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ON THE REDUCTION OF LP PROBLEM

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ABSTRACT

LP problems reduction prior to optimal solutions finding and during computational procedure is discussed in this paper. So, the notions of redundant constraints /§§ 2,3/, conditionally redundant constraints and conditionally redundant variables /§4/ are introduced.

Redundant constraints are defined as these not changing feasible solutions set $\mathcal{X}$ when removed from its definition. The constraints of the second type are those, which are not active in optimal solutions set $\mathcal{X}_0$ of LP problem.

By conditionally redundant variables we understand these having zero value in optimal solution. Removal of the redundant variables and constraints from further considerations does not effect optimal solution choice, but frequently leads to a significant reduction of LP problem size.

In §§ 2,3, sufficient conditions for redundant constraints and in §4 for conditionally redundant constraints and variables detection are discussed.

Two methods of LP problems reduction presented in [7], [9] are analysed as well.

1. INTRODUCTION [1]

Speaking about the LP problems reduction we have on mind the diminution of problem size prior to numerical procedure leading to optimal solution, if one exists. By the size of LP problem we understand the number of significant constrai /i.e. all constraints different from nonnegativity conditi defining feasible solutions set $\mathcal{X}$ and the number of deci variables in this problem.

It is well known that too big number of the constraints de set $\mathcal{X}$ is inconvenient when looking for optimal solution because it lengthens computing time and effects accuracy the results. Given an objective function and a type of optimization /maximization or minimization/, only some con traints defining the set $\mathcal{X}$, define also optimal solutio set $\mathcal{X}_o$ $(\mathcal{X}_o \subset \mathcal{X})$. Neglecting LP constraints /or some of th not defining $\mathcal{X}_o$ set can reasonably shorten the computati work leading to $\mathcal{X}_o$ determination.

The special place among all constraints defining set $\mathcal{X}$ occupied by these which do not influence $\mathcal{X}$, while removed

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[1] This report is, in a large part, based on papers preser by the author and others at SGPiS seminar on "Numerical thods of large scale optimization models", chaired by W.Grabowski. The seminar was sponsored by the Institut Planning at the Council of Ministers, and its results published in /6/. Some unpublished theorems and notion troduced by dr W. Dubnicki are used in this report as

the problem. Such constraints we call <u>redundant.</u> We shall introduce the notion of <u>conditionally redundant constraints</u> i.e. the constraints of $\mathfrak{X}$ not determining the set $\mathfrak{X}_o$.

