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# MEASURABLE-LIPSCHITZ SELECTIONS AND SET-VALUED INTEGRAL EQUATIONS OF FRACTIONAL ORDER

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Abstract. In this paper we study the sufficient conditions for the existence of measurable-Lipschitz selection for the set-valued function $F : I \times R \to R$ in the two cases when $F$ has convex or nonconvex values.
An application we prove the existence of an integrable solution for the set-valued integral equation

$$x(t) \in g(t) + \int_0^t k(t,s)F(s,x(s))ds, \ t \in [0,1].$$

The set-valued integral equation of fractional-order

$$x(t) \ \in g(t) \ + \ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \ F(s,x(s))ds, \ t \in [0,1] \text{ and } \beta \in (0,1)$$

will be given as an example.

## 1. Introduction

The existence of measurable-Lipschitz selection for the set valued map $F$ has been studied by V.V. Chistyakov and A. Nowak in [3], when $F$ is set-valued map from $T \times X$ into $Y$, where ($X$ is an interval (open, closed, half-closed, bounded or not) on the real line $R$ and $(Y, d)$ be a metric space with metric $d$). P. Bettiol and H. Frankowska (see [3]) assumed that $F$ is measurable-Lipschitz set-valued map and proved some properties of the set of solutions to the differential inclusion

$$x'(t) \in F(t, x(t)), \ x(t) \in K.$$

Myelkebir Aitalioubrahim in [1] prove the existence theorem of Boundary value problem of second order (with Neumann Boundary conditions), where $F$ is measurable in the first argument and Lipschitz in the second argument.
Mireille Broucke and Ari Arapostathis in [5] show that given any finite set of trajectories of a Lipschitz differential inclusion (where $F$ is measurable in the first argument and Lipschitz in the second argument) there exists a continuous selection from the set of its solutions that interpolates the given trajectories. In addition, we present a result on lipschitzian selections.

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In the sequel we prove the existence of measurable-Lipschitz selection for the set-valued function $F : I \times R \to R$ in the two cases when $F$ has convex or nonconvex values.
In section (4) as an application we prove the existence of an integrable solution $x$ for the set-valued integral equation

$$x(t) \in g(t) + \int_0^t k(t,s)\ F(s,x(s))ds,\ t \in [0,1].$$

An example we study the existence of integrable solution $x$ for the following set-valued integral equation of fractional-order

$$x(t) \in g(t) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ F(s,x(s))ds,\ t \in [0,1] \text{ and } \beta \in (0,1).$$

## 2. Preliminaries

In this section we establish our notations and we recall some basic definitions and known results used in the proof of our results here.
Let $L^1 = L^1[0,1]$ be the class of equivalent integrable function on the interval $I = [0,1]$ with the usual norm

$$\|x(t)\| = \int_0^1 |x(s)|\, ds.$$

$P(Y)$ denoted to the family of nonempty subsets of $Y$.
$P_{cl}(Y)$ denoted to the family of nonempty closed subsets of $Y$.
$P_{cl,bd}(Y)$ denoted to the family of nonempty closed, bounded subsets of $Y$.
Let $(X,d)$ be a metric space and let $A \subseteq X,\ x \in X$ and $d(x,A) = \inf\{d(a,x);\ a \in A\}$.
**Definition 1** For any $A, B \in P_{cl,bd}(X)$, the Hausdorff distance is defined by

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right\}.$$

**Definition 2** Let $(X,d)$ and $(Y,\rho)$ be two metric spaces and let $T : X \to P_{cl}(Y)$ be a multivalued mapping. Then $T$ is called Lipschitz multivalued mapping ($c-$Lipschitz) if there exists a constant $c > 0$ such that for each $x, y \in X$ we have

$$H(T(x),T(y)) \leq c\ d(x,y)$$

where $H$ is the Hausdorff metric.
The constant $c$ is called the Lipschitz constant of $T$. In particular if $c < 1$, then $T$ is called contraction multivalued mapping on $X$.
**Definition 3** Let $(T,\Sigma)$ be a measurable space and $X$ be a topological space, a multivalued function $F : T \to X$ is measurable if for each open set $O$ in $X$ the set

$$F^{-1}(O) = \left\{t \in T;\ F(t) \bigcap O \neq \emptyset\right\}$$

is measurable (i.e. $F^{-1}(O) \in \Sigma$).
**Definition 4** A Polish space is a separable completely metrizable topological space, and a Suslin space is the image of a Polish space under a continuous mapping.
**Remark:** An example of Suslin space is a separable completely metrizable topological space.

**Theorem** (Yankov-Von Neumann selection theorem)
If $(\Omega, \Sigma, \mu)$ complete measure space, $X$ is Souslin space and

$$F : \Omega \to P(X)$$

is multifunction, such that

$$Gr(F) \in \Sigma \times \beta(X)$$

(where $\beta(X)$ denote the Borel $\sigma-$field on $X$). Then $F$ admits a measurable selection.

