UNITED ARAB REPUBLIC

THE INSTITUTE OF NATIONAL PLANNING

التخطيط القومى

Memo. No. 397

ITERATIVE PRICE AND QUANTITY DETERMINATION FOR SHORT-RUN PRODUCTION AND FOREIGN TRADE PLANNING

by

Tom Kronsjö

10, February 1964

CAIRO
3, MOHAMED MAZHAR - ZAMALEK

> Today, more than ever before in the history of science, theoretical formulation goes hand in hand with computational feasibility.
>
> R. Bellman and S. Dreyfus

Iterative Price and Quantity Determination for Short-run Production and Foreign Trade Planning

By
Tom Kronsjö
Institute of National Planning
Operations Research Center
Cairo - Egypt.

Economic interrelations of importance for short-run production and foreign trade planning may, as a first approximation [1], be described by a very large linear programming model. The great size of this model necessitates, if it should become managable for practical use, special analysis of its equational structure and exploitation of its special features. This study will be centered on the utilization of the structural properties of the equations describing

---

1) The author wishes to express his sincere gratitude to Dr. Alfred Zauberman, London School of Economics and Political Science, for stimulating discussions and proposals in regard to the elaboration of this mathematical appendix and to Dr. Salah Hamid, Director of the Operations Research Center of the Institute of National Planning, Cairo, for enabling the undertaking of this study as part of the center's research activities for the preparation of the Egyptian Five Year Plan.

the relationships of foreign trade and production variables.

This paper may be seen as a continuation of the discussion by W. Trzeciakowski, J. Mycielski, K. Rey, J. Głowacki, W. Piaszczyński in Poland, by A. Nagy, T. Liptak, A. Marton, M. Tardos in Hungary, by V. Pugachev, V. Volkonskij, Yu. Chernyak, A. Modin in the Soviet Union and by P. Pigot in France as well as of earlier contributions by the author. (Cf. par. 15 and literature references at the end.)

Unfortunately, an older generation of economists, both in the East and in the West, seems to have difficulty in understanding the importance of the challenges of these problems, and that mathematical and computational analyses will become as important to the economist in the future as it is today to the mechanist or the physicist.

A Survey of the Model Analyzed

The table at the end of this mathematical appendix may be unfolded during the reading.

1. Variables

Quantity variables are denoted as follows:

Production levels of various industrial branches (i) with internal procerses (j) are denoted by the vectors x_i (with elements x_{ij}). x_i (i=0,1, ,m)

Export and import variables are denoted by the vectors y_i, each of which embraces certain commodity numbers and the relevant markets for the commodities in question(thus similarly with elements y_{ij}) y_i $(i=0,1, ,m)$

<u>Price variables</u> are denoted as follows:

The feasible prices of the various foreign currency resources are denoted by the vector v (with elements j) v

The feasible prices of the various commodities used or produced by the industrial branches (i) are denoted by the vectors u_i (with elements u_{ij}) u_i $(i=0,1, ,m)$

The feasible prices of the production capacity vectors (cf. $-\bar{x}_i$ in par. 3 below) are denoted by k (with elements k_{ij}) k_i $(i=0,1, ,m)$

The feasible prices of the export and import constraint vectors (cf. $-\bar{y}_i$ in par. 3 below) are denoted by h_i(with elements h_{ij}) h_i $(j=0,1, ,m)$

<u>The iteration</u> is denoted by the variables r,s,t. Subiterations (s) within an iteration (r) by rs, etc.

Auxiliary variables in master (i.e. coordinating) problems are denoted by z. Though the same name (z) is used in various masters they are not identical. z_r, z_{irs}, z_t etc.

2. Equations or Inequality Constraints

The balance of payments constraints

(1) $C_o y_o + \ldots + C_i y_i + \ldots + C_m y_m = bp$

where $C_i (i=0,1, \ ,m)$ are matrices of the foreign prices obtained or paid for export or import quantities to various markets (and possibly including certain conditions for the commodity structure as determined by trade agreements).

bp is the vector of net requirement of foreign currency holdings (and possible of the trade composition). Its elements are bp_j.

