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STUDY ON MULTI-ORDER FRACTIONAL FOKKER-PLANCK EQUATION BY VARIATIONAL ITERATION METHOD

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Abstract. The aim of the present paper is to investigate the application of the variational iteration method for solving the multi-fractional linear and nonlinear Fokker-Planck equation and some similar equations. Some examples including fractional forward Kolmogorov equation, fractional backward Kolmogorov equation and fractional anisotropic Fokker-Planck equation are provided to verify the effectiveness of the method.

1. Introduction

In the last past decades, the fractional differential equations appear more and more frequently in different research areas and engineering applications[1, 2, 3], such as anomalous transport in disordered systems, some percolations in porous media, and the diffusion of biological populations. But these nonlinear fractional differential equation are difficult to get their exact solutions[4, 5, 6]. An effective method for solving such equations is needed. The variational iteration method first introduced by He[7, 8]for solving linear or nonlinear partial differential equations. The method, well addressed(see [9]-[14]), has been employed to solve a large variety of linear and nonlinear problems with approximations converging rapidly to accurate solutions. The method has many advantages over the classical technique mainly, it provides an efficient numerical solution with high accuracy and minimal calculations. The Fokker-Planck equation arises in various fields in natural science, including solid-state physics, quantum optics, chemical physics, theoretical biology and circuit theory. The Fokker-Planck equation was first used by Fokker and Planck[15] to investigate the Brownian motion of particles, and was later rigorously derived by Kolmogorov. If a small particle of mass m is immersed in a fluid, the equation of motion for the distribution function $w(x,t)$ is given by $\partial w(x,t)/\partial t = \gamma \partial w/\partial v + \gamma KT/m \partial^2 w/\partial v^2$, where $v$is the velocity for the Brownian motion of a small particle, γ is the fraction constant, K is Boltzmann's constant and T is the temperature of the fluid. This is the simplest types of Fokker-Planck equation. In this paper, we extend the variational iteration method to multi-fractional Fokker-Planck equation, the time-space fractional forward Kolmogorov equation

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can be expressed as follows

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=[-\frac{\partial^{\beta}A(x,t)}{\partial x^{\beta}}+\frac{\partial^{2\beta}B(x,t)}{\partial x^{2\beta}}]u(x,t), \tag{1}$$

with the initial condition

$$u(x,0)=\varphi(x), x\in R \tag{2}$$

where $u(x,t)$is an unkown function, $A(x,t)$ and $B(x,t)$ are called diffusion and drift coefficients. There also exists another type of this equation which is called time-space fractional backward Kolmogorov equation as

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=[-A(x,t)\frac{\partial^{\beta}}{\partial x^{\beta}}+B(x,t)\frac{\partial^{2\beta}}{\partial x^{2\beta}}]u(x,t), \tag{3}$$

A generalization of Eq.(1) to variables of, $x_1, x_2, \cdots, x_N$, yields to time-space fractional anisotropic Fokker-Planck equation:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=[-\sum_{i=1}^{N}\frac{\partial^{\beta_i}A_i(x,t)}{\partial x_i^{\beta_i}}+\sum_{i,j=1}^{N}\frac{\partial^{2r}B_{i,j}(x,t)}{\partial x_i^{r}\partial x_j^{r}}]u(x,t), \tag{4}$$

where time fractional derivatives and space fractional derivatives are described in Caputo sense, when the fractional parameter are all equal to one, the fractional equation reduces to the classical equations. Tatari et al.[16] obtained an exact solution of Fokker-Planck equation using the Adomian decomposition method.. Yildirim[17] introduced the solutions of the Fokker-Planck equation by the homotopy perturbation method. Odibat et al[18] studied the numerical solution of fractional forward Kolmogorov equation by variational iteration method and Adomian decomposition method.

This paper is devoted to study the fractional forward Kolmogorov equation, fractional backward Kolmogorov equation and fractional anisotropic Fokker-Planck equation. Our work here stems mainly from variational iteration method, that has been widely used in applied sciences, which is capable of handing a wider class of diffusion problems. Numerical solutions of multi-fractional Fokker-Planck equations shall be presented to demonstrate the effectiveness of the algorithm.

