the method of lower and upper solutions is extended to deal with certain nonlinear fractional boundary value problem of order $3 < \delta < 4$. Two well-defined monotone sequences of lower and upper solutions which converge uniformly to actual solution of the problem are presented. The convergence of these sequences is verified numerically through one example and a result on the existence of positive solutions is obtained.

1. Introduction

Boundary value problems with fractional order (BVPF) have many applications in economics, engineering and physical sciences. They considered as generalization of boundary value problems to non-integral order. The importance of such problems comes from their various applications, and the fact that they are used to model certain phenomena which can't be modeled by equations with natural derivatives, see [10, 21]. For extensive literature and results, we refer the readers to [13, 18, 23, 24] and the references therein. Since finding exact solutions of such problems is difficult task, developing efficient numerical and analytical techniques for fractional differential equations has attracted many authors in recent years. The basic theory of fractional differential equations involving Riemann-Liouville fractional derivative with $0 < q < 1$, has been investigated in [14]. The crucial task in the analytical treatment is to prove the existence and uniqueness of solutions. Such results are obtained in [6, 8, 9, 15, 16, 17] for fractional differential problems with Riemann-Liouville fractional derivative, and in [7, 26] for problems with Caputo's fractional derivative. The analysis is based on the Laplace transform, some fixed point results and the method of lower and upper solutions. A good survey about the developments of the existence results can be found in [1, 12].

The method of lower and upper solutions has been considered as one of the effective tools in studying elliptic and parabolic boundary value problems with natural derivatives. It has been used to study multiplicity of solutions, to explore

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existence and uniqueness of solutions, as well as, to obtain accurate numerical solutions [2, 3, 25]. For extensive survey we refer the readers to [22]. By means of lower and upper solutions method and fixed point theorems, Liang et al. [16] proved the existence of positive solutions of the following nonlinear boundary value problem of fractional order

$$\begin{aligned} D_{0+}^{\delta}y(x)+f(x,y) &= 0, \quad 0<x<1, \;\; 3<\delta\leq 4, \\ y(0) &= y'(0)=y''(0)=y''(1)=0, \end{aligned}$$

where $f\in C([0,1]\times[0,\infty),(0,\infty))$ and D_{0+}^{δ} is the Riemann-Liouville fractional derivative. Numerical solutions of the above problem in integro-differential form have been considered by many authors. For instance, the well-known Adomian decomposition method has been implemented to obtain numerical and analytical solutions for $\delta=4$ in [11] and for $3<\delta\leq 4$ in [19]. Also, the efficiency of the variational iteration method and homotopy perturbation method is proved for such problems in [20].

Devoted by the above works, the purpose of this article is to extend the maximum principle and the method of lower and upper solutions for the fourth order fractional boundary value problem

$$D^{\delta}y(x)+f(x,y,y'') = 0, \quad 0<x<1, \;\; 3<\delta<4, \tag{1}$$

$$y(0) = a_1, y(1)=b_1, \tag{2}$$

$$y''(0)-\mu_1\, y'''(0) = a_2, y''(1)+\mu_2\, y'''(1)=b_2, \tag{3}$$

where $f\in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $a_1,a_2,b_1,b_2\in\mathbb{R},\mu_1,\mu_2\geq 0$, and D^{δ} is the left Caputo fractional derivative of order δ. We transform the problem into a system of two differential equations, one of fractional order and the other one with natural order. By generalizing the recent results in [4] for a fractional boundary value problem, and the ones in [22] for elliptic systems with natural order, we obtain two monotone sequences of pairs of lower and upper solutions that converge uniformly to actual solutions of the problem. These sequences are used to obtain accurate numerical results of the problem, as well. It is worth to mention that the well known results for elliptic systems can't be generalized for fractional systems without the use of recent results obtained in [5].

We organize this paper as follows. In section 2, we present some preliminary definitions and lemmas and establish a new positivity result for fractional order derivative. In Section 3, we present an algorithm to construct the monotone sequences of lower and upper pairs of solutions. We then prove the convergence of these sequences to actual solutions of the problem. In Section 4, we present some numerical results and finally some concluding remarks are presented in Section 5.

2. Preliminary results

In this section, we present the definition of and some results about the Caputo fractional derivative. We then define a pair of lower and upper solution of the problem and present a positivity result which will be used through the text.

Definition 2.1. *Let $y\in(C[0,1],\mathbb{R})$ and $\zeta>0$. The left Riemann-Liouville fractional integral of order ζ is defined by*

$$I^{\zeta}y(x)=\frac{1}{\Gamma(\zeta)}\int_0^x \frac{y(t)}{(x-t)^{1-\zeta}}\;dt, x>0$$

where $\Gamma(\zeta) = \int_0^{+\infty} t^{\zeta-1}e^{-t}dt$ *is the well-known Euler Gamma function.*

Definition 2.2. *Let* $y \in (C^n[0,1],\mathbb{R})$ *and* $\zeta > 0$. *The left Caputo fractional derivative of order* ζ *is defined by*

$$D^\zeta y(x) = I^{n-\zeta}\left(y^{(n)}(x)\right) = \frac{1}{\Gamma(n-\zeta)}\int_0^x \frac{y^{(n)}(t)}{(x-t)^{\zeta-n+1}}dt,$$

where $n = [\zeta]+1$, *and* $[\zeta]$ *is the greatest integer number of* ζ.

The relations between the Caputo fractional derivative and the Riemann-Liouville fractional integral are given in the following lemma.

