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Interpolation formulas

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1. Introduction

Interpolation has been said to be the art of reading between the lines of tabulated values of a function. We may now make a distinction between interpolation and extrapolation. The latter is the art of reading before the first line or after the last line of a tabulated function. More specifically, we may define interpolation as the process of finding the values of a function for any value of the independent variable within an interval for which some values are given and extrapolation as the process of finding the values outside of this interval.

The process of interpolation becomes very important in advanced mathematics when dealing with functions which either are not known at every value of the independent variable within an interval, or the expression of which is so complicated that the evaluation of the function is prohibitive. It is then that the function is replaced by a simple function which assumes the known values of the given function and from which the other values may be computed to the desired degree of accuracy. This is the broader sense of interpolation.

In precise mathematical language we are concerned with a function, $y = y(x)$, whose values $y_o, y_1, \dots, y_n$, are known for the values $x_o, x_1, \dots, x_n$ of the independent variable. Interpolation now seeks to replace $y(x)$ by a simpler function, $P(x)$, which has the same values as $y(x)$ for $x_0, x_1, \ldots, x_n$ and from which other values can easily be calculated. The function $P(x)$ is said to be an interpolating formula or interpolating function. In many engineering application this function is called a smoothing function.

A desired characteristic of interpolating functions is that they be simple. Consequently, the most frequently employed forms are the polynomial and the finite trignometric series. In these cases we refer to the process as polynomial interpolation or trignometric interpolation. The latter is used if the given values indicate that the function is periodic. The interpolating function can, of course, be arbitrarily chosen and can take any form; thus it could be exponential, logarithmic, etc. One such form which is frequently used is that of a rational fraction. However, it should always be as simple as possible.

The use of the polynomial and trignometric series is based on Weierstrass' Theorems

I. Every function, $F(x)$, which is continuous in an interval (a,b) can be represented there, to any degree of accuracy, by a polynomial $P(x)$, i.e.,

$$|F(x) - P(x)| < \varepsilon$$

for all $a < x < b$ and where ε is any preasigned positive quantity.

II. Every continuous function, $F(x)$, of period 2Π can be represented by a finite trignometric series:

$$T(x) = a_o + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx +$$
$$b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx$$

such that

$$\left| F(x) - T(x) \right| < \varepsilon$$

for $a < x < b$ and $\varepsilon > 0$

2 - General Interpolation Formulas

Arbitrarily Spaced Data

2.1 Interpolation Polynomials

A polynomial of degree n

$$(1) \qquad y(x) = a_o + a_1 x + a_2 x^2 + \dots + a_n x^n = \sum_{k=o}^{n} a_k x^k$$

has n + 1 coefficients and it can be required to pass through the n + 1 points (x_i, y_i) $i = 0, 1, 2, \dots, n$ with $x_i \neq x_j$. To find the polynomial we see that we can always solve for the coefficients a_k by Cramers rule or by any other method.

Putting the values (x_i, y_i) $i = o, 1, 2, \dots, n$ into the polynomial (1), we have

$$y_i = \sum_{k=o}^{n} a_k x_i^k \quad , i = o, 1, 2, \ldots , n$$

this means

$$(2) \quad \begin{cases} y_o = a_o + a_1 x_o + a_2 x_o^2 + \ldots + a_n x_o^n \\ y_1 = a_o + a_1 x_1 + a_2 x_1^2 + \ldots + a_n x_1^n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n = a_o + a_1 x_n + a_2 x_n^2 + \ldots + a_n x_n^n \end{cases}$$

The determinant of the unknowns a_k of these n + 1 linear equations is the Vandermonde determinant:

$$(3) \quad D = \begin{vmatrix} 1 & x_o & x_o^2 & \ldots & x_o^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \ldots & & & & \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{vmatrix}$$

$$= (x_1 - x_o)(x_2 - x_o)(x_3 - x_o) \ldots (x_n - x_o)$$
$$(x_2 - x_1)(_3 - x_1) \ldots (x_n - x_1)$$
$$(x_3 - x_2) \ldots (x_n - x_2)$$
$$\ldots\ldots$$
$$(x_n - x_{n-1})$$

$$= \prod_{i > k} (x_i - x_k)$$

Thus the Vandermonde determinant vanishes if and only if any two of the x_i coincide.