2. Fractional Calculus

There are several approaches to define the fractional calculus, e.g. Riemann-Liouville, Gruünwald-Letnikow, Caputo, and Generalized Functions approach. Rie mann-Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for real world physical problems since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet, Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.

Definition 1. The Riemann-Liouville fractional integral operator $J^{\alpha}(\alpha\geq 0)$ of a function $f(t)$, is defined as

$$J^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau, \quad (\alpha\geq 0) \tag{5}$$

where $\Gamma(\cdot)$ is the well-known gamma function, and some properties of the operator J^{α} are as follows

$$J^{\alpha}J^{\beta}f(t)=J^{\alpha+\beta}f(t), \quad (\alpha\geq 0,\beta\geq 0) \tag{6}$$

$$J^{\alpha}t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)}t^{\alpha+\gamma}, \quad (\gamma \geq -1) \tag{7}$$

Definition 2. The Caputo fractional derivative D^{α} of a function $f(t)$ is defined as

$$_0D_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(t)d\tau}{(t-\tau)^{\alpha+1-n}}, \quad (n-1 < Re(\alpha) \leq n, n \in N) \tag{8}$$

the following are two basic properties of the Caputo fractional derivative

$$_0D_t^{\alpha}t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}t^{\beta-\alpha}, \tag{9}$$

$$(J^{\alpha}D^{\alpha})f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{t^k}{k!}, \tag{10}$$

we have chosen to the Caputo fractional derivative because it allows traditional initial and boundary conditions to be included in the formulation of the problem. And some other properties of fractional derivative can be found in [1, 2].

3. Description of the method

The variational iteration method which provides an analytical approximate solution is applied to various nonlinear problems[9-14]. To solve the multi-fractional Fokker-Planck equation by means of variation iteration method, we take Eq.(1) as an example, and rewrite Eq.(1) in the form

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = [-\frac{\partial^{\beta}A(x,t)u(x,t)}{\partial x^{\beta}} + \frac{\partial^{2\beta}B(x,t)u(x,t)}{\partial x^{2\beta}}], t > 0, x > 0 \tag{11}$$

where $0 < \alpha \leq 1, 0 < \beta \leq 1$, the correction functional for Eq.(11) can be approximately expressed as follows:

$$u_{n+1} = u_n + \int_0^t \lambda(\xi)(\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial^{\beta}A(x,\xi)\tilde{u}_n(x,\xi)}{\partial x^{\beta}} - \frac{\partial^{2\beta}B(x,\xi)\tilde{u}(x,\xi)}{\partial x^{2\beta}})d\xi, \tag{12}$$

where $\lambda(\xi)$ is a general Lagrange multiplier, which can be identified optimally via variational theory, here $\tilde{u}_n(x,\xi)$ is considered as restricted variations, making the above functional stationary,

$$\delta u_{n+1} = \delta u_n + \delta\int_0^t \lambda(\xi)(\frac{\partial u_n}{\partial \xi} + \frac{\partial^{\beta}A\tilde{u}_n}{\partial x^{\beta}} - \frac{\partial^{2\beta}B\tilde{u}}{\partial x^{2\beta}})d\xi, \tag{13}$$

yields the following Lagrange multiplier

$$\lambda(\xi) = -1, \tag{14}$$

therefore, we obtain the following iteration formula:

$$u_{n+1} = u_n - \int_0^t (\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial^{\beta}A(x,\xi)\tilde{u}_n(x,\xi)}{\partial x^{\beta}} - \frac{\partial^{2\beta}B(x,\xi)\tilde{u}(x,\xi)}{\partial x^{2\beta}})d\xi, \tag{15}$$

we take initial condition $u(x,0) = \varphi(x)$ as the initial approximations $u_0(x,t)$, then the approximations $u_n(x,t)$,for $n \geq 1$, can be completely determined. Finally, we approximate the solution $u(x,t) = \lim_{n\to\infty} u_n(x,t)$ by the Nth term $u_N(x,t)$.