Lemma 2.1. *For* $y(x) \in C^n([0,1],\mathbb{R})$, $\zeta > 0$ *and* $n = [\zeta]+1$, *we have*

(1) $D^\zeta(I^\zeta y(x)) = y(x)$, *and*
(2) $I^\zeta(D^\zeta y(x)) = y(x) - \sum\limits_{k=0}^{n-1} c_k x^k$, *where* $c_k = \frac{y^{(k)}(0^+)}{k!}$.

For the proof of the above results and more details about the definition and properties of the fractional derivative, the reader is referred to [13] and [24].
By substituting $y_1(x) = y(x)$ and $y_2(x) = -y_1''(x)$, and using the fact that $D^\delta y = D^{\delta-2}D^2 y$, the problem (1-3) is reduced to

$$\begin{aligned} D^2y_1(x)+y_2(x) &= 0, 0<x<1 && (4)\\ D^\alpha y_2(x)+g(x,y_1,y_2) &= 0, \quad 0<x<1, \quad 1<\alpha<2, && (5)\\ y_1(0) &= e_1, y_1(1)=e_2, && (6)\\ y_2(0)-\mu_1\ y_2'(0) &= e_3, y_2(1)+\mu_2\ y_2'(1)=e_4, && (7) \end{aligned}$$

where $e_1 = a_1, e_2 = b_1, e_3 = -a_2, e_4 = -b_2, \alpha = \delta-2$ and $g(x,y_1,y_2) = -f(x,y_1,-y_2)$.
We have the following definition of lower and upper pairs of solutions.

Definition 2.3. *A pair of functions* $(v_1,v_2) \in C^2([0,1],\mathbb{R})^2$ *is called a lower solution of the problem (4-7), if they satisfy the following inequalities*

$$\begin{aligned} D^2v_1(x)+v_2(x) &\geq 0, \ 0<x<1 && (8)\\ D^\alpha v_2(x)+g(x,v_1,v_2) &\geq 0, \ 0<x<1, \ 1<\alpha<2, && (9)\\ v_1(0) &\leq e_1, v_1(1)\leq e_2, && (10)\\ \text{and} \qquad v_2(0)-\mu_1\ v_2'(0) &\leq e_3, v_2(1)+\mu_2\ v_2'(1)\leq e_4. && (11) \end{aligned}$$

Analogously, A pair of functions $(w_1,w_2) \in C^2([0,1],\mathbb{R})^2$ *is called an upper solution of the problem (4-7), if they satisfy the reversed inequalities. In addition, if*

$$v_1(x) \leq w_1(x), \text{ and } v_2(x) \leq w_2(x) \text{ for all } x \in [0,1],$$

then we say that (v_1,v_2) *and* (w_1,w_2) *are ordered pairs of lower and upper solutions.*

The following positivity results will be used throughout the text.

Lemma 2.2. *(Positivity Lemma) Let* $z(x) \in C^2([0,1],\mathbb{R}), r(x) \in (C[0,1],\mathbb{R})$ *and* $r(x) < 0, \forall x \in [0,1]$. *Then* $z(x) \geq 0$ *in* $[0,1]$ *provided that one of the following hold.*

(A1) $z''(x) \leq 0$, $0<x<1$, *and* $z(0), z(1) \geq 0$.

(A2)

$$D^\alpha z(x) + r(x)z(x) \le 0, \quad 0 < x < 1, \quad 1 < \alpha < 2, \tag{12}$$
$$z(0) - \nu_1 z'(0) \ge 0,\ z(1) + \nu_2 z'(1) \ge 0,$$

where $\nu_1, \nu_2 \ge 0$ *and* $\nu_1 \ge \frac{1}{\alpha - 1}$.

Proof. The proof of $(A1)$ is a simple result of the maximum principle. To prove (A2) assume by contradiction the conclusion is false, then $z(x)$ has absolute minimum at x_0 with $z(x_0) < 0$. Let $x_0 \in (0,1)$ then $z'(x_0) = 0$. By Theorem 2.1 of [5] there holds

$$\Gamma(2-\alpha)(D^\alpha z)(x_0) \ge x_0^{-\alpha}\Bigg((\alpha-1)(z(0) - z(x_0)) - x_0 z'(0)\Bigg). \tag{13}$$

In the following we prove that $(D^\alpha z)(x_0) \ge 0$. We consider two cases (i) $z'(0) \le 0$ and (ii) $z'(0) > 0$. If $z'(0) \le 0$, then the result is clear from Eq. (13), since $1 < \alpha < 2$, $z(0) \ge z(x_0)$ and $0 < x_0 < 1$. We shall prove the result for $z'(0) > 0$. Since $\nu_1(\alpha - 1) \ge 1$ and applying the boundary condition $z(0) \ge \nu_1 z'(0)$, we have

$$(\alpha-1)\Bigg(z(0) - z(x_0)\Bigg) \ge (\alpha-1)\Bigg(\nu_1 z'(0) - z(x_0)\Bigg) \ge z'(0) - (\alpha-1)z(x_0).$$

