# UNITED ARAB REPUBLIC

## THE INSTITUTE OF NATIONAL PLANNING
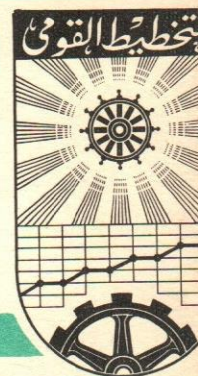
# Contents

# CHAPTER 1

## The real roots of an equation

This chapter deals with those methods which are applicable to
finding the real roots of the equation

$$f(x) = 0 \qquad\qquad (1.1)$$

where $f(x)$ is any piecewise continuous function of x having
numerical coefficients, whether a polynomial or transendental
function.

The finding of the real roots of $f(x)$ can in general be divided
into two parts. The first part has as its goal the finding of
an approximate value of the root. The second part makes use of
this approximate knowledge of the root to obtain the root out
to the desired number of significant figures.

## 1.1 Finding an approximate value of a real root

Sometimes, one may have a good guess as to the approximate
value of the root; but in general it will be necessary to
use one of the following methods.

### a. Graphical Methods

Generally speaking, the best method of finding the approximate
value of a root is to plot the function

$$y = f(x) \qquad\qquad (1.2)$$

and determine approximately the points at which this plot
crosses the x axis. At these points $y=0$, and hence the

corresponding values of x, by Eq.(1.2), satisfy Eq.(1.1) and
are therefore real roots of that equation.

In some cases it is preferable to write the equation in the form

$$f_1(x) = f_2(x) \qquad\qquad (1.3)$$

in which case we plot the two functions

$$y_1 = f_1(x) \quad \text{and} \quad y_2 = f_2(x) \qquad\qquad (1.4)$$

The abscissas of their points of intersection obviously satisfy
Eq.(1.3) and hence are the real roots of this equation.

b.  Analytical Methods

There are two common analytical methods for finding the
approximate value of the root of an equation.

One is to find a simpler equation that has a root approxima-
tely equal to the required root of the given equation.  This
can often be done by neglecting a term known to be small.
The second method makes use of the following theorem.

Theorem: If $f(x)$ is a real function that is continuous
between $x=a$ and $x=b$, where a and b are real numbers, and
if $f(a)$ and $f(b)$ are of opposite signs, then there is at
least one real root between a and b.

1.2  Method of False Position

Suppose that $f(a)$ and $f(b)$ are
of opposite signs, the plot $f(x)$
crosses the x axis between $x=a$
and $x=b$ and that, therefore,
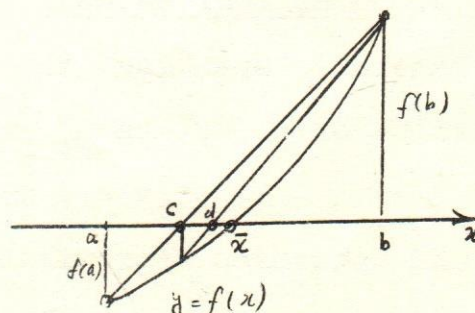


a root $x=\bar{x}$ lies between these limits.  Fig.1.1. Method of Flase
Position

The root $x=\bar{x}$ will be approximately given by $x=c$:

$$c = a + \frac{f(a)}{f(a)-f(b)} \ (b-a) \qquad\qquad (1.5)$$

Since $f(a)$ and $f(b)$ are of opposite signs, $f(c)$ must be opposite in sign to one of them; therefore it is possible to apply the approximation again for a still better prediction of the value of $\bar{x}$. Therefore a better approximation