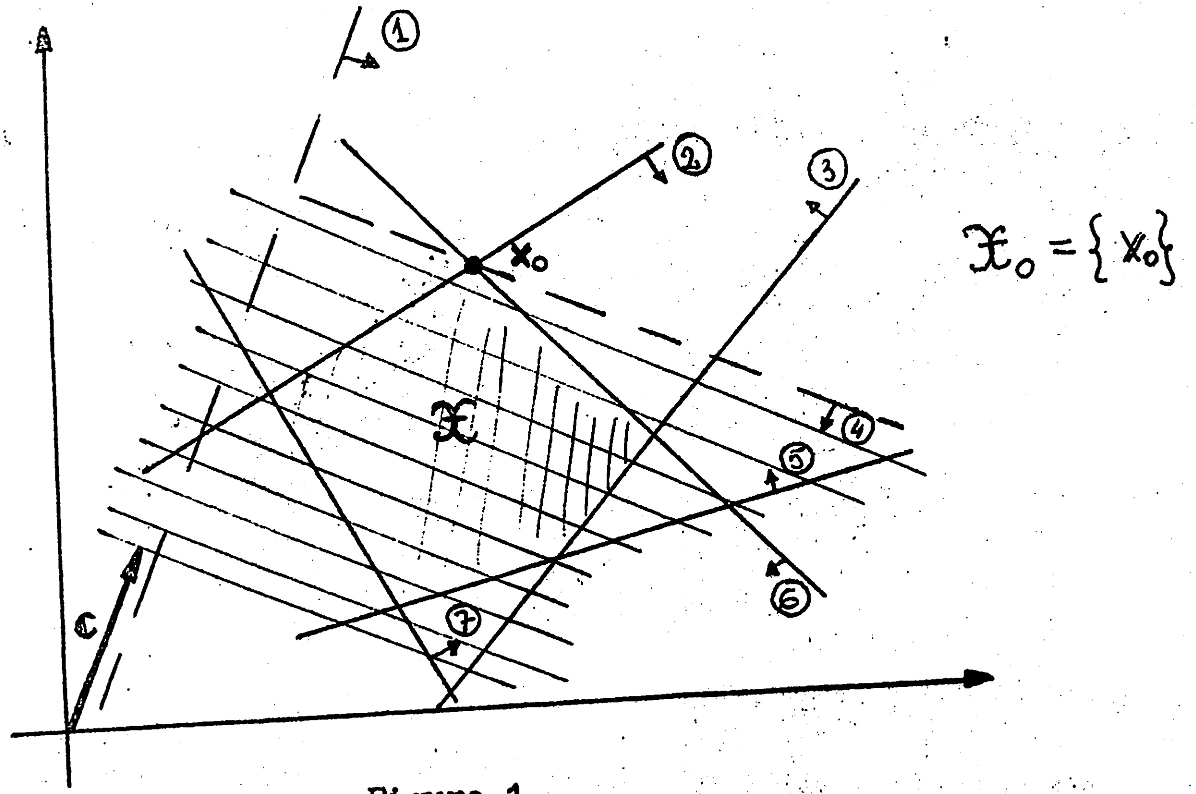

Figure 1

Fig. 1 presents set $\mathfrak{X}$ of some LP problem, where objective function is maximized. Constraints of this problem, numbered 1 and 4 are redundant. Conditionally redundant are constraints numbered 1, 3, 5, 7.

Let's assume that LP problem will be solved by some specific type of simplex method. In this method nonnegativity conditions are automatically satisfied, so they are not introduced into computer storage, and they do not increase the size of LP problem.

Considering redundant or conditionally redundant constraints, we shall take into account all but nonnegativity conditions. In some cases nonnegativity conditions can play important role in the determination of $\mathcal{X}$ or $\mathcal{X}_0$, therefore it is reflected in the notion of redundant or conditionally redundant variable.

2. Redundant inequalities

A. Our further considerations will deal with the following LP problem , formulated here in canonical form:

$$cx \longrightarrow \max \qquad /1/$$

$$\text{s.t.} \quad a_i x \leq b_i \quad (i=1,\dots,m) \qquad /2/$$

$$x \geq 0, \qquad /3/$$

where x, c, a_i are vectors in R^n .

Canonical formulation of LP problem is equivalent to standard and mixed forms, what results from the equivalence of transformations:

$$a_i x \leq b_i \equiv \begin{pmatrix} a_i x + x_{n+i} = b_i \\ x_{n+i} \geq 0 \end{pmatrix} \qquad /4a/$$

and

$$a_i x = b_i \equiv \begin{pmatrix} a_i x \leq b_i \\ -a_i x \leq -b_i \end{pmatrix} \qquad /4b/$$

Writing /1/ - /3/ form, we want to stress that in initial formulation of LP problem, all constraints different from /3/ were written as inequalities. It will be shown further, that differ rules govern redundancy determination in case of inequalities, than in case of equations. The following notation results from /1/ - /3/ formulation:

$$R_+^n = \{x \mid x \in R^n, x \geqslant 0\},\ J = \{1,2,\dots,m\},\ J_{/k} = J - \{k\},$$

$$\mathcal{X} = \{x \mid x \in R_+^n,\ a_i x \leqslant b_i \quad (i \in J)\},$$

$$\mathcal{X}_{/k} = \{x \mid x \in R_+^n,\ a_i x \leqslant b_i \quad (i \in J_{/k})\},$$

$$S_k = \{x \mid x \in R_+^n,\ a_k x \leqslant b_k\} \text{ and } k \in J,$$

$$T_k = \{x \mid x \in R_+^n,\ x \in \mathcal{X}_{/k} - S_k\},$$

From the above definitions it follows immediately:

$$\mathcal{X} \subset \mathcal{X}_{/k} \text{ and } \mathcal{X} \subset S_k. \qquad /5/$$