The above result together with $\alpha > 1, x_0 \in (0,1), z(0) < 0$ and $z'(0) > 0$ imply

$$(\alpha-1)(z(0)-z(x_0))-x_0z'(0) \ge z'(0)-(\alpha-1)z(x_0)-x_0z'(0) = z'(0)(1-x_0)-(\alpha-1)z(x_0) \ge 0,$$

and $(D^\alpha z)(x_0) \ge 0$. The above results together with $r(x) < 0$ imply

$$D^\alpha z(x_0) + r(x_0)z(x_0) > 0,$$

which contradicts (12).
If $x_0 = 0$, by simple maximum principle, $z'(0^+) \ge 0$. Applying the boundary condition $z(0) - \nu_1 z'(0) \ge 0$, yields $z(0) = z(x_0) \ge 0$ and a contradiction is reached. Similarly, if $x_0 = 1$, simple maximum principle implies $z'(1^-) \le 0$. The boundary condition $z(1) + \nu_2 z'(1^-) \ge 0$ yields $z(1) = z(x_0) \ge 0$ and a contradiction is reached. □

3. Monotone sequences of lower and upper solutions

In this section, we construct two monotone sequences of lower and upper pairs of solutions to problem (4-7). We then use these sequences to establish an existence result and to construct solutions for the problem.
Given ordered lower and upper pairs of solutions to problem (4-7), $V = (v_1, v_2)$ and $W = (w_1, w_2)$, respectively, we define the set

$$[V, W] = \left\{H = (h_1, h_2) \in C^2([0,1], \mathbb{R})^2 : v_1 \le h_1 \le w_1, v_2 \le h_2 \le w_2\right\}.$$

In the following we assume that the nonlinear term $g(x, u_1, u_2)$ satisfies the following conditions on $[V, W]$.

- (R1) $g(x, h_1, h_2)$ is nondecreasing with respect to h_1, that is, $\frac{\partial g}{\partial h_1}(x, h_1, h_2) \ge 0$, for all $H = (h_1, h_2) \in [V, W]$.
- (R2) There exists a positive constant c such that

$$-c \le \frac{\partial g}{\partial h_2}(x, h_1, h_2), \text{ for all } H = (h_1, h_2) \in [V, W]. \tag{14}$$

Next, we present the main theorem in this paper which describes how to construct the monotone sequences of lower and upper pairs of solutions.

Theorem 3.1. *Assume that the conditions (R1) and (R2) hold and consider the iterative sequence* $U^{(k)} = (u_1^{(k)}, u_2^{(k)})$ *defined by*

$$\begin{aligned} -D^2 u_1^{(k)}(x) &= u_2^{(k-1)}(x), \quad 0<x<1 && (15)\\ -D^\alpha u_2^{(k)} + c\, u_2^{(k)} &= c\, u_2^{(k-1)} + g(x, u_1^{(k-1)}, u_2^{(k-1)}),\ 0<x<1, 1<\alpha<2 && (16)\\ u_1^{(k)}(0) &= \gamma_1(k), \quad u_1^{(k)}(1) = \gamma_2(k), && (17)\\ u_2^{(k)}(0) - \mu_1 D u_2^{(k)}(0) &= \gamma_3(k), \quad u_2^{(k)}(1) + \mu_2 D u_2^{(k)}(1) = \gamma_4(k). && (18)\end{aligned}$$

Then for $\mu_1 \geq \frac{1}{\alpha-1}$ *we have*

(1) *If* $U^{(0)} = V = (v_1, v_2)$ *and* $\{\gamma_i(k),\ i = 1,2,3,4, k \geq 0\}$ *is increasing sequence with* $\gamma_i(k) \leq e_i$, *then* $U^{(k)} = V^{(k)} = (v_1^{(k)}, v_2^{(k)})$ *is an increasing sequence of lower pairs of solutions to problem (4-7).*
(2) *If* $U^{(0)} = W = (w_1, w_2)$ *and* $\{\gamma_i(k),\ i = 1,2,3,4, k \geq 0\}$ *is a decreasing sequence with* $\gamma_i(k) \geq e_i$ *then* $U^{(k)} = W^{(k)} = (w_1^{(k)}, w_2^{(k)})$ *is a decreasing sequence of upper pairs of solutions to problem (4-7). Moreover,*
(3) $v_1^{(k)} \leq w_1^{(k)}$ *and* $v_2^{(k)} \leq w_2^{(k)}$ *for all* $k \geq 0$.

Proof. (1) First, we apply induction arguments to show that $U^{(k)} = (v_1^{(k)}, v_2^{(k)})$ is an increasing sequence. From Equations (15-18) we have

$$\begin{aligned} -D^2 v_1^{(1)}(x) &= v_2^{(0)}(x), && (19)\\ v_1^{(1)}(0) &= \gamma_1(1), v_1^{(1)}(1) = \gamma_2(1), \end{aligned}$$

and

$$\begin{aligned} -D^\alpha v_2^{(1)} + c\, v_2^{(1)} &= c v_2^{(0)} + g(x, v_1^{(0)}, v_2^{(0)}), && (20)\\ v_2^{(1)}(0) - \mu_1 D v_2^{(1)}(0) &= \gamma_3(1), v_2^{(1)}(1) + \mu_2 D v_2^{(1)}(1) = \gamma_4(1). \end{aligned}$$

Since $V = (v_1^{(0)}, v_2^{(0)})$ is a pair of lower solution, we have

$$\begin{aligned} &D^2 v_1^{(0)}(x) + v_2^{(0)}(x) \geq 0, && (21)\\ &v_1^{(0)}(0) = \gamma_1(0) \leq e_1,\ v_1^{(0)}(1) = \gamma_2(0) \leq e_2, \end{aligned}$$

and

$$\begin{aligned} &D^\alpha v_2^{(0)} + g(x, v_1^{(0)}, v_2^{(0)}) \geq 0, && (22)\\ &v_2^{(0)}(0) - \mu_1 D v_2^{(0)}(0) = \gamma_3(0) \leq e_3,\ v_2^{(0)}(1) + \mu_2 D v_2^{(0)}(1) = \gamma_4(0) \leq e_4. \end{aligned}$$