$$d = c + \frac{f(c)}{f(c)-f(b)} \ (b-c) \qquad\qquad (1.6)$$

to the root $x=\bar{x}$ is obtained by replacing a in Eq.(1.5) by c. The method of false position is very simple in principle as it merely replaces the plot of $f(x)$ between any two points a and b by its chord. The simplicity and complete generality of this method make it a very powerful tool for computation.
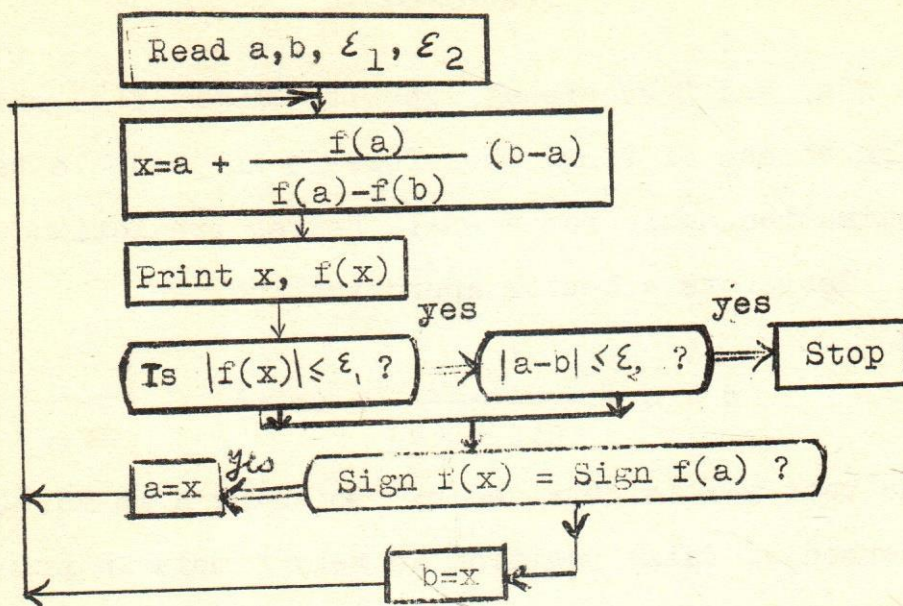
FLOW CHART

$f(x) = 0$

$\bar{x} = a + \frac{f(a)}{f(a)-f(b)} \ (b-a)$

Read a,b,

## FLOW CHART

$$f(x) = 0 \quad , \quad X = a + \frac{f(a)}{f(a)-f(b)} (b-a)$$

```
┌─────────────────────────┐
│  Read a,b, ε₁, ε₂       │
└─────────────────────────┘
           ↓
┌─────────────────────────┐
│ x=a + ──f(a)── (b-a)    │
│       f(a)-f(b)         │
└─────────────────────────┘
           ↓
┌─────────────────────────┐
│  Print x, f(x)          │
└─────────────────────────┘
           ↓
( Is |f(x)| ≤ ε₁ ? )  →yes→  ( |a-b| ≤ ε₂ ? )  →yes→  [ Stop ]
           ↓                        ↓
  [a=x] ←yes← ( Sign f(x) = Sign f(a) ? )
                        ↓
                     [ b=x ]
```
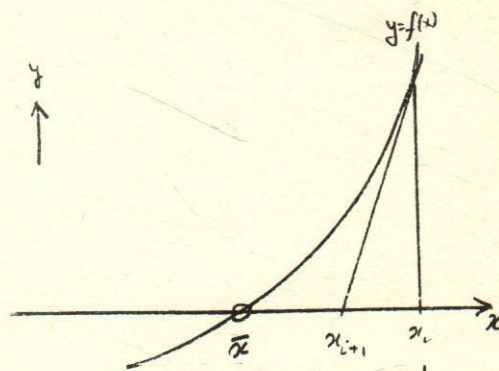
Test: $f(x) = xe^x - 2 = 0$

$a=0, \; b=1$

Ans. $x = 0.8526055$

## 1.3 The Newton-Raphson Method

In Fig.1.2, $x_{i+1}$ is determined by drawing a tangent to the curve at $x_i$ and extending it until it inter-sects the x axis at the point $(x_{i+1}, 0)$. The slope of the tangent is given by

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$



Newton-Raphson Method
Fig. 1.2.