Equations (1) and (2) can be regarded as n + 2 homogeneous equations in the n + 2 quantities $-1, a_o, a_1, \ldots, a_n$. Hence their determinant must vanish:

$$(4) \quad \begin{vmatrix} y(x) & 1 & x & x^2 & \ldots & x^n \\ y_o & 1 & x_o & x_o^2 & \ldots & x_o^n \\ y_1 & 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & & & & & \\ y_n & 1 & x_n & x_n^2 & \ldots & x_n^n \end{vmatrix} = 0$$

This can be regarded as an equation in y(x).

If we expand this determinant by elements of the first row, the first term will be D y(x), and every other term will be equal to some x^j multiplied by its cofactor, which is a constant. Hence when we solve for y(x), we shall have y(x) expressed as a linear combination of the functions x^j in just the form

$$y = a_o + a_1 + \ldots + a_n x^n$$

as required. Also if we set $x = x_i$ in (4) and subtract the i + 2 from the first row we get

$$D\,(y(x_i) - y_i) = 0$$

which shows that $y(x_i) = y_i$, and the values of y at the points x_i agree with those of f(x).

It is possible that the coefficients of x^n is zero and that the polynomial is of degree less than n. To cover this detail, the statement that a polynomial is of degree n often means of degree n or less. In the trivial case when all the y_i are equal the polynomial is of degree zero, and y(x) = C (constnat).

2.2. The lagrange Method of Interpolation

The basic idea behind the method is first to find a polynomial which takes on the value 1 at a particular sample point and the value 0 at all the other sample points i.e.

$$L_i(x) = \begin{cases} 1 & \text{for } x = x_i \\ 0 & \text{for } x = x_j \quad j \neq i \end{cases}$$

It is easy to see that the function

$$(5) \quad L_i(x) = \frac{(x-x_o)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_o)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$$

$$= \prod_{j=o}^{n} \frac{(x-x_j)}{(x_i-x_j)} = \prod_{\substack{j=o \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

(where the prime on the product means " excluding the ith value") is such a polynomial of degree n_i it is 1 when $x = x_i$ and 0 when $x = x_j \quad j \neq i$.

The polynomial $L_i(x)\ y_i$ takes on the value y_i at the sample point x_i and is zero at all other sample point. It then follows the well, known lagrange formula:

$$y(x) = L_o\ (x)\ y_o + L_1(x)\ y_1 + \dots + L_n(x)\ y_n = \sum_{i=o}^{n} L_i(x) y_i,$$

where

$$L_o(x) = \frac{(x-x_1)\ (x-x_2)\ (x-x_3)\ \dots\ (x-x_n)}{(x_o-x_n)(x_o-x_2)(x_o-x_3)\ \dots\ (x_o-x_n)}$$

$$L_1(x) = \frac{(x-x_o)\ (x-x_2)\ (x-x_3)\ \dots\ (x-x_4)}{(x_1-x_o)(x_1-x_2)(x_1-x_3)\ \dots\ (x_1-x_4)}$$

$$L_2(x) = \frac{(x-x_o)\ (x-x_1)\ (x-x_3)\ \dots\ (x-x_n)}{(x_2-x_o)(x_2-x_1)(x_2-x_3)\ \dots\ (x_2-x_n)}$$

. . . .

$$L_n(x) = \frac{(x-x_o)\ (x-x_1)\ (x-x_2)\ \dots\ (x-x_{n-1})}{(x_n-x_o)(x_n-x_1)(x_n-x_2)\ \dots\ (x_n-x_{n-1})}$$

With other method we can write the solution of (4) in the form:

$$D\, y(x) = \begin{vmatrix} o & 1 & x & x^2 & \cdots & x^n \\ y_o & 1 & x_o & x_o^2 & & x_o^n \\ y_1 & 1 & x_1 & x_1^2 & & x_1^n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ y_n & 1 & x_n & x_n^2 & & x_n^n \end{vmatrix} = - D^{*}$$