4. Approximate solutions of the multi-fractional equations

In order to access the advantages and the accuracy of the variational iteration method presented in this paper for multi-fractional Fokker-Planck equation, we have applied it to the following several problems. All the results are calculated by using the symbolic calculus software Mathematica.

Case 1: In this case, we consider $A(x,t,u) = \frac{4u}{x} - \frac{x}{3}$, $B(x,t,u) = u$ and the time-space fractional forward Kolmogorov equation as follows:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = [-\frac{\partial^{\beta}(\frac{4u}{x} - \frac{x}{3})u}{\partial x^{\beta}} + \frac{\partial^{2\beta} u^2}{\partial x^{2\beta}}], \tag{16}$$

subject to the initial condition

$$u(x,0) = x^2, \tag{17}$$

according to the formula (15), the iteration formula for Eq.(16) is given by

$$u_{n+1} = u_n - \int_0^t (\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial^{\beta}(\frac{4u_n}{x} - \frac{x}{3})u_n}{\partial x^{\beta}} - \frac{\partial^{2\beta} u_n^2}{\partial x^{2\beta}}) d\xi, \tag{18}$$

by the above variational iteration formula, begin with $u_0 = x^2$ we can obtain the following approximations

$$u_0 = x^2, \tag{19}$$

$$u_1 = x^2 - \frac{22tx^{3-\beta}}{\Gamma(4-\beta)} + \frac{24tx^{4-2\beta}}{\Gamma(5-2\beta)}, \tag{20}$$

$$\begin{aligned} u_2 = x^2 &- \frac{768t^3x^{7-5\beta}\Gamma(8-4\beta)}{\Gamma(8-5\beta)\Gamma^2(5-2\beta)} + \frac{192t^3x^{8-6\beta}\Gamma(9-4\beta)}{\Gamma(9-6\beta)\Gamma^2(5-2\beta)}\frac{48tx^{4-2\beta}}{\Gamma(5-2\beta)} \\ &- \frac{24t^{2-\alpha}x^{4-2\beta}}{\Gamma(3-\alpha)\Gamma(5-2\beta)} - \frac{92t^2x^{5-3\beta}\Gamma(6-2\beta)}{\Gamma 6-3\beta)\Gamma(5-2\beta)} + \frac{24t^2x^{6-4\beta}\Gamma(7-2\beta)}{\Gamma(7-4\beta)\Gamma(5-2\beta)} \\ &- \frac{1936t^3x^{5-3\beta}\Gamma(6-2\beta)}{3\Gamma(6-3\beta)\Gamma^2(4-\beta)} - \frac{44tx^{3-\beta}}{\Gamma(4-\beta)} + + \frac{484t^3x^{6-4\beta}\Gamma(7-2\beta)}{3\Gamma(7-4\beta)\Gamma^2(4-\beta)} \\ &+ \frac{22t^{2-\alpha}x^{3-\beta}}{\Gamma(3-\beta)\Gamma(4-\beta)} + \frac{1408t^3x^{6-4\beta}\Gamma(7-3\beta)}{\Gamma(7-4\beta)\Gamma(5-2\beta)\Gamma(4-\beta)} - \frac{352t^3x^{7-5\beta}\Gamma(8-3\beta)}{\Gamma(8-5\beta)\Gamma(5-2\beta)\Gamma(4-\beta)} \\ &+ \frac{253t^2x^{4-2\beta}\Gamma(5-\beta)}{3\Gamma(5-2\beta)\Gamma(4-\beta)} - \frac{22t^2x^{5-3\beta}\Gamma(6-\beta)}{\Gamma(6-3\beta)\Gamma(4-\beta)} \end{aligned} \tag{21}$$

and so on, in the same manner the rest of components of the iteration formula (18) can be obtained using the Mathematica package. When fractional derivatives $\alpha = 1, \beta = 1$, the exact solution of the Eq.(16) was given in [17] using homotopy perturbation method. and the approximate solution of Eq.(16) is

$$u_0 = x^2,$$
$$u_1 = x^2(1+t),$$
$$u_1 = x^2(1+t+\frac{t^2}{2!}),$$
$$\vdots$$

If the fractional derivative $\beta = 1$, the approximate solution of the Eq.(16) is

$$u_0 = x^2,$$
$$u_1 = x^2(1+t),$$

$$u_1 = x^2(1 + t + \frac{t^2}{2!} - \frac{t^{2-\alpha}}{\Gamma(3-\alpha)}),$$

$$\vdots$$

which was given in [18].