Solving for $x_{i+1}$, we obtain the Newton-Raphson formula:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{1.7}$$

To determine the rate of convergence of the iterates in the Newton-Raphson method, we expand $f(x)$ about one of the iterates $x_i$ using Taylor's series with a remainder

$$f(x) = f(x_i)+(x-x_i)f'(x_i)+ \tfrac{1}{2}(x-x_i)^2 f''(\xi)$$

where $\xi$ lies between $x$ and $x_i$. Substituting $x=\bar{x}$ and solving for $\bar{x}$ in the second term on the right-hand side, we have

$$\bar{x} = x_i - \frac{f(x_i)}{f'(x_i)} - (\bar{x}-x_i)^2 \frac{f''(\xi)}{2f'(x_i)}$$

therefore, by Eq. (1.7)

$$x_{i+1}-\bar{x} = (x_i-\bar{x})^2 \frac{f''(\xi)}{2f'(x_i)} \qquad (1.8)$$

Since $x_j-\bar{x}$ is the error in the jth iterate to the root, Eq.(1.8) states that each iteration squares the error and then multiplies it by the factor $f''(\xi)/2f'(x_i)$, where $\xi$ lies somewhere between $x_i$ and the root $\bar{x}$. It is clear from the above that the convergence will be poor it $f''/2f'$ is large in the neighborhood of the root. This will usually happen if $f(x)$ does not cross the x axis at a sufficiently steep angle.
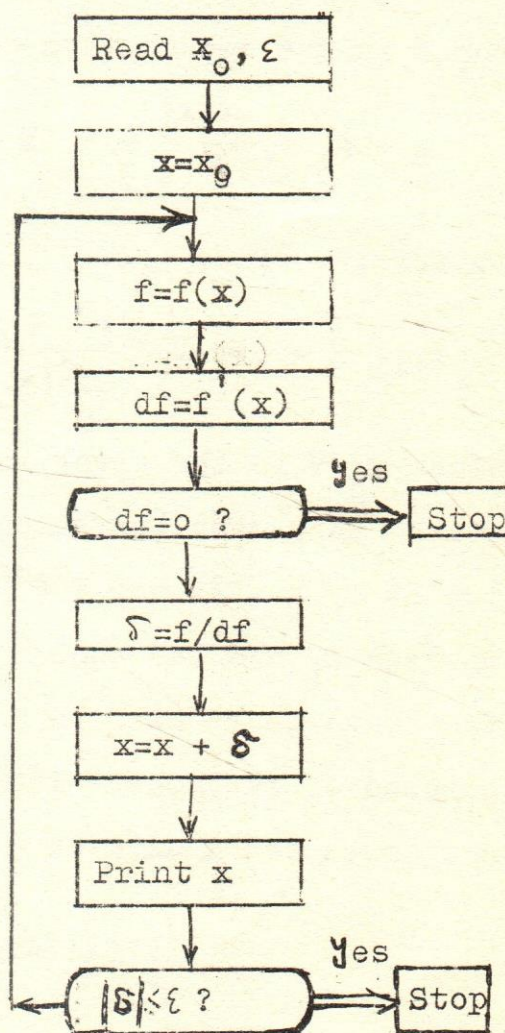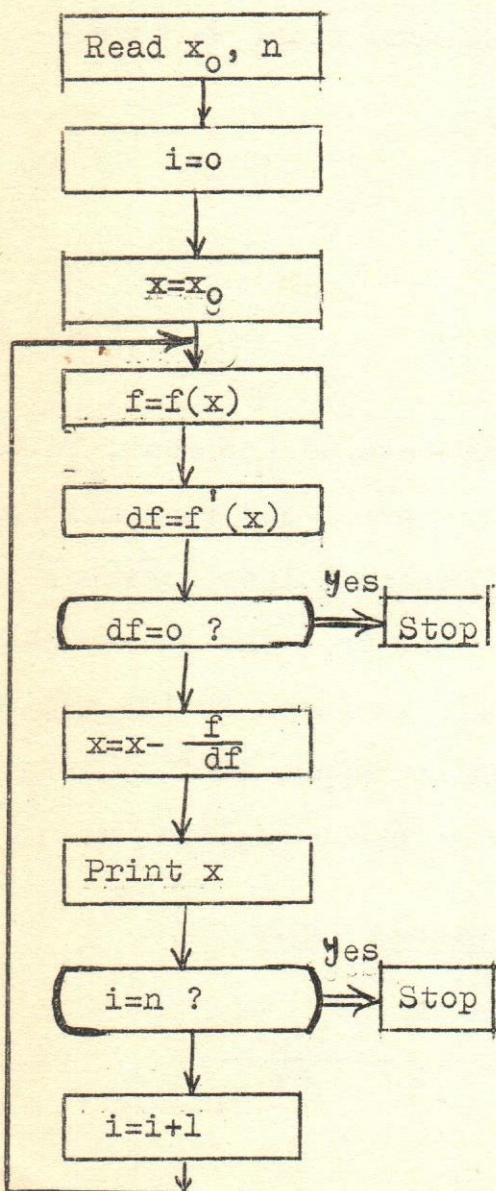
### Iteration-Method of higher Order