Let $z_1 = v_1^{(1)} - v_1^{(0)}$ and by substituting Eq. (19) in (21) we have $D^2 z_1 \leq 0$, with $z_1(0) = \gamma_1(1) - \gamma_1(0) \geq 0$, and $z_1(1) = \gamma_2(1) - \gamma_2(0) \geq 0$. Applying the Positivity Lemma we have $z_1 \geq 0$, and hence $v_1^{(1)} \geq v_1^{(0)}$.
Let $z_2 = v_2^{(1)} - v_2^{(0)}$ and by substituting Eq. (20) in (22), we have $D^\alpha z_2 - z_2 \leq 0$, with $z_2(0) - \mu_1 D z_2(0) = \gamma_3(1) - \gamma_3(0) \geq 0$, and $z_2(1) + \mu_2 D z_2(1) = \gamma_4(1) - \gamma_4(0) \geq 0$. Applying the Positivity Lemma we have $z_2 \geq 0$ and hence $v_2^{(1)} \geq v_2^{(0)}$. Thus the result is proved for $k = 1$. Assume that the result is true for $k = n$, that is;

$$v_1^{(k)} \geq v_1^{(k-1)} \text{ and } v_2^{(k)} \geq v_2^{(k-1)}, \text{ for } k = 0, 1, \cdots, n.$$

From Eq.'s (15) and (16), we have

$$-D^2 v_1^{(n)} = v_2^{(n-1)}, \tag{23}$$
$$-D^2 v_1^{(n+1)} = v_2^{(n)}, \tag{24}$$
$$-D^\alpha v_2^{(n)} + c v_2^{(n)} = c v_2^{(n-1)} + g(x, v_1^{(n-1)}, v_2^{(n-1)}), \tag{25}$$

and

$$-D^\alpha v_2^{(n+1)} + c v_2^{(n+1)} = c v_2^{(n)} + g(x, v_1^{(n)}, v_2^{(n)}). \tag{26}$$

By subtracting Eq. (24) from Eq. (23) and Eq. (26) from Eq. (25) we have

$$\begin{aligned} D^2(v_1^{(n+1)} - v_1^{(n)}) &= v_2^{(n-1)} - v_2^{(n)}, \\ D^\alpha(v_2^{(n+1)} - v_2^{(n)}) - c(v_2^{(n+1)} - v_2^{(n)}) &= c(v_2^{(n-1)} - v_2^{(n)}) + g(x, v_1^{(n-1)}, v_2^{(n-1)}) \\ &\quad - g(x, v_1^{(n)}, v_2^{(n)}). \end{aligned}$$

Let $z_1(x) = v_1^{(n+1)}(x) - v_1^{(n)}(x)$ and $z_2(x) = v_2^{(n+1)}(x) - v_2^{(n)}(x)$. Then, using the induction hypotheses, the conditions (R1) and (R2), and the Mean Value theorem, we have

$$\begin{aligned} D^2 z_1 &= v_2^{(n-1)}(x) - v_2^{(n)} \le 0, \\ D^\alpha z_2 - c\ z_2 &= c(v_2^{(n-1)} - v_2^{(n)}) + (v_1^{(n-1)} - v_1^{(n)})\frac{\partial g}{\partial y_1}(\rho_1) + (v_2^{(n-1)} - v_2^{(n)})\frac{\partial g}{\partial y_2}(\rho_2), \\ &= (v_2^{(n-1)} - v_2^{(n)})\left(c + \frac{\partial g}{\partial y_2}(\rho_2)\right) + (v_1^{(n-1)} - v_1^{(n)})\frac{\partial g}{\partial y_1}(\rho_1) \le 0, \end{aligned}$$

for some $\rho_1 = \mu v_1^{(n-1)} + (1-\mu) v_1^{(n)}$, $\rho_2 = \nu v_2^{(n-1)} + (1-\nu) v_2^{(n)}$ and $0 \le \mu, \nu \le 1$. Since the sequence $\{\gamma_i(k),\ i = 1,2,3,4, k \ge 0\}$ is increasing we have

$$z_1(0), z_1(1), z_2(0) - \mu_1 D z_2(0), z_2(1) + \mu_2 D z_2(1) \ge 0,$$

and hence by the Positivity Lemma, $z_1, z_2 \ge 0$, and the result is proved for $k = n+1$.
Second, we prove that $(v_1^{(k)}, v_2^{(k)})$, $k \ge 0$ is a pair of lower solution. Since the sequence $\{v_2^{(k)}\}$ is increasing and $-D^2 v_1^{(k)} = v_2^{(k-1)}$, we have

$$D^2 v_1^{(k)} + v_2^{(k)} = -v_2^{(k-1)}(x) + v_2^{(k)}(x) \ge 0,$$

which together with $v_1^{(k)}(0) = \gamma_1(k) \le e_1$ and $v_1^{(k)}(1) = \gamma_2(k) \le e_2$, proves that $v_1^{(k)}$ is a lower solution. From Eq. (16) we have

$$D^\alpha v_2^{(k)} = c(v_2^{(k)} - v_2^{(k-1)}) - g(x, v_1^{(k-1)}, v_2^{(k-1)}).$$