Case 2: In this case, we consider $A(x,t,u) = -(x+1)$, $B(x,t,u) = x^2 e^t$ and the space fractional backward Kolmogorov equation as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{(x+1)\partial^\beta u}{\partial x^\beta} + \frac{(x^2 e^t)\partial^{2\beta} u}{\partial x^{2\beta}}, \tag{22}$$

subject to the initial condition

$$u(x,0) = x + 1, \tag{23}$$

according to the formula (15), the iteration formula for Eq.(22) is given by

$$u_{n+1} = u_n - \int_0^t (\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{(x+1)\partial^\beta u_n}{\partial x^\beta} - \frac{(x^2 e^\xi)\partial^{2\beta} u_n}{\partial x^{2\beta}})d\xi, \tag{24}$$

by the above variational iteration formula, begin with $u_0 = x+1$ we can obtain the following approximations

$$u_0 = x + 1, \tag{25}$$

$$u_1 = 1 + x - \frac{2x^{3-2\beta}}{\Gamma(2-2\beta)} + \frac{2e^t x^{3-2\beta}}{\Gamma(2-2\beta)} + \frac{tx^{1-\beta}}{\Gamma(2-\beta)} + \frac{tx^{2-\beta}}{\Gamma(2-\beta)}, \tag{26}$$

$$\begin{aligned}
u_2 = 1 + x &+ \frac{6x^{5-4\beta}}{\Gamma(4-4\beta)} - \frac{12e^t x^{5-4\beta}}{\Gamma(4-4\beta)} + \frac{6e^t x^{5-4\beta}}{\Gamma(4-4\beta)} - \frac{10x^{5-4\beta}\beta}{\Gamma(4-4\beta)} + \frac{20e^t x^{5-4\beta}\beta}{\Gamma(4-4\beta)} \\
&- \frac{10e^{2t} x^{5-4\beta}\beta}{\Gamma(4-4\beta)} + \frac{4x^{5-4\beta}\beta^2}{\Gamma(4-4\beta)} - \frac{8e^t x^{5-4\beta}\beta^2}{\Gamma(4-4\beta)} + \frac{4e^{2t} x^{5-4\beta}\beta^2}{\Gamma(4-4\beta)} + \frac{x^{3-3\beta}\beta}{\Gamma(2-3\beta)} - \frac{e^t x^{3-3\beta}}{\Gamma(2-3\beta)} \\
&+ \frac{e^t t x^{3-3\beta}}{\Gamma(2-3\beta)} + \frac{2x^{4-3\beta}\beta}{\Gamma(3-3\beta)} - \frac{2e^t x^{4-3\beta}}{\Gamma(3-3\beta)} + \frac{2e^t t x^{4-3\beta}}{\Gamma(3-3\beta)} - \frac{x^{4-3\beta}\beta}{\Gamma(3-3\beta)} + \frac{e^t x^{4-3\beta}\beta}{\Gamma(3-3\beta)} \\
&+ \frac{tx^{2-\beta}}{\Gamma(2-\beta)} - \frac{e^t t x^{4-3\beta}\beta}{\Gamma(3-3\beta)} + \frac{t^2 x^{1-2\beta}}{2\Gamma(2-2\beta)} + \frac{t^2 x^{2-2\beta}}{2\Gamma(2-2\beta)} - \frac{x^{3-2\beta}}{\Gamma(2-2\beta)} + \frac{e^t x^{3-2\beta}}{\Gamma(2-2\beta)} + \frac{tx^{1-\beta}}{\Gamma(2-\beta)} \\
&+ \frac{2e^t x^{3-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)} - \frac{2x^{3-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)} - \frac{2tx^{3-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)} - \frac{2x^{4-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)} \\
&+ \frac{2e^t x^{4-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)} - \frac{2tx^{4-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)} + \frac{t^2 x^{2-2\beta}\Gamma(3-\beta)}{2\Gamma(3-2\beta)} + \frac{t^2 x^{3-2\beta}\Gamma(3-\beta)}{2\Gamma(3-2\beta)}
\end{aligned} \tag{27}$$