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \left(1 + \tfrac{1}{2}\frac{f''(x_i)f(x_i)}{f'^2(x_i)}\right) \qquad (1.9)$$

## FLOW CHART

$$x_{i+1} = x_i - \underbrace{\frac{f(x_i)}{f'(x_i)}}_{\delta}$$

Another method

| | |
|---|---|
| **Read $x_0$, n** | **Read $x_0$, $\varepsilon$** |
| i=0 | x=$x_0$ |
| x=$x_0$ | f=f(x) |
| f=f(x) | df=f'(x) |
| df=f'(x) | df=o ?  →yes→ Stop |
| df=o ?  →yes→ Stop | $\delta$=f/df |
| $x=x-\dfrac{f}{df}$ | x=x + $\delta$ |
| Print x | Print x |
| i=n ?  →yes→ Stop | $|\delta| \leq \varepsilon$ ?  →yes→ Stop |
| i=i+1 | |

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}\left(1+ \frac{1}{2}\frac{f''(x_i)f(x_i)}{f'^2(x_i)}\right)$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad}_{\delta}$$

```
┌─────────────────────┐
│    Read x_0, δ      │
└─────────────────────┘
           │
           ▼
┌─────────────────────┐
│      x = x_0        │
└─────────────────────┘
           │
           ▼
┌─────────────────────┐
│      f = f(x)       │
└─────────────────────┘
           │
           ▼
┌─────────────────────┐
│     df = f'(x)      │
└─────────────────────┘
           │
           ▼                    yes
(     f = 0 ?     )───────────► ┌──────┐
           │                    │ Stop │
           │                    └──────┘
           ▼
┌─────────────────────┐
│    ddF = f''(x)     │
└─────────────────────┘
           │
           ▼
┌─────────────────────────────┐
│  δ = -f/df (1 + f.ddf/2(df)²)│
└─────────────────────────────┘
           │
           ▼
┌─────────────────────┐
│      x = x + δ      │
└─────────────────────┘
           │
           ▼
┌─────────────────────┐
│       Print x       │
└─────────────────────┘
           │
           ▼                    yes
(     |δ| ≤ ε     )───────────► ┌──────┐
                                │ Stop │
                                └──────┘
```