The commodity balances which state that import - export + production - use in production should equal the requirement vectors b_i with elements b_{ij}:

$$(2) \quad B_{oo} y_o \quad + A_{o1} x_1 + \ldots + A_{oi} x_i + \ldots + A_{om} x_m + A_{oo} x_o = b_o$$

$$B_{11} y_1 \quad + A_{11} x_1 \quad + A_{1o} x_o = b_1$$

.

$$B_{ii}y_i \qquad\qquad +A_{ii}x_i \qquad\qquad + A_{io}x_o=b_i$$

$$B_{mm}y_m \qquad\qquad + A_{mm}x_m \quad + A_{mo}x_o=b_m$$

The important assumption has been made here that the production structure may be characterized by the following four features:

i) Certain commodities, e.g. labour, electricity and water, are inputs to or outputs from all branches of production, as depicted by the first row in (2) of the o-group of equations.

ii) Certain production processes require inputs from almost all branches of production, as may be the case for the chemical industry. Such production processes are grouped together in the penultimate column of the x_o activities.

iii) Except for the common input or output commodities defined above and those processes which use inputs from almost all industrial branches, the industries are supposed to be groupable in branches which only use or produce mutually exclusive groups of commodities, e.g. the textile industries producing only commodities belonging to a "textile" commodity group, the mechanical industry only those belonging to a "mechanical" group as depicted by the matrices A_{ii}.

iv) Special constraints on the export and import variables such as balance of payments (and possibly on the balancing of certain commodities as determined by trade agreements) are supposed to be included <u>in the matrices C_i</u>.

3. Bounds

The <u>production level</u> vectors have to be within the capacity <u>bounds</u>

(1) $0 \leq x_i \leq \bar{\bar{x}}_i$ (i=0,1, ,m)

Similarly, the <u>export and import</u> vectors have to be within their corresponding marketing <u>bounds</u>

(2) $0 \leq y_i \leq \bar{\bar{y}}_i$ (i=0,1, ,m)

We wish to state all the conditions of the original problem in the form of $\geq$ or $=$ as we will then obtain all feasible price solutions to the corresponding dual as non-negative magnitudes. We therefore multiply the right hand part of the above conditions by (-1) and get

(3) $- x_i \geq -\bar{\bar{x}}_i$ (i=0,1, ,m)

and

(4) $- y_i \geq -\bar{\bar{y}}_i$ (i=0,1, ,m)

4. Preference Function

The <u>preference function</u> is formally defined by the expression

(1) Min $g_o y_o + \ldots + g_i y_i + \ldots + g_m y_m + f_1 x_1 + \ldots + f_i x_i + \ldots f_m x_m + f_o x_o$

No discussion is made on the actual coefficients.

5. Summary of the Model

$$g_o y_o + g_1 y_1 + \ldots + g_i y_i + \ldots + g_m y_m + f_1 x_1 + \ldots + f_i x_i + \ldots + f_m x_m + f_o x_o = \text{Min}$$

$$C_o y_o + C_1 y_1 + \ldots + C_i y_i + \ldots + C_m y_m = bp$$

$$B_{oo} y_o + A_{o1} x_1 + \ldots + A_{oi} x_i + \ldots + A_{om} x_m + A_{oo} x_o = b_o$$

