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Lectures on Production Theory and Techniques

by

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Economic Expert

1965

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PREFACE

This is the first draft of my lectures on 'Production Theory and Techniques' prepared on the suggestion of Professor Bent Hansen with whom I had the good fortune to teach the so-called Advanced Economic Theory. As Prof. Hansen so rightly pointed out, Production Theory forms the backbone of all Growth Theory and the Growth Models differ from each other according to the Production Function they use or imply. (There cannot be any growth theory without production theory). Thus the Production Theory occupies the same or even more crucial position in the Post World War II 'Growth Era' as the Consumption Theory did in the Inter-War 'Business Cycles Era'. Unfortunately the pure economists cannot be expected to be at their best in Production Theory as they, along with statisticians, were in Consumption Theory. Any really worthwhile invention or discovery in Production Theory is more likely to come from engineers and physicists rather than from economists or econometricians. And it is for nought that the former group has been often more successful in programming and planning than the latter in the recent years.

I am extemely grateful to Prof. Hansen for his continuous ecouragement and suggestions. He has kindly glanced through all the manuscripts and pointed out gross errors. Many smaller errors remain which I discovered later. Also in one or two cases the derivation is incomplete, I shall correct and complete them if and when I revise these lectures.

I am not giving any bibliography. For this, reference may be made to the following two excellent survey articles, the first of which I was able to avail of, but the second I could not, as it appeared too late.

1) A.A.Walters: "Production and Cost Functions: An Econometric survey" Econometrica 1963.

2) F.H.Hahn and R.C. Mathews: "The Theory of Economic Growth: "A Survey" The Economic Journal 1964

My thanks are due to Mrs.A.Habib for her kindly undertaking the typing of all the Masters and the appropriate insertions and deletions which were required quite often and also to Miss Faten Fouad for typing my hand written manuscripts. If some errors remain, they are due to my own negligence.

A.Qayum.

Dr. A. Qayum For Internal Use

MICRO-PHYSICAL PRODUCTION FUNCTION.

(I) Definition of production function, 1. (II) Production surface, plane sections, average and marginal productivity, product elasticity, 2. (III) Isoquants and isoclines, 5. (IV) Elasticity of substitution, 8. (V) Homogeneity of production functions, 11. (VI) Varying returns to factors and varying returns to scale, 14. (VII) Product transformation curve, 18. (VIII) A numerical illustration, 20.

I

Production function is a technically extremal concept. It implies that either the maximum amount of output is produced with the given amounts of inputs or the given amount of output is produced with the minimum amount of each of the inputs, the other inputs being given. Symbolically we can express a production function as

$$x = f(u, v, w, \ldots \ldots \ldots), \tag{1}$$

where x is the amount of output X and u, v, w, ... are the amounts of inputs U, V, W, ... etc. There are two crucial assumptions about the techniques and the inputs which play a great role in the development of the theory of production. One is the assumption that the techniques are continuously variable which means that the inputs can be combined in any proportion we desire in producing the output, i.e., f is a continuous function in u, v, w, ... In reality this is not always true. The techniques or processes of production are not continuously variable, that is, the inputs cannot be combined in any proportion desired, to produce a certain output. This means that in reality f is not continuous. It must, however, be stressed that there is a difference between the variability in the combination of the commodity inputs and variability in the combination of the basic factors of production, labour and capital. If we retrace the variability in combination of inputs in the production of final output to the variability in the combination of inputs in producing the inputs which will go to produce the final output and so backwards till we reach the stage when the original factors labour and capital are combined to produce inputs at the first stage, we can visualize easily that the variability in the combination of the basic factors of production, labour and capital is likely to be much greater than that in the combination of inputs in producing the final inputs. It is not suggested here that labour and capital are continuously variable or perfectly substitutable but it is maintained that the reality can be roughly approximated by assuming that a continuous variability exists in the combination of factors. Because of this and because of the fact that the bulk of the theory so far developed and almost all of the theory developed earlier is based on the assumption of continuous variability, we shall discuss the theory in a major part of this series of lectures on the assumption of continuous variability of factor combination. In the concluding lectures we shall discuss the case when the processes of production are not continuously variable, but discrete.