Thus

$$y(x) = - D^{-1} D^{*}$$

which coincides with (1) if we expand along the first row, but has the form,

(7) $$y(x) = \sum_{i=o}^{n} L_i(x)\, y_i$$

when we expand along the first column. The L_i are themselves polynomials with coefficients which depend only upon the x_j. These polynomials are

(8) $$L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

They can be obtained by direct expansion of the determinant, or we can verify that they satisfy the necessary conditions if we not that $L_i(x_j) = \delta_{ij}$

with δ_{ij} the Kronecker δ. From this it follows that with $L_i(x)$ defined by (8) and $y(x)$ by (7) we have $y(x_i) = y_i$.

Note:

The Lagrange interpolation formula is not very practical for computations. Its form can be modified somewhat to make it more tractable.

Consider the special case when all the $y_i = 1$. Then $y(x) = 1$ for all x, that is

$$L_o(x) + L_1(x) + \ldots + L_n(x) = 1$$

is an identity.

we may now divide the right-hand side of the Lagrange formula (7)

$$y(x) = \sum_{i=o}^{n} L_i(x) \; y_i$$

by

(9) $$\sum_{i=o}^{n} L_i(x) = 1$$

and defining

(10) $$u_i = \frac{1}{(x-x_i) \prod\limits_{\substack{i \\ (i \neq j)}} (x_i - x_j)}$$

we get,

when we divide numerator and denomiator by

$\prod_j (x-x_j)$:

$$(11) \qquad y(x) = \frac{\sum_{i=o}^{n} u_i\, y_i}{\sum_{i=o}^{n} u_i}$$

This is sometimes called the "barycentric formula" and is easier to use than the Lagrange formula. See the flow chart (1).

2.3 Newton's general interpolation formula

Newton's interpolation formula, which we now develop, is simply another way of writing the interpolating polynomial. It is useful because the number of points being used can easily be increased or decreased without repeating all the computation. Let the polynomial passing through the n + 1 points (x_i, y_i) , i=o,1,...,n be

$$(12) \qquad y(x) = \gamma_o + \gamma_1 (x-x_o) + \gamma_2 (x-x_o)(x-x_1) + \dots + \gamma_n (x-x_o)(x-x_1)\dots(x-x_{n-1})$$

It is now desired to determine the coefficients

γ_i, $i = o, 1, \ldots, n$

$$(13)\begin{cases} x = x_o : & y_o = \gamma_o \\ x = x_1 : & y_1 = \gamma_o + \gamma_1 (x_1 - x_o) \\ x = x_2 : & y_2 = \gamma_o + \gamma_1 (x_2 - x_o) + \gamma_2 (x_2 - x_o)(x_2 - x_1) \\ \ldots & \ldots \\ x = x_n : & y_n = \gamma_o + \gamma_1 (x_n - x_o) + \gamma_2 (x_n - x_o)(x_n - x_1) \\ & \quad + \ldots + \gamma_n (x_n - x_o)(x_n - x_1) \ldots (x_n - x_{n-1}) \end{cases}$$

we can solve (13) for the coefficients γ_i as follows:

$$\gamma_o = y_o,$$

$$\gamma_1 = \frac{y_1 - y_o}{x_1 - x_o} = [x_1 \; x_o]$$

$$\gamma_2 = \frac{y_2 - y_o - \frac{y_1 - y_o}{x_1 - x_o}(x_2 - x_o)}{(x_2 - x_o)(x_2 - x_1)} = \frac{\frac{y_2 - y_o}{x_2 - x_1} - \frac{y_1 - y_o}{x_1 - x_o}\,\frac{x_2 - x_o}{x_2 - x_1}}{x_2 - x_o}$$

$$= \left\{ \frac{y_2 - y_1}{x_2 - x_1} + \frac{y_1 - y_o}{x_2 - x_1} - \frac{y_1 - y_o}{x_1 - x_o}\,\frac{x_2 - x_o}{x_2 - x_1} \right\} / (x_2 - x_o)$$