Hence, applying the Mean Value theorem and using the fact that the sequences $\{v_1^{(k)}\}$ and $\{v_2^{(k)}\}$ are increasing we have

$$\begin{aligned} D^\alpha v_2^{(k)} + g(x, v_1^{(k)}, v_2^{(k)}) &= c(v_2^{(k)} - v_2^{(k-1)}) + g(x, v_1^{(k)}, v_2^{(k)}) - g(x, v_1^{(k-1)}, v_2^{(k-1)}), \\ &= c(v_2^{(k)} - v_2^{(k-1)}) + \frac{\partial g}{\partial y_1}(\rho_1)(v_1^{(k)} - v_1^{(k-1)}) \\ &\quad + (v_2^{(k)} - v_2^{(k-1)})\frac{\partial g}{\partial y_2}(\rho_2), \\ &= \left(c + \frac{\partial g}{\partial y_2}(\rho_2)\right)(v_2^{(k)} - v_2^{(k-1)}) + \frac{\partial g}{\partial y_1}(\rho_1)(v_1^{(k)} - v_1^{(k-1)}) \geq 0, \end{aligned}$$

which together with $v_2^{(k)}(0) - \mu_1 D v_2^{(0)}(0) = \gamma_3(k) \leq e_3$ and $v_2^{(k)}(1) + \mu_2 D v_2^{(0)}(1) = \gamma_4(k) \leq e_4$, proves that $v_2^{(k)}$ is a lower solution. Here $\rho_1 = \eta v_1^{(k-1)} + (1-\eta) v_1^{(k)}$ and $\rho_2 = \zeta v_2^{(k-1)} + (1-\zeta) v_2^{(k)}$ for some $0 \leq \eta, \zeta \leq 1$.

(2) The proof is similar to that of (1). First we apply induction arguments to prove that the two sequences $\{w_1^{(k)}\}$ and $\{w_2^{(k)}\}$ are decreasing. Then, we use these results to show that $(w_1^{(k)}, w_2^{(k)})$ is a pair of upper solution for each $k \geq 0$.

(3) Since $V = (v_1^{(0)}, v_2^{(0)})$ and $W = (w_1^{(0)}, w_2^{(0)})$ are ordered pairs of lower and upper solutions, we have $v_1^{(0)} \leq w_1^{(0)}$ and $v_2^{(0)} \leq w_2^{(0)}$. Hence, the result is true for $n = 0$. Assume the result is true for $k = n$, that is; $v_1^{(k)} \leq w_1^{(k)}$ and $v_2^{(k)} \leq w_2^{(k)},\ k = 0, 1, \cdots, n$. We have

$$-D^2 v_1^{(n+1)} = v_2^{(n)} \text{ and } -D^2 w_1^{(n+1)} = w_2^{(n)}.$$

Hence,

$$D^2(w_1^{(n+1)} - v_1^{(n+1)}) = v_2^{(n)} - w_2^{(n)} \leq 0,$$

which together with $w_1^{(n+1)}(0) \geq v_1^{(n+1)}(0)$ and $w_1^{(n+1)}(1) \geq v_1^{(n+1)}(1)$, proves that $w_1^{(n+1)} - v_1^{(n+1)} \geq 0$, and hence; the result is proved for $k = n+1$. Similarly, since

$$-D^\alpha v_2^{(n+1)} + c v_2^{(n+1)} = c v_2^{(n)} + g(x, v_1^{(n)}, v_2^{(n)}),$$

and

$$-D^\alpha w_2^{(n+1)} + c w_2^{(n+1)} = c w_2^{(n)} + g(x, w_1^{(n)}, w_2^{(n)}),$$

we have

$$D^\alpha(w_2^{(n+1)} - v_2^{(n+1)}) - c(w_2^{(n+1)} - v_2^{(n+1)}) = c(v_2^{(n)} - w_2^{(n)}) + g(x, v_1^{(n)}, v_2^{(n)}) - g(x, w_1^{(n)}, w_2^{(n)}).$$

Let $z = w_2^{(n+1)} - v_2^{(n+1)}$ and using the Mean Value theorem, we get

$$D^\alpha z - cz = (c + \frac{\partial g}{\partial y_1}(\rho_1))(v_2^{(n)} - w_2^{(n)}) + \frac{\partial g}{\partial y_2}(\rho_2)(v_1^{(n)} - w_1^{(n)}),$$

where $\rho_1 = \eta v_1^{(n)} + (1-\eta) v_1^{(n+1)}$ and $\rho_2 = \zeta v_2^{(n)} + (1-\zeta) v_2^{(n+1)}$ for some $0 \leq \eta, \zeta \leq 1$. Using the induction hypothesis and the conditions (R1) and (R2), we have

$$D^\alpha z - cz \leq 0,$$

which together with $z(0) - \mu_1 D z(0) \geq 0$ and $z(1) + \mu_2 D z(1) \geq 0$ proves that $z \geq 0$. Thus $w_2^{(n+1)} \geq v_2^{(n+1)}$ and the proof is complete.

□

We state the convergence results of the two sequences in the following theorem.