and so on, in the same manner the rest of components of the iteration formula (30) can be obtained using the Mathematica package. When fractional derivatives $\beta = 1$, the exact solution of the Eq.(22) was given in [17] using homotopy perturbation method. and the approximate solution of Eq.(22) is

$$u_0 = x + 1,$$

$$u_1 = (x+1)(1+t),$$

$$u_2 = (x+1)(1 + t + \frac{t^2}{2!}),$$

$$\vdots$$

Case 3: We will consider $N = 2, A_1(x,y) = x, A_2(x,y) = 5y, B_{1,1}(x,y) = x^2, B_{1,2}(x,y) = 1, B_{2,1}(x,y) = 1, B_{2,2}(x,y) = y^2$, and the multi-fractional Fokker-Planck equation as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -\frac{\partial^\beta (xu)}{\partial x^\beta} + \frac{\partial^{2\beta}(x^2 u)}{\partial x^{2\beta}} - \frac{\partial^\gamma (5yu)}{\partial y^\gamma} + \frac{\partial^{2\gamma}(y^2 u)}{\partial y^{2\gamma}} + \frac{\partial^{2m}(u)}{\partial x^m \partial y^m} + \frac{\partial^{2n}(u)}{\partial y^n \partial x^n} \quad (28)$$

subject to the initial condition

$$u(x,y,0) = x, \quad (29)$$

similar to the formula (3.5), the iteration formula for Eq.(22) is given by

$$u_{n+1} = u_n - \int_0^t \left(\frac{\partial^\alpha u}{\partial \xi^\alpha} + \frac{\partial^\beta (xu)}{\partial x^\beta} - \frac{\partial^{2\beta}(x^2 u)}{\partial x^{2\beta}} + \frac{\partial^\gamma (5yu)}{\partial y^\gamma} - \frac{\partial^{2\gamma}(y^2 u)}{\partial y^{2\gamma}} - \frac{\partial^{2m}(u)}{\partial x^m \partial y^m} - \frac{\partial^{2n}(u)}{\partial y^n \partial x^n}\right) d\xi, \quad (30)$$

by the above variational iteration formula, begin with$u_0 = x + 1$, we can obtain the following approximations

$$u_0 = x, \quad (31)$$

$$u_1 = x + \frac{6tx^{3-2\beta}}{\Gamma(4-2\beta)} - \frac{2tx^{2-\beta}}{\Gamma(3-\beta)} + \frac{2txy^{2-2\gamma}}{\Gamma(3-2\gamma)} - \frac{5ty^{1-\gamma}}{\Gamma(2-\gamma)}, \quad (32)$$