## 3. Measurable-Lipschitz selections

In this section we give some various sufficient conditions for the existence of measurable-Lipschitz selection of the set-valued function $F$ in the two cases when $F$ has convex and nonconvex values.

**Definition 5** We say that $F$ is measurable-Lipschitz if $F(.,x)$ is measurable for all $x \in X$ and $F(t,.)$ is Lipschitz for all $t \in I$ .

For the compact convex set-valued functions we have the following selection theorem.

**Theorem 1** Let $F : I \times R \to P(R)$ be a nonempty compact convex set-valued function satisfies the following conditions:

(i) $F(.,x)$ is measurable in $I$ for every $x \in R$,
(ii) $F(t,.)$ is $c-$Lipschitz set-valued function for each fixed $t \in I$,
(iii) $F(t,0)$ is integrable in the sense that ($\forall a \in F(t,0)$ implies that $a$ is integrable) and there exists an integrable function $m$ , such that $|F(t,0)| \leq m(t)$.

Then there exists a selection $f$ of $F$ satisfies the following conditions:

(1) $f(.,x)$ is measurable for every $x \in R$,
(2) $f(t,.)$ is $c-$Lipschitz function,
(3) $f(t,0)$ is integrable function.

**Proof** By Corollary [7. Co. 2] we have $F(t,.)$ has $c-$Lipschitz selection $v(.)$. Now we define the set of all $c-$Lipschitz selections of the function $F(t,.)$ by the set valued function

$$G(t) = \{v \in C(R,R) \ : \ v \ is \ Lipschitz \ selection \ of \ F(t,.)\} .$$

Now we prove that $G$ has a measurable selection.
Let $\alpha : I \to C(R,R)$ defined by

$$\alpha(t) = \sup \left\{ \frac{H(F(t,x), F(t,y))}{|x-y|}, \ x, y \in R, \ x \neq y \right\}$$

from the definition of $\alpha$ we have $\alpha$ is measurable, because $\alpha$ can be written as

$$\alpha(t) = \sup \{\varphi(t,x,y), \ x, y \in R, \ x \neq y\}$$

such that $\varphi(t,x,y) = \frac{H(F(t,x),F(t,y))}{|x-y|}$, thus $\varphi$ is measurable in $t$ and by the continuity of $\varphi$ in $(x,y)$, we have

$$\alpha(t) = \sup \{\varphi(t,x,y), \ x, y \in R \ and \ rational, \ x \neq y\}$$

hence $\alpha$ is measurable.
Let $\beta : C(R,R) \to R \bigcup \{+\infty\}$ defined as follows

$$\beta(v) = \sup \left\{ \frac{|v(x) - v(y)|}{|x-y|},\ x, y \in R,\ x \neq y \right\}$$

for each $x, y \in R,\ x \neq y$, we have the function

$$(v, x, y) \mapsto \frac{|(v(x) - v(y)|}{|x-y|}$$

is continuous in $(v, x, y)$. Therefore by the continuity of this function in $(x, y)$, we can write $\beta$ by

$$\beta(v) = \sup \left\{ \frac{|(v(x) - v(y))|}{|x-y|},\ x, y \in R \ and\ rational,\ x \neq y \right\}$$

therefore the function

$$v \mapsto \frac{|(v(x) - v(y)|}{|x-y|}$$

is continuous and hence lower semicontinuous.
We define also $\Gamma(t,u) = \sup \{d(u(x), F(t,x)),\ x \in R\}$,
then the function $(t, u, x) \mapsto d(u(x), F(t,x))$ is measurable in $t$ and continuous in $u$ and $x$.
Thus, for each fixed $x \in R$ it is $\Sigma \times \beta(C(X,Y))-$measurable.
We have $\Gamma(t,u) = \sup \{d(u(x), F(t,x)),\ x \ is \ rational\}$ by virtue of the continuity in $x$. Consequently, $\Gamma$ is product-measurable. Note that for each fixed $t \in I$, $\Gamma(t,.)$ is lower semicontinuous, being the supremum of continuous functions $u \mapsto d(u(x), F(t,x)),\ x \in R$. Now we set $\gamma(t,v) = \sup \{\Gamma(t,v),\ \beta(v) - \alpha(t)\}$ hence $\gamma$ is product measurable and lower semicontinuous in $v$.
We have

$$Gr(G) = \{(t,v) \in I \times C(R,R) :\ \gamma(t,v) \leq 0\} = \gamma^{-1}(] - \infty, 0])$$