Theorem 3.2. *Assume that the conditions (R1) and (R2) hold, and consider the two iterative sequences* $V^{(k)} = (v_1^{(k)}, v_2^{(k)})$ *and* $W^{(k)} = (w_1^{(k)}, w_2^{(k)})$ *obtained from Eq.'s (15-18) with* $U^{(0)} = V = (v_1, v_2)$ *and* $U^{(0)} = W = (w_1, w_2)$*, respectively. Then for* $\mu_1 \geq \frac{1}{\alpha-1}$

(1) *the two sequences converge uniformly to* $V^* = (v_1^*, v_2^*)$ *and* $W^* = (w_1^*, w_2^*)$*, respectively with* $v_1^* \leq w_1^*$ *and* $v_2^* \leq w_2^*$*. Moreover,*
(2) *for any solution* $Y = (y_1, y_2) \in [V, W]$ *of (4-7), we have* $v_1^* \leq y_1 \leq w_1^*$ *and* $v_2^* \leq y_2 \leq w_2^*$*, i.e;* $Y \in [V^*, W^*]$*.*

Proof. (1) The two sequences $\{v_1^{(k)}\}$ and $\{v_2^{(k)}\}$ are increasing and bounded above by $w_1^{(0)}$ and $w_2^{(0)}$, respectively. Hence, they converge to v_1^* and v_2^*, respectively. Since they are sequences of continuous functions defined on a compact set $[0, 1]$, then by Dini's theorem, the convergence is uniform. By applying similar arguments, the sequences $\{w_1^{(k)}\}$ and $\{w_2^{(k)}\}$ converge uniformly to w_1^* and w_2^*, respectively. Since $v_1^{(k)} \leq w_1^{(k)}$ and $v_2^{(k)} \leq w_2^{(k)}$, for each $k \geq 0$, then $v_1^* \leq w_1^*$ and $v_2^* \leq w_2^*$, which completes the proof.

(2) It is enough to show that $v_1^{(k)} \leq y_1 \leq w_1^{(k)}$ and $v_2^{(k)} \leq y_2 \leq w_2^{(k)}$, for each $k \geq 0$. We apply induction arguments to show that $v_1^{(k)} \leq y_1$ and $v_2^{(k)} \leq y_2$. Similar arguments can be used to prove that $y_1 \leq w_1^{(k)}$ and $y_2 \leq w_2^{(k)}$. Since $Y \in [V, W]$, then the result is true for $k = 0$. Assume the result is true for $k = n$, that is, $v_1^{(k)} \leq y_1 \leq w_1^{(k)}$ and $v_2^{(k)} \leq y_2 \leq w_2^{(k)}$, for each $k = 0, 1, \cdots, n$. We have

$$-D^2 v_1^{(n+1)} = v_2^{(n)} \text{ and } D^2 y_1 + y_2 = 0.$$

By adding the above equations and using the induction hypothesis, we have

$$D^2(y_1 - v_1^{(n+1)}) = v_2^{(n)} - y_2 \leq 0,$$

which together with $y_1(0) \geq v_1^{(n+1)}(0)$ and $y_1(1) \geq v_1^{(n+1)}(1)$ proves that $y_1 \geq v_1^{(n+1)}$, and the result is true for $k = n + 1$.
By adding Equtions (5) and (26) we have

$$D^\alpha(y_2 - v_2^{(n+1)}) + cv_2^{(n+1)} = cv_2^{(n)} + g(x, v_1^{(n)}, v_2^{(n)}) - g(x, y_1, y_2).$$

Subtracting $c(y_2 - v_2^{(n+1)})$ from both sides and using the Mean Value theorem, the above equation reduces to

$$\begin{aligned} D^\alpha(y_2 - v_2^{(n+1)}) - c(y_2 - v_2^{(n+1)}) &= -c(y_2 - v_2^{(n+1)}) + c(v_2^{(n)} - v_2^{(n+1)}) + \\ &\quad \frac{\partial g}{\partial y_1}(\rho_1)(v_1^{(n)} - y_1) + \frac{\partial g}{\partial y_2}(\rho_2)(v_2^{(n)} - y_2) \\ &= \left(c + \frac{\partial g}{\partial y_2}(\rho_2)\right)(v_2^{(n)} - y_2) + \frac{\partial g}{\partial y_1}(\rho_1)(v_1^{(n)} - y_1), \end{aligned}$$

where $\rho_1 = \eta v_1^{(n)} + (1-\eta) y_1$ and $\rho_2 = \zeta v_2^{(n)} + (1-\zeta) y_2$ for some $0 \leq \eta, \zeta \leq 1$. Applying (R1), (R2) and the induction hypothesis, we get

$$D^\alpha(y_2 - v_2^{(n+1)}) - c(y_2 - v_2^{(n+1)}) \leq 0,$$

which together with

$$y_2(0)-\mu_1 Dy_2(0) \geq v_2^{(n+1)}(0)-\mu_1 Dv_2^{(n+1)}(0) \text{ and } y_2(1)+\mu_2 Dy_2(1) \geq v_2^{(n+1)}(1)+\mu_2 Dv_2^{(n+1)}(1),$$

proves that $y_2 \geq v_2^{(n+1)}$, and the result is true for $k = n+1$.

□

The existence of solution of the problem (4-7) is established in the following theorem.