$\delta = -\dfrac{f}{df}\left(1+\dfrac{f.ddf}{2(df)^2}\right)$

$|\delta| \le \varepsilon$
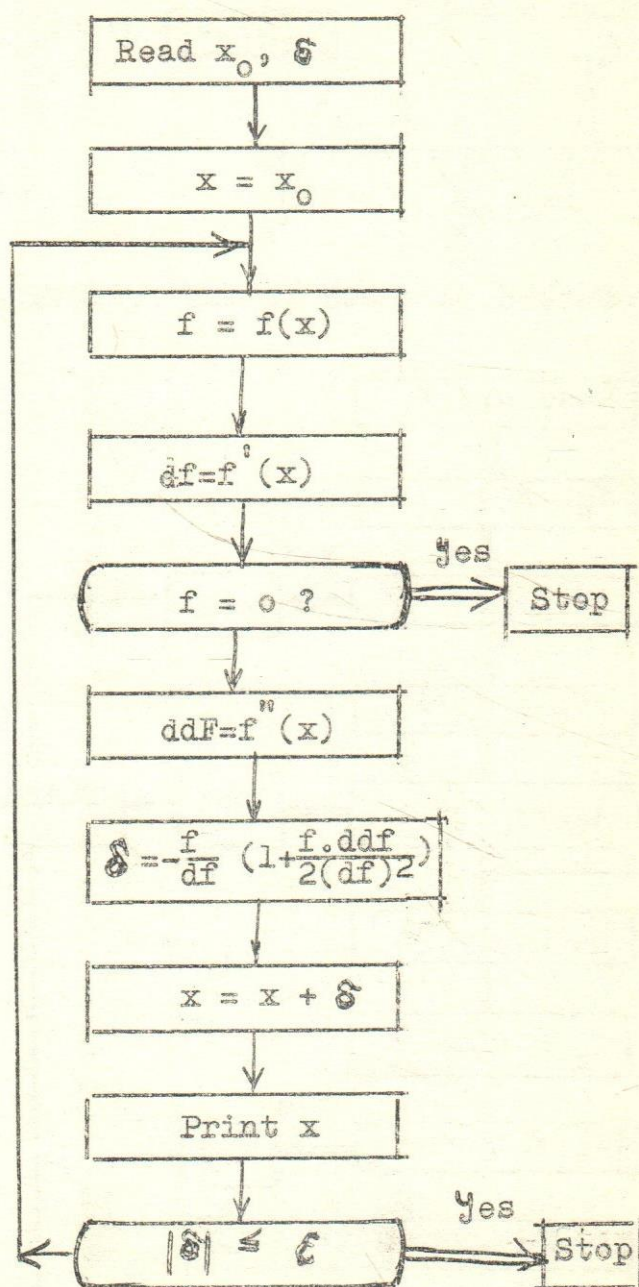
## 1.4   The Half - Method

If $f(x)$ is a real function that is continuous between $x=u$ and $x=v$, where u and