The second crucial assumption relating to the development of the theory is that the inputs are continuously divisible. This is again not a wholly realistic assumption. However, as regards the divisibility, the commodity inputs fare better than some of the factors such as built capital, entrepreneurship and organizational capacity. The assumption of divisibility as regards the commodity inputs is largely realistic, but very tough problems are faced in connection with the non-divisibility of the factors of production mentioned above. In the earlier discussions, we shall assume that the factors and inputs are continuously divisible, we shall take up the problem of non-divisibility and rigidity of factors in later lectures.

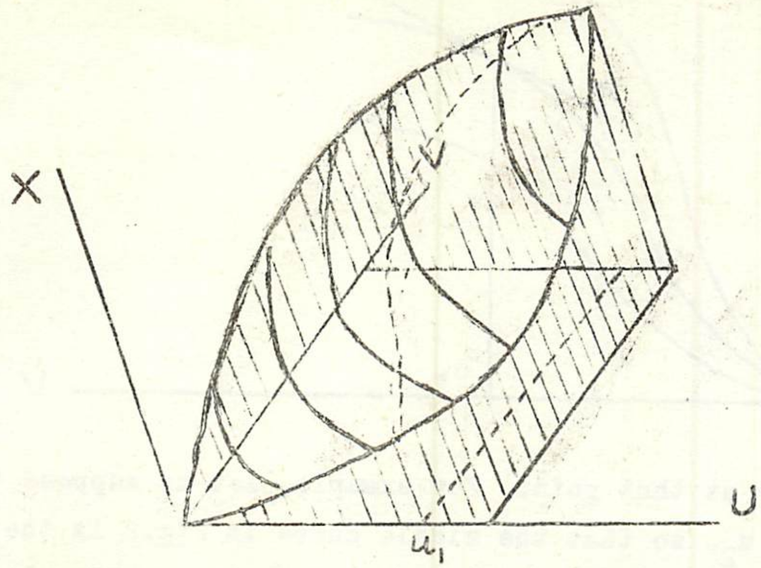
II

Our discussion will be much simplified if we include two factors U and V in the production function and this will also enable us to use graphic illustrations. Let the production function, then be

(2) $x = f(u, v)$

Plotting for different values of u and v taken along the axes OU and OV and for x taken along OX vertically, we can construct a production surface, as shown below.

Fig.1.



The production surface can assume numerous forms. For the sake of simplicity we suppose that it takes the three dimentional form as shown in Fig.1. By using the method of plane sections, we can derive several useful results from (2). If we keep one of the factors, say U, constant at u, we can study the variation of X with respect to the other factor V. This is shown by the vertical section of the production surface by a plane parpendicular at u_1, u_2, etc., the resulting curves are given by

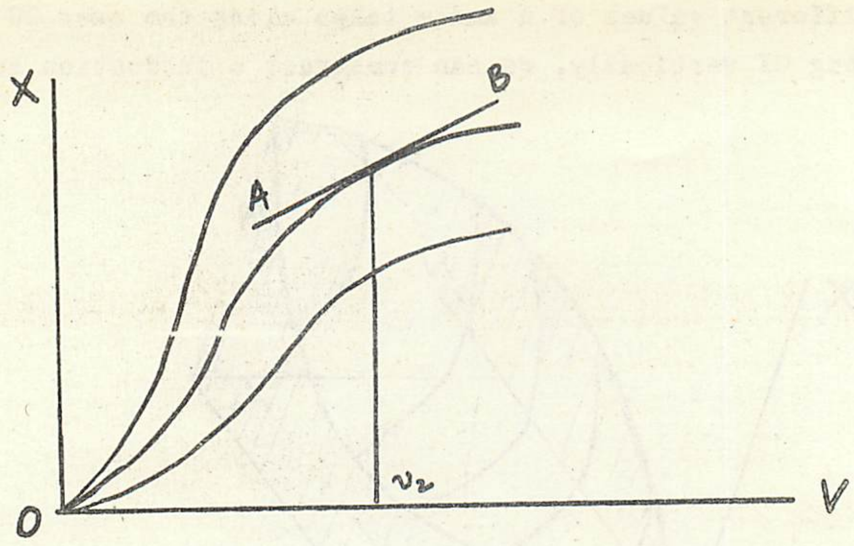
(3) $$x = f(u_1, v) = \emptyset_1(v)$$

(4) $$x = f(u_2, v) = \emptyset_2(v)$$

The curves for different values of u can be represented in the two dimensional plane v,x as in Fig.2. The curves in Fig.2 express output x in relation to variable v when u is held fixed in different quantities. Similarly we can hold v fixed in different quantities and demonstrate the variation of x in relation to u. In Fig.1, the dotted lines represent a vertical plane section at u_1 giving respectively the foot of the plane and the curve in which it cuts the surface. Curves like the dotted one in Fig.1 can be represented in two dimensional plane as done in Fig.2.