Theorem 3.3. *Assume that the conditions (R1) and (R2) hold, and consider the two iterative sequences $V^{(k)} = (v_1^{(k)}, v_2^{(k)})$ and $W^{(k)} = (w_1^{(k)}, w_2^{(k)})$ obtained from Eq.'s (15-18) with $U^{(0)} = V = (v_1, v_2)$ and $U^{(0)} = W = (w_1, w_2)$, respectively. If $\mu_1 \geq \frac{1}{\alpha-1}$ and $\gamma_i(k) = e_i$ for $i = 1, 2, 3, 4$ and $k \geq 0$ then the problem (4-7) possesses two solutions $V^* = (v_1^*, v_2^*)$ and $W^* = (w_1^*, w_2^*)$ in $[V, W]$ with $v_1 \leq v_1^* \leq w_1^* \leq w_1$ and $v_2 \leq v_2^* \leq w_2^* \leq w_2$.*

Proof. We have

$$-\lim_{k\to\infty} D^2 v_1^{(k)} = \lim_{k\to\infty} v_2^{(k-1)}, \tag{27}$$

and

$$-\lim_{k\to\infty} D^\alpha v_2^{(k)} + c\lim_{k\to\infty} v_2^{(k)} = c\lim_{k\to\infty} v_2^{(k-1)} + \lim_{k\to\infty} g(x, v_1^{(k-1)}, v_2^{(k-1)}). \tag{28}$$

Since $\{v_1^{(k)}\}$ and $\{v_2^{(k)}\}$ converge uniformly and using the result in [4], we have

$$\lim_{k\to\infty} D^2 v_1^{(k)} = D^2\left(\lim_{k\to\infty} v_1^{(k)}\right) = D^2 v_1^*,$$

and

$$\lim_{k\to\infty} D^\alpha v_2^{(k)} = D^\alpha\left(\lim_{k\to\infty} v_2^{(k)}\right) = D^\alpha v_2^*.$$

The continuity of $g(x, y_1, y_2)$ yields

$$\lim_{k\to\infty} g(x, v_1^{(k-1)}, v_2^{(k-1)}) = g\left(x, \lim_{k\to\infty} v_1^{(k-1)}, \lim_{k\to\infty} v_2^{(k-1)}\right) = g(x, v_1^*, v_2^*).$$

By substituting the above results in Eq.'s (27) and (28), we get

$$-D^2 v_1^* = v_2^*, \text{ and } D^\alpha v_2^* = g(x, v_1^*, v_2^*).$$

Since $\gamma_i(k) = e_i$, by the continuity of the boundary conditions v_1^* and v_2^* satisfy the boundary conditions (6-7). Thus, (v_1^*, v_2^*) is a solution of (4-7).
By applying similar arguments we obtain (w_1^*, w_2^*) is also a solution of (4-7). The inequalities $v_1 \leq v_1^* \leq w_1^* \leq w_1$ and $v_2 \leq v_2^* \leq w_2^* \leq w_2$, are obtained in Theorem 3.2.

□

We refer to the solutions $V^* = (v_1^*, v_2^*)$ and $W^* = (w_1^*, w_2^*)$ as the maximal and minimal solutions, respectively.

Remark 1. *The question whether these two solutions are the same is left as an open problem. This question has been solved for elliptic systems with natural order using the Green's identity which is still not generalized for fractional derivatives. However, we leave it to the numerical results as a practical criterion to ensure the convergence of the two sequences to a common limit and hence the results of uniqueness of solutions is obtained.*

4. Numerical Results

In this section, we apply the analysis described in the previous sections on one example to show the validity of our procedure. Based on Theorem 3.1 and 3.2, we start our procedure by initial ordered lower and upper pairs of solutions $V = (v_1^{(0)}, v_2^{(0)})$ and $W = (w_1^{(0)}, w_2^{(0)})$, respectively. Then, we generate new lower and upper pairs of solutions by solving the system of boundary value problems (3.2)-(3.3) with the following boundary conditions

$$u_1^{(k)}(0) = e_1,\ u_1^{(k)}(1) = e_2,\ u_2^{(k)}(0) - \mu_1 Du_2^{(k)}(0) = e_3, \text{and } u_2^{(k)}(1) + \mu_2 Du_2^{(k)}(1) = e_4.$$

Discritize the interval $[0, 1]$ with the nodes $x_i = ih$, $h = \frac{1}{N}$ for some positive integer N. Let $u_1^{(k,j)} = u_1^{(k)}(x_j), u_2^{(k,j)} = u_2^{(k)}(x_j)$, and $g^{(k,j)} = g(x_j, u_1^{(k,j)}, u_2^{(k,j)})$ for $j = 0, 1, ..., N$. We approximate Equation (3.2) by the finite difference method as follows:

$$-\frac{u_1^{(k,j+1)} - 2u_1^{(k,j)} + u_1^{(k,j-1)}}{h^2} = u_2^{(k-1,j)},$$

or

$$\begin{aligned} u_1^{(k,j+1)} &= 2u_1^{(k,j)} - u_1^{(k,j-1)} - h^2 u_2^{(k-1,j)}, k = 1, 2, ..., N-1, \\ u_1^{(k,0)} &= e_1, u_1^{(k,1)} = \omega_1. \end{aligned}$$

To find the unknown ω_1, we use the linear shooting method by setting $u_1^{(k,N)} = e_2$. To solve Equation (3.3), we first apply I^α for both sides to get

$$-u_2^{(k)}(x) + e_3 + (\mu_1 + x)\omega_2 = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{-cu_2^{(k)}(t) + cu_2^{(k-1)}(t) + g(t, u_1^{(k-1)}(t), u_2^{(k-1)}(t))}{(x-t)^{1-\alpha}} dt.$$