hence

$$Gr(G) \in \Sigma \times \beta(C(R,R))$$

and therefore $G$ satisfies the Yankov-Von Neumann selection theorem, hence there exists a measurable selection $g$ of $G$.
Now we define $f(t,x) = g(t)(x)$, observe that $f(t,x) \in F(t,x),\ \forall (t,x) \in I \times R$, first $f(t,.)$ is clearly Lipschitz ($f(t,.) = g(t)(.)$ is a Lipschitz selection of $F(t,.)$), let $U_y \subseteq R$ be open set, we set

$$U = \{v \in C(R,R),\ v(x) \in U_y\}.$$

We have $U$ is open in $C(R,R)$, so that

$$\begin{aligned} (f(.,x))^{-1}(U_y) &= \{t \in I :\ f(t,x) \in U_y\} = \{t \in I :\ g(t)(x) \in U_y\} \\ &= \{t \in I :\ g(t) \in U\} = g^{-1}(U) \in \Sigma. \end{aligned}$$

Hence $t \mapsto f(t,x)$ is measurable for each fixed $x \in R$.
Now $f(t,0) \in F(t,0),\ \forall t \in I$, then by assumption (iii) $\exists\ m$ (integrable) such that $f(t,0) \leq m(t)$, which implies that $f(t,0)$ is integrable.

For the compact set-valued functions, as a result of [3], we have the following theorem.
**Theorem 2** Let $F : I \times R \to P_{cp}(R)$ be measurable-Lipschitz multifunction.

Then $F$ has a measurable-Lipschitz selection $f : I \times R \to R$.
Moreover, if there exists an integrable function $m$, such that $|F(t,0)| \leq m(t)$, where

$$|F(t,x)| = \sup\{|v|,\ v(t) \in F(t,x),\ t \in I\},$$

then $f(t,0)$ is integrable.
**Proof** (see [3]).

For the set-valued functions with closed valued ( not necessarily convex ) we have the following selection theorem.
**Theorem 3** Let $F : I \times R \to P_{cl}(R)$ be a set-valued function satisfies the following conditions:

(i) $F(.,x)$ is measurable in $I$ for every $x \in R$.
(ii) $F(t,.)$ is $c-$Lipschitz set-valued function for each fixed $t \in I$.
(iii) $F(t,0)$ is integrable in the sense that ($\forall a \in F(t,0)$ implies that $a$ is integrable) and there exists an integrable function $m$, such that $|F(t,0)| \leq m(t)$.

Then there exists a selection $f$ of $F$ satisfies the following conditions:

(1) $f(.,x)$ is measurable for every $x \in R$.
(2) $f(t,.)$ is $c-$Lipschitz function.
(3) $f(t,0)$ is integrable.

**Proof** By Theorem [8, Th. 2] we have $F(t,.)$ has $k-$Lipschitz selection $v(.)$. Now we define new set valued function

$$G(t) = \{v \in C(R,R) :\ v \ is\ Lipschitz\ selection\ of\ F(t,.)\}$$

Now by the same way as in Theorem 1 we can prove that there exist selection $f$ of $F$ satisfies the conditions (1)-(3)

## 4. Set-valued integral equation

Consider the integral equation

$$x(t) \ = \ g(t) \ + \ \int_0^t \ k(t,s)\ f(s,x(s))ds \tag{1}$$

with the following assumptions

(1) $g : I \to R$ is integrable on $I$,
(2) $k(.,.)$ is measurable in the two variable and there exists $M > 0$ such that such that $\int_0^1 |k(t,s)|\ dt \ \leq M,\ s \in I$.
(3) $f(.,x)$ is measurable for each fixed $x \in R$.
(4) $f(t,.)$ is $c-$Lipschitz for each fixed $t \in I$.
(5) $f(t,0) \ = \ a(t)$ is integrable.
(6) $Mc < 1$.

**Theorem 4** Assume that the assumptions (1)-(6) are satisfied. Then the integral equation (1) has a unique integrable solution $x$.
**Proof** Let us define the operator $G$ by

$$(Gx)(t) = g(t) + \int_0^t k(t,s) f(s,x(s))ds,$$

then the integral equation (1) can be written as

$$x(t) \ = \ (Gx)(t).$$

From assumption (4) we have

$$|f(t,x) - f(t,y) \ \leq \ c\ |x-y| \ \Rightarrow \ |f(t,x) - f(t,0)| \ \leq \ c\ |x|$$

which proves that

$$|f(t,x)| \ \leq \ |a(t)| \ + \ c\ |x|$$

Now, let $x$ be integrable function, then

$$||Gx|| \ \leq \ ||g|| \ + \ \int_0^1 \ \int_0^t \ |k(t,s)\ |f(s,x(s))|dsdt$$

this implies that

$$||Gx|| \ \leq \ ||g|| \ + \ \int_0^1 \ \int_0^t \ |k(t,s)|\ [a(s) \ + \ c\ |x(s)|]\ dsdt$$

$$||Gx|| \ \leq \ ||g|| \ + \ \int_0^1 \ [a(s) \ + \ c\ |x(s)|] \ \int_s^1 \ |k(t,s)|dtds$$

$$||Gx|| \ \leq \ ||g|| \ + \ M\ [||a|| \ + \ c\ ||x||]$$

which proves that $G : L^1 \to L^1$.
Now from our assumptions we have

$$\|Gx - Gy\| \leq \int_0^1 \int_0^t |k(t,s)|\, |f(s,x(s)) - f(s,y(s))|\, dsdt.$$