$$B_{11} y_1 + A_{11} x_1 + A_{1o} x_o = b_1$$

$$B_{ii} y_i + A_{ii} x_i + A_{io} x_o = b_i$$

$$B_{mm} y_m + A_{mm} x_m + A_{mo} x_o = b_m$$

$$-y_o \geqq -\bar{y}_o$$

$$-y_1 \geqq -\bar{y}_1$$

$$-y_i \geqq -\bar{y}_i$$

$$-y_m \geqq -\bar{y}_m$$

$$-x_1 \geqq -\bar{x}_1$$

$$-x_i \geqq -\bar{x}_i$$

$$-x_m \geqq -\bar{x}_m$$

$$-x_o \geqq -\bar{x}_o$$

$x_i \geqq 0$, $y_i \geqq 0$ $(i = 0,1, \; ,m)$

6. The Dual Formulation

Instead of considering the original formulation, it will be useful at various calculation stages to deal with the dual:

$$v + b_o' u_o + b_1' u_1 + .. + b_i' u_i + .. + b_m' u_m - \bar{y}_o' h_o - \bar{y}_1' h_1 - .. - \bar{y}_i' h_i - .. - \bar{y}_m' h_m - \bar{x}_1' k_1 - .. - \bar{x}_i' k_i - .. - \bar{x}_m' k_m - \bar{x}_o' k_o = \text{Max}$$

$$A_{oo}' u_o + A_{1o}' u_1 + .. + A_{io}' u_i + .. + A_{mo}' u_m - k_o \leq f_o'$$

$$A_{om}' u_o + A_{mm}' u_m - k_m \leq f_m'$$

$$A_{oi}' u_o + A_{ii}' u_i - k_i \leq f_i'$$

$$A_{o1}' u_o + A_{11}' u_1 - k_1 \leq f_1'$$

$$v + B_{mm}' u_m - h_m \leq g_m'$$

$$v + B_{ii}' u_i - h_i \leq g_i'$$

$$v + B_{11}' u_1 - h_1 \leq g_1'$$

$$v + B_{oo}' u_o - h_o \leq g_o'$$

$$v \geq 0$$

$$u_i \geq 0 \, , \; h_i \geq 0 , \; k_i \geq 0 \qquad (i = 0, 1, \; , m)$$

(' denotes transposition of a vector or a matrix)

7. Parameters of Action

will in the following be both the quantity variables x_i, y_i $(i=0,1, ,m)$ and the price variables v , u_i, h_i and k_i $(i=0,1, ,m)$, as well as the auxiliary variables z_r, z_{irs}, etc.

8. A Graphic Picture

of the equation system, the variables, constants, prices and preference coefficients is given in Table 4 at page 56.

The pluses and minuses denote +1 and -1, respectively. The reformulated bounds (cf. 3.3, 3.4) are found in the lower half of the table.

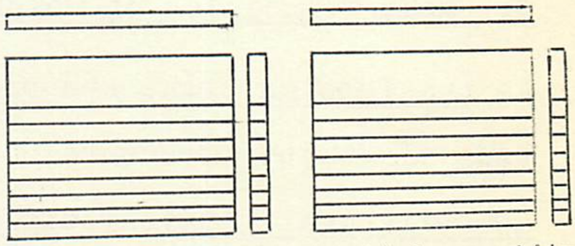
9. Main Principles of Solution

As the foreign trade and production variables are subject to very different structural constraints, they should be treated as being qualitatively different, and different methods of calculation should consequently be employed in solving them. This will be an important theme of the following exposition. Another, will be that of breaking the problem into smaller more rapidly solved subproblems, the solution of which are coordinated at various levels.

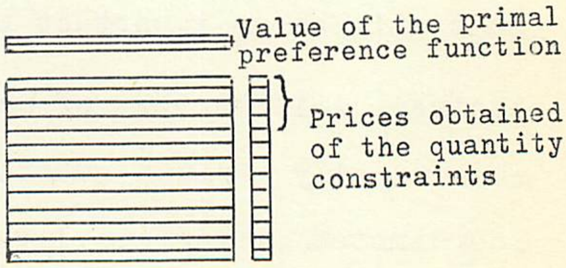
10. A General Survey of the Iterative Procedure

An attempt will now be made to give a birds-eye view of the general course of the solution process. The linear programme will be symbolized by a large rectangle together with two narrow ones. If some quantity solution is inserted and leads to the fulfilment of some equations the satisfied equations are indicated by horizontal lines. If some price solution to the dual problem is inserted, the satisfied price equations (columns) of the dual are indicated by vertical lines.