Definition 1

Constraint "k" of system /2/ is redundant in $\mathcal{X}$ iff

$$\mathcal{X}_{/k} \subset S_k.$$

From the def.1 and relation /5/ the following holds:

Theorem 1

The following relations:

/i/ $\mathcal{X}_{/k} \subset S_k$,

/ii/ $\mathcal{X}_{/k} \subset \mathcal{X}$,

/iii/ $\mathcal{X}_{/k} = \mathcal{X}$,

can be used in the redundancy definition of "k" constraint, because they are mutually equivalent.

In the definition of the set $\mathcal{X}$, it is convenient to introduce the notion opposite to redundant constraint, i.e. essential constraint. We assume that the set $\mathcal{X}$ will be changed by the removal of a essential constraint.

Definition 2

The constraint "k" in system /2/ is essential in $\mathcal{X}$, iff

$$\mathcal{X}_{/k} \not\subset S_k .$$

From the definition 1, theorem 1 and definition 2 we get:

Theorem 2

The following relations:

/i/ $\mathcal{X}_{/k} \not\subset S_k$,

/ii/ $\mathcal{X}_{/k} \neq \mathcal{X}$,

/iii/ $\mathcal{J}_k \neq \emptyset$,

can be used in the number "k" essential constraint definition, because they are mutually equivalent.

From the numerical point of view, the following theorems are more important than theorem 1 and 2[2/].

Theorem 3.

"k" constraint in system /2/ is essential, iff

$$\mathcal{J}_k(t) = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \;\middle|\; \begin{array}{l} a_i x \leqslant t b_i \quad i \in J_{/k} \\ a_k x > t b_k \\ x \geqslant 0,\ t \geqslant 0 \end{array} \right\} \neq \emptyset$$

[2/] Theorem 3, as well as necessary condition in theorem 4, were formulated and proved by W.Dubnicki at SGPiS seminar and they will be published. Proof of the sufficient condition in theorem 4 is given by the author, as a modification of this condition, as presented in /4/.

Theorem 4

The constraint "k" in system /2/ is redundant in $\mathcal{X}$, iff:

$$V_{/k} = \left\{ v=[v_i]_{i\in J_{/k}} \;\middle|\; \begin{array}{l} a_k \leq \sum\limits_{i\in J_{/k}} v_i a_i \\ b_k \geqslant \sum\limits_{i\in J_{/k}} v_i b_i \\ \quad v \geq 0 \end{array} \right\} \neq \emptyset$$

Having in mind the numerical importance of the sufficient conditions in theorem 4, here we present its proof.

Let's assume that $V_{/k} \neq \emptyset$. For any $x \in \mathcal{X}_{/k}$ and $v \in V_{/k}$ we get

$1^\circ \quad a_k x \leq \left(\sum\limits_{i\in J_{/k}} v_i a_i\right) x$

$2^\circ \quad \sum\limits_{i\in J_{/k}} v_i (a_i x) \leq \sum\limits_{i\in J_{/k}} v_i b_i$

$3^\circ \quad \sum\limits_{i\in J_{/k}} v_i b_i \leq b_k$

From 1°, 2°, 3° we have $(x\in\mathcal{X}_{/k} \Rightarrow x \in S_k) \Rightarrow \mathcal{X}_{/k} \subset S_k$), so the constraint "k" is redundant in $\mathcal{X}$. The necessary (and sufficient) condition in theorem 4 describes a rule of practical determination constraint "k" redundancy in $\mathcal{X}$.