v are real numbers, and if $f(u)$ and $f(v)$ are of opposite signs, then there is at least one real root between u and v.
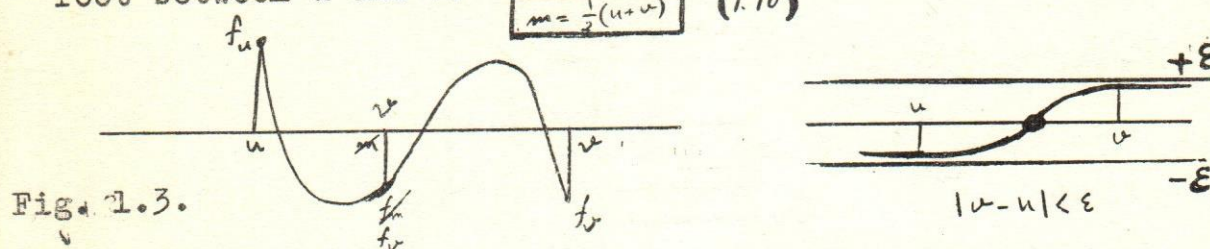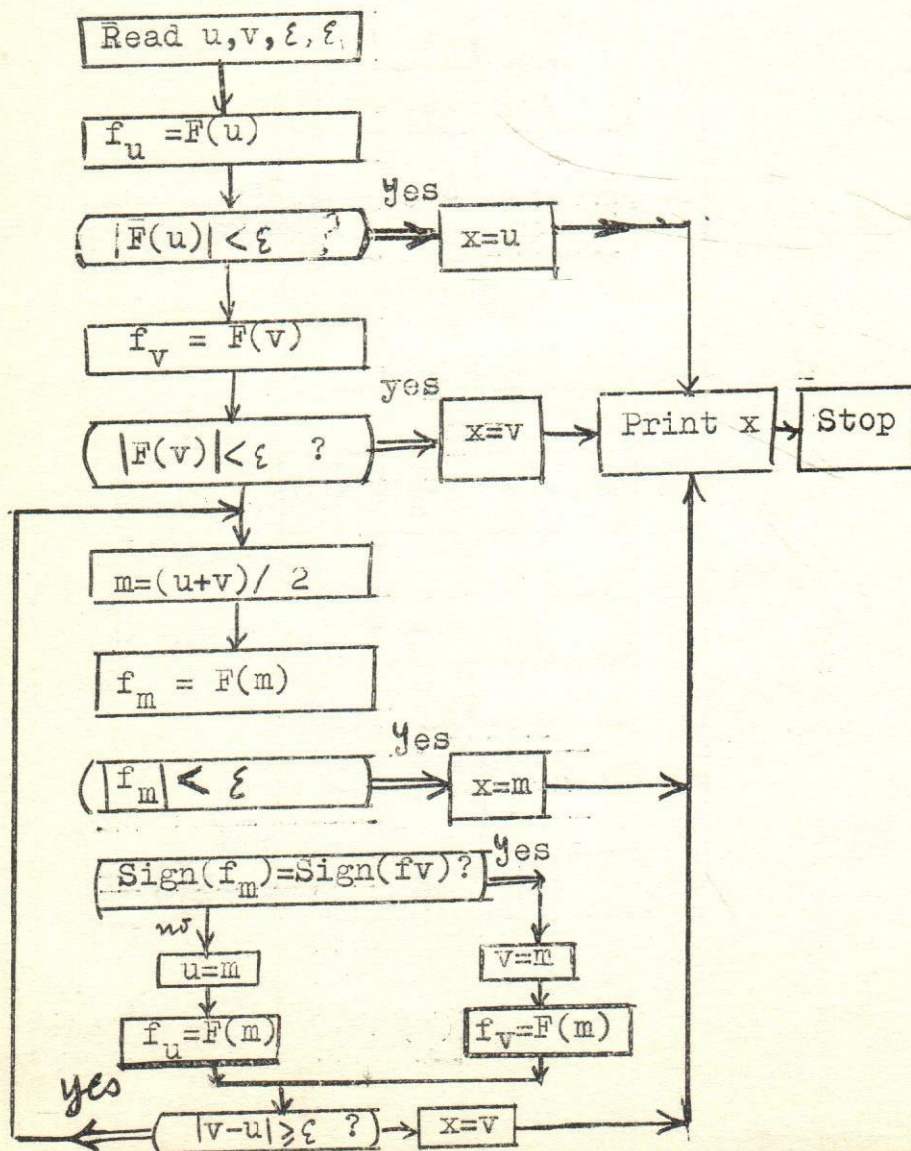
$$m = \tfrac{1}{2}(u+v) \quad (1.10)$$