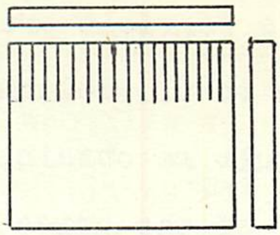
1. Various known quantity solutions satisfying part of the equations (shown by lines in the figures) are combined



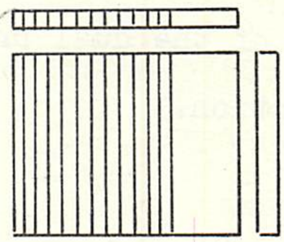
2. to a quantity solution satisfying all quantity equation (rows). As a result prices of the earlier unfulfilled equations are determined as is also an estimate of the minimum of the primal preference function.



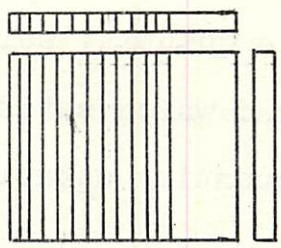
3. The feasible prices obtained are inserted in the dual problem



4. and a feasible price solution of the left columns is obtained. (The price equations of the right columns have been temporarily disregarded as they are difficult to satisfy).

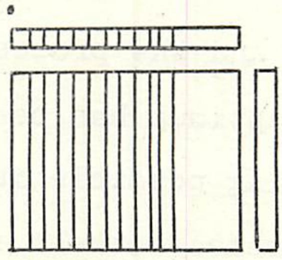


5. This feasible price solution to the left columns is combined with earlier known price solutions which likewise satisfy only the left columns



Partial price solution 1

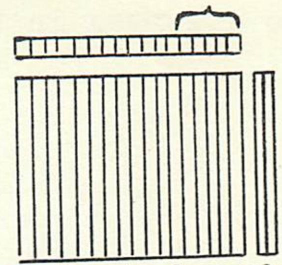
.
.
.



Partial price solution r

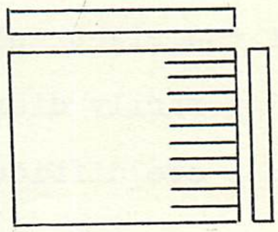
6. to give a feasible price solution to all columns. As a result we obtain the "prices" of the price equations, i.e., quantities as well as an estimate of the maximum of the dual preference function.

Quantities of price constraints obtained

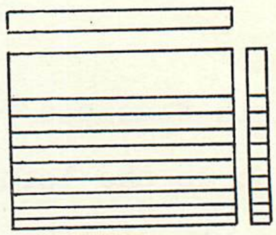


Value of the dual preference function

7. These quantity solutions are inserted in the right part of the quantity equations and



8. feasible quantity solutions are found which satisfy the lower quantity equations.



Partial quantity solution r+1

RETURN TO 1. The partial quantity solution is combined with the earlier known ones and the process repeated until the optimum has been found or the remaining possible improvement i.e. the difference between the minimum (cf. point 2.) and the maximum estimates (cf. point 6.) is less than a certain tolerance.

11. The Favourable Properties of Some Subproblems Involving B_{ii} Matrices

The way in which we will solve the general problem will partly be based upon exploiting the favourable structure of the B_{ii} matrices. We will mainly have to deal with two different types of subproblems involving these matrices.

The first of these we may name

11.A. The Export and Import Quantity Problem

which will be of the type

$$(1) \qquad \text{Min} \quad gy$$
$$Cy = bp$$
$$By = b$$
$$-y \geqq -\bar{y}$$
$$y \geqq 0$$

In dealing with a problem of this type, we introduce the indexes c for commodity, d for incompletely convertible currency block or district and a for trade activity, i.e. either export (E) or import (I).