Corollary 1

If the following system of inequalities has a solution

$$\left\{ \begin{array}{l} \sum\limits_{i\in J_{/k}} a_i v_i \geqslant a_k \iff \left(\sum\limits_{i\in J_{/k}} a_{ij} v_i \geqslant a_{kj} \quad (j=1,\dots,n) \right) \\ \sum\limits_{i\in J_{/k}} b_i v_i \leq b_k \\ \qquad\qquad v_i \geqslant 0 \quad (i \in J_{/k}) \end{array} \right. \qquad /6/$$

then, constraint "k" is redundant in $\mathcal{X}$.

Solution of the above system of linear inequalities for v_i can be reached by solving of the LP problem

$$z \longrightarrow \max \qquad /7/$$

s.t.

$$\begin{cases} \sum_{i\in J_{/k}} a_i v_i \geqslant a_k \\ \sum_{i\in J_{/k}} b_i v_i + z = b_k \end{cases} \qquad /8/$$

$$v_i \geqslant 0 \quad (i \in J_{/k}).$$

If in the optimal solution $[\bar{v}^T \bar{z}]$ of /7/ - /8/ we have $\bar{z} \geqslant 0$, so $\bar{v} \in V_{/k}$, what indicates the redundancy of "k" constraint. The problem /7/ - /8/ is solved until basic solution satisfying

$$[v^T, z] \geqslant [0^T, 0]$$

is met.

If in the optimal solution of /7/ - /8/ $\bar{z} < 0$, than according to a necessary condition of theorem 4, the constraint "k" is not redundant.

B. The set $V_{/k}$ is defined by n + 1 inequalities. Taking into account the system /2/, generating the problem /7/-/8/, usually it is true that $n > m$. Hence, m constraints in system /2/ created numerical difficulties, there is no usefulness in applying problem /7/ - /8/.

Fortunately the situation is not so bad. Considering a solution of system /6/ it is possible to assume that some $v_i = 0$, what practically means a possibility to reduce the number of inequalities in system /2/, searched for a redundant constraint.

It is easy to prove,

Theorem 5

If a linear inequality is redundant in relation to a given set of m linear inequalities, it is also redundant in relation to this set expanded by p additional inequalities.

Utilizing theorem 5 we present some modifications of corollary 1, resulting in an efficient numerical procedure of a redundancy determination, applied in relation to the whole or partial system /2/. Let $J(k) \subset J_{/k}$, and r be a quantity of subset $J(k)$ ($r = \operatorname{card} J(k)$).

Theorem 4 - I

If for some $J(k)$ the following inequalities hold

$$\begin{cases} a_{kj} \leqslant \frac{1}{r} \sum\limits_{i \in J(k)} a_{ij} & (j=1,\dots,n) \\ b_k \geqslant \frac{1}{r} \sum\limits_{i \in J(k)} b_i , \end{cases} \qquad /9/$$

then constraint "k" from system /2/ is redundant in $\mathcal{X}$.

To prove the theorem 4-I it is enough to see, that it is a special case of corollary 1, where $v \in V_{/k}$ is a vector with coordinates

$$v_i = \begin{cases} \frac{1}{r} & \text{for } i \in J(k) \\ 0 & \text{for } i \in J_{/k} - J(k). \end{cases}$$

Theorem 4-II

If for some $J(k)$ the following inequalities hold

$$\begin{cases} a_{kj} \leqslant \sum\limits_{i \in J(k)} a_{ij} & (j=1,\dots,n) \\ b_k \geqslant \sum\limits_{i \in J(k)} b_i . \end{cases} \qquad /10/$$

then constraint "k" from system /2/ is redundant in $\mathcal{X}$.