This implies that

$$\|Gx - Gy\| \ \leq \ c\ \int_0^1 \int_0^t |k(t,s)|\, |x(s) - y(s)|\, dsdt$$

and

$$\|Gx - Gy\| \ \leq \ c\ \int_0^1 |x(s) - y(s)| \int_s^1 |k(t,s)|\, dtds.$$

Therefore

$$\|Gx - Gy\| \ \leq \ c\ M \ \ \|x - y\|$$

whenever $x,\ y$ are integrable. Then from Assumption (6) and the Banach Contraction Mapping Principle we deduce that $G$ has unique fixed point $x$ and therefore the integral equation (1) has a unique integrable solution $x$.

Now we present some existence theorems for the solution to the set-valued integral equation

$$x(t) \ \in \ g(t) \ + \ \int_0^t \ k(t,s)\ F(s,x(s))ds. \qquad (2)$$

Consider the following assumptions

(1) $g : I \to R$ is integrable on $I$,
(2) $k(.,.)$ is measurable in the two variable and there exists $M \ > \ 0$ such that such that $\int_0^1 \ |k(t,s)|\ dt \ \leq M,\ s \in I$,
(3) $F$ is Caratheodory-$c-$Lipschitz set-valued function, with compact, convex values,

(4) $F(t,0)$ is integrable in the sense that ($\forall a \in F(t,0)$ implies that $a$ is integrable) and there exists an integrable function $m$ , such that $|F(t,0)| \leq m(t)$,

(5) $M\ c\ < 1$.

**Theorem 5** If the assumptions (1)-(5) are satisfied, then set-valued integral equation (2) admits an integrable solution $x$.

**Proof** From Theorem 1 and assumption (3) we deduce that the set-valued function $F$ admits a measurable-Lipschitz selection $f$ and assumption (4) implies that $f(t,0)$ is integrable. Applying Theorem 4 we deduce that the integral equation (1)

$$x(t) \ = \ g(t) \ + \int_0^t \ k(t,s) \ f(t,x(s))ds$$

has a unique integrable solution $x$ and hence the set-valued integral equation (2) admits an integrable solution in $x$.

Using Theorems 2 or 3, the result of Theorem 5 can be obtained if we replace condition (3), in Theorem 5, by one of the following conditions:

(a) $F$ is set-valued function with compact values and satisfies
   (1) $F(.,x)$ is measurable in $[0,1]$ for every $x \in R$,
   (2) $F(t,.)$ is $k-$Lipschitz set-valued function for each fixed $t \in I$.

(b) $F$ is set-valued function with closed values and satisfies
   (1) $F(.,x)$ is measurable in $[0,1]$ for every $x \in R$,
   (2) $F(t,.)$ is $k-$Lipschitz set-valued function for each fixed $t \in I$.

## 5. Set-valued integral equation of fractional-order

**Definition 6** The fractional-order integral of the function $f \in L^1[a,b]$ of order $\beta > 0$ is defined by (see [12])

$$I_a^\beta \ f(t) \ = \ \int_a^t \ \frac{(t \ - \ s)^{\beta \ - \ 1}}{\Gamma(\beta)} \ f(s) \ ds.$$

Consider now the set-valued integral equation of fractional-order

$$x(t) \ \in g(t) \ + \int_0^t \ \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \ F(s,x(s))ds, \ t \in [0,1] \text{ and } \beta \in (0,1). \qquad (3)$$

**Corollary 1** If the assumptions 1 and 3 - 5 of Theorem 5 are satisfied, then the set-valued integral equation (3) has an integrable solution $x$.

**Proof** Let $k(t,s) \ = \ \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}$, then we find that it is measurable in the two variable and

$$\int_s^1 \ K(t,s)dt = \int_s^1 \ \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} dt = \frac{(t-s)^\beta}{\Gamma(\beta)\beta} |_s^1 = \frac{(1-s)^\beta}{\Gamma(\beta+1)}.$$

Therefore

$$\int_s^1 \ |k(t,s)|dt \ = \ \frac{(1-s)^\beta}{\Gamma(\beta+1)} \ \leq \ \frac{1}{\Gamma(\beta+1)} \ = \ M$$

and hence the assumptions of theorem 5 are satisfied and therefore the set-valued integral equation (3) admits an integrable solution $x$.

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