Slopes of the individual input-output curves shown in Fig.2 at points corresponding to different values of v give the physical marginal

Fig. 2.



productivity of v at that point. For example, let us suppose that the value of u is fixed at u_2, so that the middle curve in Fig. 2 is the relevant curve giving the variation in output x with the variable v. The slopes of tangents at different points of this curve give the marginal productivity of v at its corresponding values. In Fig. 2 the slope of AB gives the marginal productivity of v at its value equal to v_2. Similarly we can draw input-output curves relating to the variable u, with v held fixed at certain levels, by drawing a plane section perpendicular to the v axis at points corresponding to those levels. Symbolically the marginal productivity of v at v_2 when u is fixed at u_2 is

$$(5) \qquad \frac{\partial x}{\partial v} = \frac{d\phi_2(v)}{dv} = \phi'_2(v_2) \quad , \quad (u = u_2)$$

Similarly we can express the marginal productivity of u.

Another quantity which may be of some interest especially in analysis relating to product imputation, is production elasticity. Equations (3) or (4) express the variation of X with changes in V when U is fixed at u_1 or u_2. From these we can derive the elasticity of production with respect to v, the formula of elasticity of production is

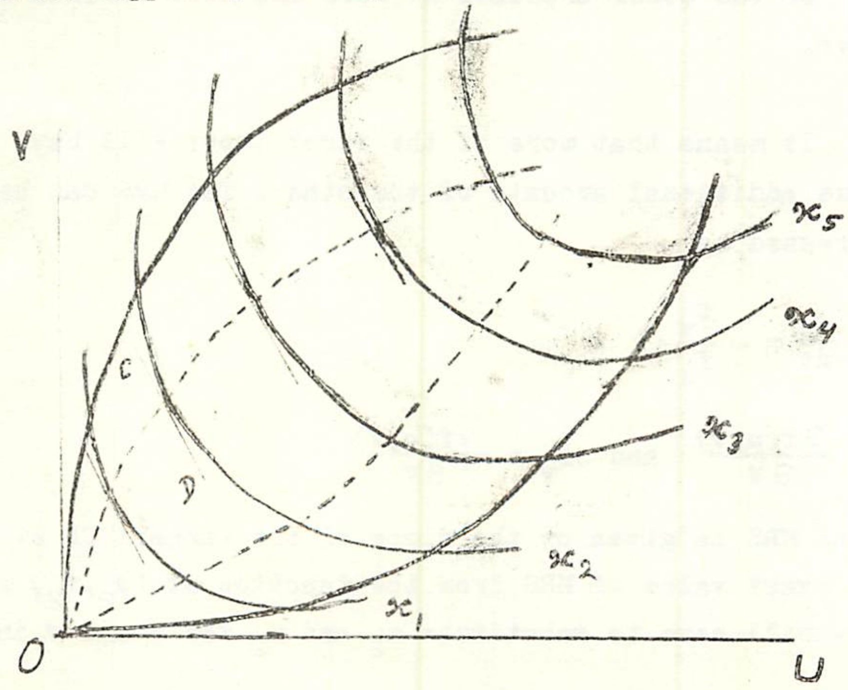
$$(6) \qquad e = \frac{dx}{x} \Big/ \frac{dv}{v} = \frac{v}{x} \cdot \frac{dx}{dv}$$

$e \gtrless 1$, according as increasing, constant or decreasing returns to factor V prevail in production. And similarly for factor U.

III

A second relationship of great importance is obtained through intersecting the production surface not by plane sections perpendicular to the coordinates representing the inputs but by a plane section perpendicular to the ordinate representing the output, i.e., to OX in Fig.1. The plane section will intersect the surface in a curve, each point of which will represent a combination of the two inputs yielding the same level of output, i.e. the level of output at which the perpendicular plane-section has been erected. By drawing plane-sections perpendicular to the vertical output axis at different level, we can have a set of curves, every point on each of which represents a different combination of inputs yielding the same level of output. In Fig.3 we draw the contours of these curves by projecting them on the horizontal surface mapped by the input coordinates.