Justification of theorem 4-II is analogous to theorem 4-I. As the components of vector v , one takes

$$v_i = \begin{cases} 1 & \text{for } i \in J(k) \\ 0 & \text{for } i \in J_{/k} - J(k) \end{cases}$$

It is easy to check inequalities /9/ and /10/ and they can be used for large systems of inequalities. For theorem 4-I we use arithmetical means. It is convenient if coefficients for particular variables and RHS[*] in constraints "i" $\in J(k)$ are of the same order as those in constraint "k". Application of theorem 4-II is proposed when significant disproportions between constraint "k" coefficients and reference constraints $i \in J(k)$ are observed.

The significant property of the set $V_{/k}$ definition is a sign discretion of coefficients appearing in its inequalities, i.e. in inequalities /9/ and /10/. For positive values b_i of RHS, system /2/ can be written in a corresponding form

$$\bar{a}_i x \leq 1 \qquad (i \in J) \qquad /2'/$$

$$x \geq 0, \qquad /3'/$$

where $\bar{a}_{ij} = a_{ij} : b_i$.

Application of theorem 4-I to the system /2'/, /3'/ results in change of an inequality in /9/ to identity 1=1. So we get:

Theorem 4-III

If in the set $\mathcal{X}$ defined by /2'/, /3'/ for some $J(k)$ it holds:

$$\bar{a}_{kj} \leq \frac{1}{T} \sum_{i \in J(k)} \bar{a}_{ij} \qquad (j=1,\dots,n) \qquad /11/$$

so, the constraint "k" is redundant in $\mathcal{X}$.

If the set $J(k)$ consists of one element for given $k \in J$, then theorem 4-III can be reduced to previously proved by [redacted] W.Grabowski [2] sufficient condition of inequality "k" deletion from system /2/-/3/, because of its redundancy in reference to inequalities $i \in J(k)$, when $b_i > 0$, $b_k > 0$.

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[*] RHS - right hand side

This condition is formed by the system of inequalities:

$$\frac{a_{kj}}{b_k} \leq \frac{a_{ij}}{b_i} \quad (j=1,\dots,n) \qquad /11'/$$

or

$$\bar{a}_{kj} \leq \bar{a}_{ij} \quad (j=1,\dots,n),$$

what equals formula /11/ for r= 1.

Theorem 4-III can not be applied to delete the equations formed from inequalities, by introduction of nonnegative slack variables. Let inequalities "i" and "k" from system /2/ satisfy conditions /11 /. The corresponding equations are:

$$a_i x + x_{n+i} = b_i \; (b_i > 0),$$

$$a_k x \qquad + x_{n+k} = b_k \; (b_k > 0).$$

These equations do not satisfy conditions /11'/, though

$$0 \equiv \frac{a_{k,n+i}}{b_k} < \frac{a_{i,n+i}}{b_i} \equiv \frac{1}{b_i},$$

however

$$\frac{1}{b_k} \equiv \frac{a_{k,n+k}}{b_k} > \frac{a_{i,n+k}}{b_i} \equiv 0.$$

C. We shall discuss two other approaches when theorem 4-III, applied to the two inequalities in system /2/, does not lead to determination of the constraint "k" redundancy. Note, that theotheorems 4-I, 4-II and 4-III are the simplifications of rem 4. They form sufficient , but not necessary conditions for constraint "k" redundancy.

First case. We consider two inequalities from system /2/, numbered "k" and "i" respectively. Their coefficients do not satisfy condition /11'/, because

$$\bar{a}_{kj} \leq \bar{a}_{ij} \quad \begin{cases} j=1,\dots,n \\ j \neq p \end{cases} \qquad /a/$$

while

$$\bar{a}_{kp} > \bar{a}_{ip}. \qquad /b/$$

Therefore, one can not say that the constraint "k" is redundant. Let us try to apply corollary 1 to these inequalities, i.e. we check if exists v satisfying inequalities:

$$\begin{cases} \bar{a}_{ij} v \geqslant \bar{a}_{kj} & j=1,\dots,m, \; j \neq p \quad /c/ \\ \bar{a}_{ip} v \geqslant \bar{a}_{kp} & /d/ \\ 0 \leqslant v \leqslant 1 . & /e/ \end{cases}$$

Because of the third inequality, the above system has a solution, if the following possibilities are excluded;

1° $\bar{a}_{ip} \geqslant 0$,

2° $\bar{a}_{ip} \cdot \bar{a}_{kp} \leqslant 0$,

3° $\bar{a}_{kj} = \bar{a}_{ij} > 0$.