We then use the Trapezoidal rule to approximate the integral at x_j

$$\begin{aligned} -u_2^{(k,j)} &= -e_3 - (\mu_1 + x_j)\omega_2 + \frac{h}{2\Gamma(\alpha)} \left[\frac{-cu_2^{(k,0)} + u_2^{(k-1,0)} + g^{(k-1,0)}}{(x_j - x_0)^{1-\alpha}} \right] \\ &\quad + \frac{h}{\Gamma(\alpha)} \left[\sum_{i=1}^{j-1} \frac{-cu_2^{(k,i)} + u_2^{(k-1,i)} + g^{(k-1,i)}}{(x_j - x_i)^{1-\alpha}} \right]. \end{aligned}$$

Finally, the unknown ω_2 is obtained by setting $u_2^{(k,N)} + \mu_2 Du_2^{(k,N)} = e_4$ and using the linear shooting method. To compute $Du_2^{(k,N)}$, we implement the backward difference formula

$$Du_2^{(k,N)} = \frac{u_2^{(k,N-1)} - u_2^{(k,N-2)}}{h}.$$

Example 1: Consider the fractional boundary value problem

$$D^{7/2}y(x) - y_2(x)e^{-y_2(x)} = 0, \quad 0 < x < 1,$$

subject to

$$y(0) = 1,\ y(1) = 0,\ y''(0) - 2y'''(0) = -1, \text{and } y''(1) = -1.$$

Following to our discussion in this paper, we transform the problem to the form

$$\begin{aligned} D^2 y_1(x) + y_2(x) &= 0, \\ D^{3/2} y_2(x) - y_2(x) e^{y_2(x)} &= 0, \end{aligned}$$

subject to

$$\begin{aligned} y_1(0) &= 1, y_1(1) = 0 \\ y_2(0) - 2y_2'(0) &= 1, y_2(1) = 1. \end{aligned}$$

An initial ordered lower and upper solutions are

$$(v_1^{(0)}, v_2^{(0)}) = (2x(x-1), -1), (w_1^{(0)}, w_2^{(0)}) = (1 - x(x-1), 1.)$$

It is easy to see that $(v_1^{(0)}, v_2^{(0)}) \leq (w_1^{(0)}, w_2^{(0)})$, and the function $g(x, y_1, y_2) = -y_2(x)e^{y_2(x)}$ satisfies the two conditions (R1) and (R2) with $c = 2e$. The numerical solutions for $v_1^{(k)}$ and $w_1^{(k)}$ are plotted in Figure 1 for $k = 0, 1, 2, 3$. One can see that the upper and lower solutions become closers to each other and numerically they almost coincide after only 4 iterations. For sure more accurate bounds can be obtained by performing more iterations. Also, since $v_1^{(1)} \geq 0$ a positivity result is obtained.

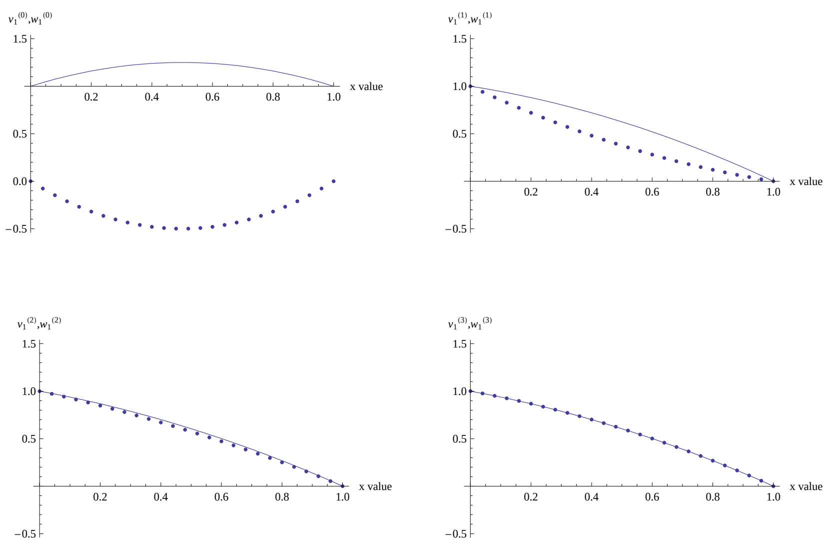

FIGURE 1. A plot of $w^{(k)}$ and $v^{(k)}$, $k = 0, 1, 2, 3$, for Example 1.

5. Concluding Remarks

In this paper we have applied comparison arguments to study a class of non-linear fractional differential equations with fractional order $3 < \delta < 4$. First, we have transformed the problem into a system of two boundary value problems, one with natural order and the other with fractional order. We then have established an existence result by introducing an increasing sequence of lower solutions that converges uniformly to a true solution (minimal solution) of the problem. Similar result is obtained by introducing a decreasing sequence of upper solutions that converges uniformly to a true solution (maximal solution) of the problem. While the existence result is established the uniqueness result is left as an open problem, but has been verified numerically in this paper.

The convergence of the lower and upper sequences to actual solution of the problem has been verified through an example. In this example, the finite difference method, the linear shooting method and the Trapezoidal rule have been used to obtain the numerical results, which indicate the rapid convergence of the lower and upper solution to the actual solution of the problem. Since our goal in this example is to show the convergence of the sequences, we used simple methods such as Trapezoidal rule. However, if one is interested in obtaining more accurate results, Simpson's rule can be implemented.

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