Fig.3.



In Fig.3 we have drawn only 5 of the infinite number of the contour lines corresponding to each point on OX in Fig.1. As each point on a curve represents the same level or amount of output, they are called isoquants or product isoquants. For example, each point on curve x_2 represents different combinations of input u and v, but gives the same level of output x_2. Symbolically, the curve x_2 can be expressed by substituting x_2 for x in (2), i.e.

(7) $$x_2 = f(u,v)$$

Similarly we can express the function for other isoquants.

The slope of an isoquant at a certain point indicates the marginal rate of substitution (or MRS for short) of one input for the other, if the output is to be maintained at the specified level. In normal cases, when the isoquants are curved as they are shown in Fig.3, the rate of substitution of one factor for the other declines as more and more of this factor is replaced by the other.

It means that more of the first input will have to be given away for the same additional amounts of the other. The MRS can be derived from (3) and is expressed as

(8) $$\frac{du}{dv} = -\frac{f_v}{f_u}$$

where $f_u = \frac{\partial f(u,v)}{\partial u}$ and $f_v = \frac{\partial f(u,v)}{\partial v}$

In Fig.3 the MRS is given by the slope of the tangent CD at point (u_2,v_2). To get the exact value of MRS from the function at (u_2,v_2) relating to isoquant x, we will have to substitute u_2 and v_2 for u and v in (8).

We can use another type of plane sections of the production surface with advantage. These planes may be taken through the vertical output axis OX at different angles between OU and OV relating to different combinations of U and V. For instance, if a plane is taken through OX at an angle 30°

from OU, amd so 60° from OV, the ridge line that we get from the intersection of this plane and the production surface gives the growth of output when U and V are combined in proportion 2 to 1 at each magnitude of production.

Thus, if a point at the foot of the plane is (u_1, v_1), then any other point at the foot of this plane through OX can be expressed as $(\lambda u_1, \lambda v_1)$ and the production function corresponding to these points giving a proportionate change in inputs can be written as

$$(9) \qquad x = f(\lambda u_1, \lambda v_1) = \Theta(\lambda)$$

(9) is a function of the variable λ only. We can get different functions like (9) by taking planes through OX and passing through points other than (u_1, v_1).

As indicated above, the slope of an individual isoquant varies from one point to the other. If we join the points on each of the isoquants at which the respective isoquants have the same slope, we will get a line wich passes through all the successively higher isoquants of the set at points where they have the same slope or they are equally 'inclined'. These lines may be called isoclines. They are shown in Fig.3 by the dotted lines. These isoclines also connect the points on the production surface where the marginal rates of substitution between the factors are equal, hence they trace (and also they are called) expansion paths. That is, they show the path of expansion on which the economy will move, if the factors at the margin are combined in such a way that their rates of substitution remain unaltered. In a competitive equilibrium, when the ratio of factor prices tends to equal their marginal rate of substitution, this means that if the prices of factors are given and remain constant, then the production in the economy will expand along the corresponding isocline or the expansion path.[1] The isoclines can be expressed by

---

1) When the prices of output and, factors are given, the cost can be mimised if $\frac{dC}{dx} = \frac{d(p_u u + p_v v + b)}{dx} = 0$ or $\frac{p_u}{p_v} = -\frac{d_v}{d_u}$.

(10) $$\frac{du}{dv} = k'$$

where k' is any constant, representing the rate of substitution between the factors.

A word may also be added to the range of the isoquants that are relevant to economic analysis. In normal circumstances, product can only be maintained if less of one input is used by increasing the use of the other. Alternatively, as the substitution of one input for the other proceeds, increasingly larger additions of the first input are needed to compensate for a given reduction of the other. In other words only that portion of an isoquant is interesting and relevant, along which the isoquant is convex to the origin.[1] In Fig.3 we have shown the range of the isoquants which are convex to the origin by thick lines. At all points between these two lines, each isoquant satisfies the following:

(11) $$\frac{du}{dv} < 0 \qquad \frac{d^2u}{dv^2} > 0 .$$

IV

We have already stated that the MRS measures the rate at which one resource, say U, is substituted for the other, viz. V, in the production of output X from the given combination of resources at an isoquant. The positive value of MRS can be denoted by r, so that

(12) $$r = -\frac{du}{dv} = \frac{f_v}{f_u}$$

Now one more interesting point to determine is how fast r changes in relation to the rate of change in the ratio of the resources at an isoquant. This will