$$\begin{aligned} u_2 = {} & x + \frac{12tx^{3-2\beta}}{\Gamma(4-2\beta)} - \frac{6t^{2-\alpha}x^{3-2\beta}}{\Gamma(3-\alpha)\Gamma(4-2\beta)} - \frac{t^2x^{4-3\beta}\Gamma(5-2\beta)}{\Gamma(5-3\beta)\Gamma(4-2\beta)} + \frac{3t^2x^{5-4\beta}\Gamma(6-2\beta)}{\Gamma(6-4\beta)\Gamma(4-2\beta)} \\ & - \frac{4tx^{2-\beta}}{\Gamma(3-\beta)} + \frac{2t^{2-\alpha}x^{2-\beta}}{\Gamma(3-\alpha)\Gamma(3-\beta)} + \frac{t^2x^{3-2\beta}\Gamma(4-\beta)}{\Gamma(4-2\beta)\Gamma(3-\beta)} - \frac{t^2x^{4-3\beta}\Gamma(5-\beta)}{\Gamma(5-3\beta)\Gamma(3-\beta)} + \frac{4txy^{2-2\gamma}}{\Gamma(3-2\gamma)} \\ & - \frac{2t^{2-\alpha}xy^{2-2\gamma}}{\Gamma(3-\alpha)\Gamma(3-2\gamma)} + \frac{12t^2x^{3-2\beta}y^{2-2\gamma}}{\Gamma(4-2\beta)\Gamma(3-2\gamma)} - \frac{4t^2x^{2-\beta}y^{2-2\gamma}}{\Gamma(3-\beta)\Gamma(3-2\gamma)} - \frac{5t^2xy^{3-3\gamma}\Gamma(4-2\gamma)}{\Gamma(4-3\gamma)\Gamma(3-2\gamma)} \\ & + \frac{t^2xy^{4-4\gamma}\Gamma(5-2\gamma)}{\Gamma(5-4\gamma)\Gamma(3-2\gamma)} + \frac{t^2x^{1-m}y^{2-m-2\gamma}}{\Gamma(2-m)\Gamma(3-m-2\gamma)} + \frac{t^2x^{1-n}y^{2-n-2\gamma}}{\Gamma(2-n)\Gamma(3-n-2\gamma)} - \frac{10txy^{1-\gamma}}{\Gamma(2-\gamma)} \\ & + \frac{5t^{2-\alpha}xy^{1-\gamma}}{\Gamma(3-\alpha)\Gamma(2-\gamma)} - \frac{30t^2x^{3-2\beta}y^{1-\gamma}}{\Gamma(4-2\beta)\Gamma(2-\gamma)} + \frac{10t^2x^{2-\beta}y^{1-\gamma}}{\Gamma(3-\beta)\Gamma(2-\gamma)} + \frac{25t^2xy^{2-2\gamma}\Gamma(3-\gamma)}{\Gamma(3-2\lambda)\Gamma(2-\gamma)} \\ & - \frac{5t^2xy^{3-3\gamma}\Gamma(4-\gamma)}{2\Gamma(4-3\gamma)\Gamma(2-\lambda)} - \frac{5t^2x^{1-m}y^{1-m-\gamma}}{2\Gamma(2-m)\Gamma(2-m-\lambda)} - \frac{5t^2x^{1-n}y^{1-n-\gamma}}{2\Gamma(2-n)\Gamma(2-n-\gamma)}, \end{aligned} \quad (33)$$

and so on, in the same manner the rest of components of the iteration formula (4.15) can be obtained using the Mathematica package. When fractional derivatives$\alpha = \beta = \gamma = m = n = 1$, the exact solution of the Eq.(28)$u(x,t) = xe^t$ was given in [17] using homotopy perturbation method. and the approximate solution of Eq.(28) is

$$u_0 = x,$$

$$u_1 = x(1+t),$$

$$u_2 = x(1 + t + \frac{t^2}{2!}),$$

$$\vdots$$

Table 1 shows the approximate solutions for Eq.(28) using the variational iteration method and the exact solution $u(x,t) = xe^t$ when the value $\alpha = \beta = \gamma = m = n = 1$,it is noted that only the third-order term of the variational iteration solution was used in evaluating the approximate solutions for Table 1.

TABLE 1. Numerical values and exact solutions when $\alpha = \beta = \gamma = m = n = 1$ for Eq.(16)

t	x	numerical value by VIM	exact solution	absolute error
0.06	0.25	0.2654	0.2654	0.0000
	0.50	0.5309	0.5309	0.0000
	0.75	0.7963	0.7963	0.0000
	1.0	1.0618	1.0618	0.0000
0.2	0.25	0.305	0.3053	0.0003
	0.50	0.61	0.6107	0.0007
	0.75	0.915	0.9160	0.0010
	1.0	1.22	1.2214	0.0014
0.4	0.25	0.37	0.3729	0.0029
	0.50	0.74	0.7459	0.0059
	0.75	1.11	1.1187	0.0087
	1.0	1.48	1.49	0.0118

5. Conclusion

In this paper, approximate solutions for the fractional forward Kolmogorov equation, fractional backward Kolmogorov equation and fractional anisotropic Fokker-Planck equation have been obtained, and the variational iteration method was successfully used to these solutions. The reliability of this method and reduction in computations give this method a wider applicability. The corresponding solutions are obtained according to the recurrence relation using Mathematica.

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