Fig. 1.3.

The Half-Method is shown in the following flow-chart

Read u,v, $\varepsilon$, $\varepsilon$

$f_u = F(u)$

$|F(u)| < \varepsilon$  ?  —yes→  $x=u$

$f_v = F(v)$

$|F(v)| < \varepsilon$  ?  —yes→  $x=v$  →  Print x  →  Stop

$m = (u+v)/2$

$f_m = F(m)$

$|f_m| < \varepsilon$  —yes→  $x=m$

$Sign(f_m) = Sign(fv)$? —yes→ $v=m$

no → $u=m$

$f_u = F(m)$     $f_v = F(m)$

yes  —  $|v-u| \geqslant \varepsilon$  ? → $x=v$

## 1.5 Example

Cubic root $\sqrt[3]{a}$

$$f(x) = x^3 - a$$
$$f'(x) = 3x^2$$
$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$
$$x_{i+1} = x_i - \frac{x_i^3 - a}{3x_i^2}$$

$$\boxed{x_{i+1} = (2x_i + \frac{a}{x_i^2})\,/3}$$   (1.11)

1. Case : $0 < a \le 1$ ⟶ $x_0 = (2 + a)\,/3$

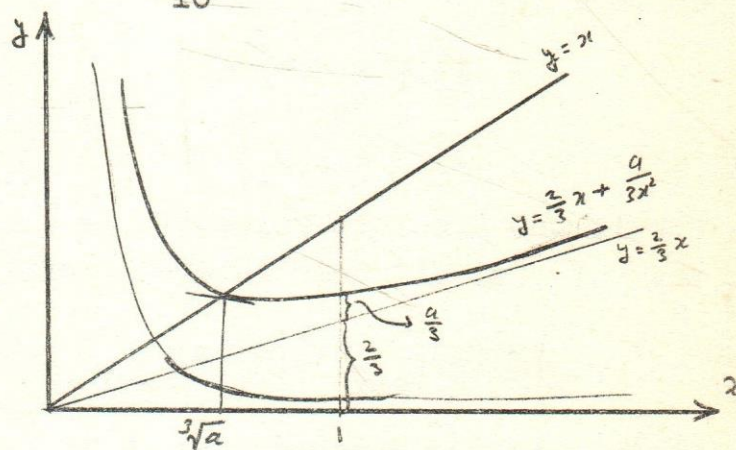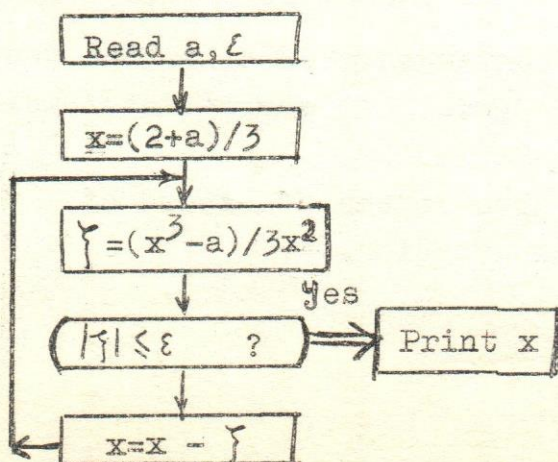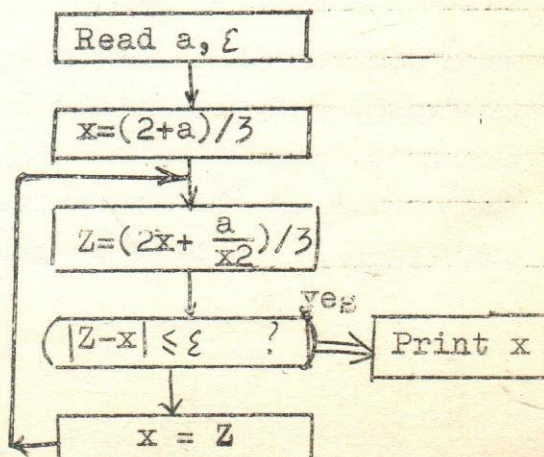2. Case : $a > 1$ ⟶ $\tilde{a} = \frac{a}{10^m}$ , m positive integer number



Fig.1.4.

Flow-chart



Other method

## 1.6  General Method of Iteration

The equation $f(x) = 0$ can be written in the from

$$g(x) = h(x) \qquad (1.12)$$

It is convenient to replace Eq. (1.12) by the set of Simultaneous equations

$$\left. \begin{array}{l} y = g(x) \\ y = h(x) \end{array} \right\} \qquad (1.13)$$

If one can solve explicity for $x$ in the second equation, these can be written

$$\left. \begin{array}{l} y = g(x) \\ x = H(y) \end{array} \right\} \qquad (1.14)$$

Suppose $x_0$ is any initial guess at the root of $f(x)$; then the iterates $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ may be defined by

$$\left. \begin{array}{l} y_{i-1} = g(x_{i-1}) \\ x_i \quad = H(y_i) \end{array} \right\} \quad i=1,2,3,\ldots$$

Now if the absolute value of the slope of $g(x)$ is less than that of $h(x)$ at their intersection, i.e. if

$$\left| g'(x) \right|_{x=\bar{x}} < \left| h'(x) \right|_{x=\bar{x}} \qquad (1.15)$$

where $\bar{x}$ is the desired root, then, for a sufficiently close guess $x_0$, $x_n \to \bar{x}$ as $n \to \infty$. If the slope of $g(x)$ is much smaller than $h(x)$, the convergence of $x_n$ to the root is rapid, and the method is a practical way of determining the root.

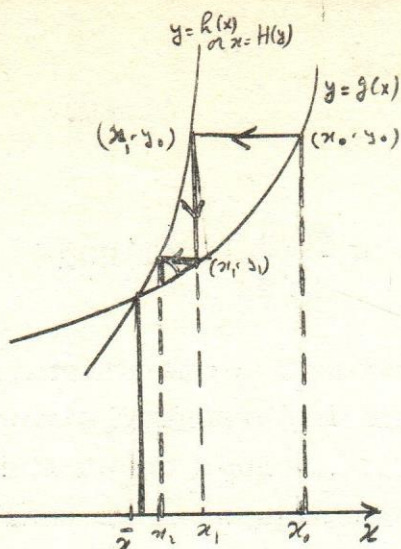The nature of the iterative process and its speed of convergence can best be shown graphically.