From inequality /a/, (c), (e), we get

$$v \geqslant \max\left\{0 ; \frac{\bar{a}_{kj}}{\bar{a}_{ij}} \,\middle|\, \bar{a}_{ij} > 0\right\} \equiv L. \qquad /12/$$

Whereas from inequalities /b/, 1° and 2° we get

$$|\bar{a}_{ip}| > |\bar{a}_{ik}| \geqslant 0$$

From /b/, /d/ and /e/ it holds

$$v \leqslant \frac{\bar{a}_{kp}}{\bar{a}_{ip}} \equiv \mathcal{U}. \qquad /13/$$

Finally, if the inequality

$$L \leqslant \mathcal{U}$$

is true, then the constraint "k" is redundant, because by the terms /12/ and /13/ system /c/ - /e/ has a solution.

Suppose now, inequality /b/ holds for more than one p. The set of such p we denote by $\mathcal{P}$. By corollary 1 the constraint "k" is redundant, when the system of inequalities

$$\bar{a}_{ij} v \geqslant \bar{a}_{kj} \quad j=1,\dots,n \, ; \; j \notin \mathcal{P} \qquad /c'/$$

$$\bar{a}_{ip} v \geqslant \bar{a}_{kp} \quad p \in \mathcal{P} \qquad /d'/$$

$$0 \leqslant v \leqslant 1 \qquad /e'/$$

has a solution. This solution exists if the following three possibilities are excluded:

1° $\bar{a}_{ip} \geqslant 0 \quad p \in \mathcal{P}$,

2° $\bar{a}_{ip} \cdot \bar{a}_{kp} \leqslant 0 \quad p \in \mathcal{P}$

3° $\bar{a}_{kj} = \bar{a}_{ij} > 0$.

In parallel way as previously, we get

$$v \geqslant \max\left\{0;\ \frac{\bar{a}_{kj}}{\bar{a}_{ij}} \,\middle|\, \bar{a}_{ij} > 0\right\} \equiv L$$

and

$$v \leqslant \min\left\{\frac{\bar{a}_{kp}}{\bar{a}_{ip}} \,\middle|\, p \in \mathcal{P}\right\} \equiv \mathcal{U}.$$

Satisfaction of inequality L ≤ U is sufficient for the constraint "k" to be redundant.

<u>Second case</u>. In the system /2'/ we analyse by double checking of /11'/ the redundancy of constraint "k" in reference to the two other constraints "i" and "h". As a result we get

$$\bar{a}_{kj} \leqslant \bar{a}_{ij} \quad \begin{cases} j = 1, \dots, n \\ j \notin \mathcal{P} \end{cases}$$

$$\bar{a}_{kj} \leqslant \bar{a}_{hj} \quad \begin{cases} j = 1, \dots, n \\ j \notin Q \end{cases}$$

$\bar{a}_{kp} > \bar{a}_{ip} \quad p \in \mathcal{P}$ and $\bar{a}_{kq} > \bar{a}_{iq}, \quad q \in Q$,

where $\mathcal{P}$ and Q are the sets of indices attached to relations not satisfying /11'/, while the constraint "k" is succesively compared to the constraint "i" and later to the constraint "h". The above presented results do not enable to define the constraint "k" as redundant. Therefore we apply corollary 1 to analyse the constraints "k", "h" and "i". So we state that the constraint "k" is redundant if the following LP problem has a solution[3].

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[3] It is enough to have any feasible solution of this problem so its consistency is sufficient.