1) It is just possible that the isoquants are convex to the origin all along their course.

give us what is called the elasticity of rate of substitution and can be denoted by σ, so that

(13) $$\sigma = \frac{d\frac{u}{v}}{\frac{u}{v}} \div \frac{dr}{r}$$

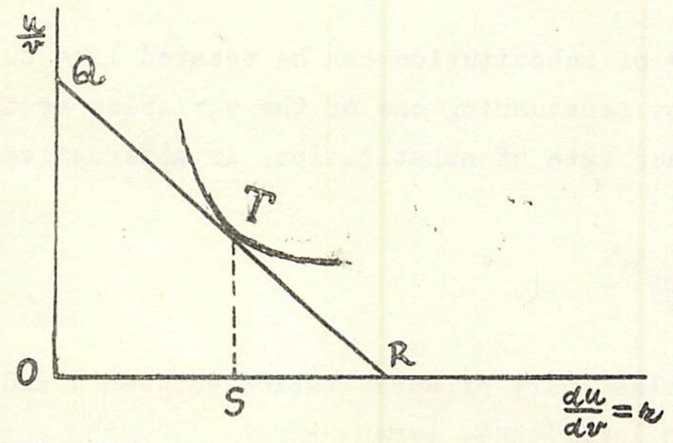
The value of σ can be expressed in an alternative form.[1)]

(14) $$\sigma = \frac{f_u f_v (uf_u + vf_v)}{-uv(f_{uu} f_v^2 - 2f_{uv} f_u f_v + f_{vv} f_u^2)}$$

(14) shows that the elasticity of substitution is symmetrical for the two resources. This can be readily seen because interchanging u and v in the right hand side of (14) does not change the expression. In other words, whether we consider substitution of U for V or V for U, we get the same value of the elasticity of substitution.

In diagrammatic terms if P is any point on the isoquant x_2, in Fig.3, then σ is the ratio of the relative change in the gradient of OP to the relative change in the gradient of the tangent at P to the isoquant, for a small movement of P along it. Alternatively we can illustrate σ by the following figure

Fig.4.



1) R.G.D. Allen, Mathematical Analysis for Economists (London 1960) p. 342. (14) can be derived immediately by substituting for $d\frac{u}{v}$ and dr in (13).
$d\frac{u}{v} = \frac{vdu - udv}{v^2}$ and $dr = \frac{\partial r}{\partial u} du + \frac{\partial r}{\partial v} dv$. But $du = -\frac{f_v}{f_u} dv = rdv$
$d\frac{u}{v} = -\frac{vr + u}{v^2} dv$ and $dr = - r(\frac{\partial r}{\partial u} - \frac{\partial r}{\partial v}) dv$.

In Fig.4 the ordinate represents the ratio of factors and the abscissa the rate of substitution between them. The curve represents the relationship between these two variables. At point T, the slope of the tangent QTR gives the marginal rate of change of $\frac{u}{v}$, the ratio of factors, with respect to $r = \frac{du}{dv}$, the marginal rate of substitution between them, so that

$$\frac{OQ}{OR} = \frac{d\frac{u}{v}}{dr}$$

Further we have $$\frac{TS}{OS} = \frac{\frac{u}{v}}{\frac{du}{dv}}$$

But $$\frac{TS}{OQ} = \frac{TR}{QR} \quad \text{and} \quad \frac{OS}{OR} = \frac{QT}{QR}$$

$$\frac{TS}{OS} = \frac{TR}{QR}\, OQ \times \frac{QR}{OR \cdot QT} = \frac{TR \cdot OQ}{OR \cdot QT}$$

$$\therefore \sigma = \frac{d\frac{u}{v}}{dr} \cdot \frac{r}{\frac{u}{v}}$$

$$= \frac{OQ}{OR} \cdot \frac{OS}{TS} = \frac{QT}{TR}$$

Thus the elasticity of substitution can be treated like the elasticity of any two variables by considering one of the variables as the ratio of the factors and the other rate of substitution. An alternative expression for σ may be

$$\frac{u}{v} = \left(\frac{du}{dv}\right)^{\sigma}$$

where σ gives the elasticity of substitution between U and V. Or we can express the same in logarithmic terms

$$\frac{d \log \frac{u}{v}}{d \log \frac{du}{dv}} = \sigma$$

which is the same as (13).