Fig 1.5  Convergence of the iterates $x_0, x_1, \cdots$ to the root $\bar{x}$ when $g(x)$ and $h(x)$ both have positive slopes.
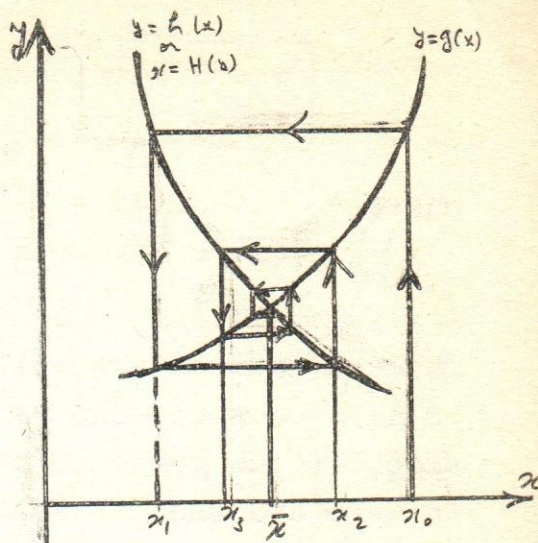
Fig 1.6  Convergence of the iterates $x_0, x_1, \cdots$, to the root $\bar{x}$ when $g(x)$ and $h(x)$ have opposite slopes.

## 1.7  Special Iterative Forms

The weakness of the iterative method is the difficulty of being able to write $f(x) = 0$ in the form

$$y = g(x)$$
$$\qquad\qquad\qquad (1.16)$$
$$y = h(x)$$

and at the same time to make sure that the first equation has a much smaller slope at the intersection and that the second equation is readily solvable for x.

One can overcome these difficulties by establishing standard methods of forming the iterative equation (1.16)

In place of $f(x) = 0$, write

$$F(x) = -\frac{f(x)}{f'(x)} = 0 \qquad\qquad (1.17)$$

This function has the same root $\bar{x}$ as $f(x) = 0$, provided $f'(\bar{x}) \neq 0$, and moreover

$$F'(\bar{x}) = -1 + \frac{f(\bar{x})\, f''(\bar{x})}{\left[f'(\bar{x})\right]^2} = -1 \qquad\qquad (1.18)$$

Therefore if one deals with $F(x)$ instead of $f(x)$, one o
obtains the iteration equations
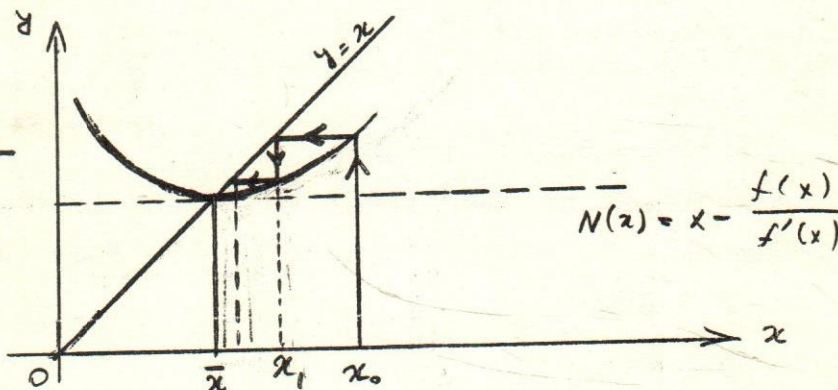
$$\boxed{\begin{array}{l} y = N(x) \\ x = y \end{array}} \qquad (1.19)$$

where

$$N(x) = F(x) + x = x - \frac{f(x)}{f'(x)} \qquad (1.20)$$

These equations are well suited for iteration, provided $f'(\bar{x})$ is not too small and is readily obtained, since $N'(\bar{x})$, the slope of the graph of the first equation at the intersection, is zero and that of the second equation is unity (see Fig 1.5)



Fig. 1.7. The iterative equivalent of the Newton-Raphson Formula.

$$N(x) = x - \frac{f(x)}{f'(x)}$$

Flow-chart

$$y = x - \frac{f(x)}{f'(x)}$$
$$x = y$$