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The Numerical Solution For The Roots of Polynomials (Part II)

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Contents

The numerical solution for the roots of polynomials

1. Introduction
2. Determination of the limits for the roots of polynomials
   2.1 Limits for real roots by Maclaurin's theorem.
   2.2 Limits for complex roots by Westerfield and Parodi
3. Approximate determination of the roots of a polynomial by means of Vieta's formulae
   3.1 Calculation of the larger roots
   3.2 Calculation of the smaller roots
4. Iteration in the complex plane
   4.1 Example
   4.2 The square root of a real number
   4.3 Flow Chart for $\sqrt{a}$ with relative error
   4.4 The square root of a complex number
   4.5 Flow-Chart for the square root of a real or complex number.
   4.6 A modification of Newtons method
5. Lin-Bairstow method for the complex roots
   5.1 Mathematical discussion
   5.2 Summary of Lin-Baristow method
   5.3 Flow-Chart for Lin-Bairstow method
6. Root-squaring method of Lobachevskii
   6.1 The basic idea of the method
   6.2 Roots all real and unequal in magnitude
   6.3 Flow-Chart
   6.4 Complex roots and roots equal in magnitude

References.

1. Introduction

In [1] we have discussed the numerical solution for the real roots of equations. The bisecting method and the false position method as shown before are very simple, complete general and always convergent. General method of iteration and other methods with its convergence were explained.

This chapter deals with those methods which are applicable to finding the roots, real as well as complex, for the polynomials, such as the iteration, Lin-Bairstow and Dandelin-Graeffe methods. For computation, every method was followed with flow-charts.

The transition from numerical analysis to programming can generally be facilitated by a flow-chart. The flow-chart is a graphic representation of the procedures and shows how the alternatives fit together. When numerical analysis is complete and the transition from mathematical language to machine language begins, the flow-chart can be an excellent device for establishing continuity.

2. Determination of the limits for the roots of a polynomial

2.1 Limits for real roots by Maclaurin's theorem

The real roots of the equation

$$a_0x^n+a_1x^{n-1}+\ldots+a_n = 0 \qquad (1)$$

where $a_0 > 0$, satisfy the inequality

$$x < 1 + \sqrt[m]{\frac{A}{a_0}} \qquad (2)$$

where m is the suffix of the first negative coefficient in the series $a_0, a_1, a_2 \ldots, a_n$, and A is the largest of the moduli of the negative coefficients.

This method allows one to determine also a lower limit for the roots. For this, it is necessary to make the substitution $x=-y$ and to multiply the equation by $(-1)^n$ in order that the first coefficient remains positive; after this we can make use once again of formula (2).

If $|a_0|$ is considerably smaller than A, formula (2) gives a widely over estimated limit. In this case the polynomial may be broken down into the sum of several polynomials, the first coefficients of which are positive, and the upper limit for each of these may be determined. The greatest of these upper limits determines the upper limit of the roots of the initial polynomial. In a lucky breaking down of the polynomial, the limits are determined a good deal more accurately than by the first method. The decomposition is usually a good one if approximately the same values are obtained for all the upper limits.

Example (1)

The roots of the equation

$$2x^9+x^7-x^4+19x^3-24x^2+11=0 \qquad (3)$$

satisfy the inequality

$$x < 1+\sqrt[5]{\frac{24}{2}} = 1+\sqrt[5]{12} \approx 2.7 .$$

Put $x=-y$ in (3) we get

$$2y^9+y^7+y^4+19y^3+24y^2-11=0$$

$$y < 1+\sqrt[9]{\frac{11}{2}} \approx 2.3$$

from which $x > -2.3$. Thus the roots of the equation lie in the interval

$$-2.3 < x < 2.7 .$$

Example (2) : To determine an upper limit for the roots of the equation :

$$x^5+12x^4-8x^3+2x^2 - 5680x+112=0$$

According to formula (2) we get :

$b=1+\sqrt[2]{5680} \approx 76.5$. Thus $x < 76.5$.

Dividing the polynomial into two added components:

$$P_1(x) = 0.1x^5 - 8x^3$$

$$P_2(x) = 0.9x^5+12x^4+2x^2-5680x+112.$$

We find upper limits for their roots:

$$b_1=1+\sqrt[2]{\frac{8}{0.1}} \approx 10 , \quad b_2=1+\sqrt[4]{\frac{5680}{0.9}} \approx 10.$$

whence $x < 10$.

Dividing the same polynomial into three added components :

$$P_1(x) = 0.2x^5 - 8x^3 ,$$

$$P_2(x) = 0.8x^5 + 2x^2 - 1680x+112,$$

$$P_3(x) = 12x^4 - 4000 x,$$

we find

$$b_1 = 1+ \sqrt{\frac{8}{0.2}} = 7.5 ,$$

$$b_2 = 1+ \sqrt[4]{\frac{1680}{0.8}} = 7.8 ,$$

$$b_3 = 1+ \sqrt[3]{\frac{4000}{12}} = 7.9 ,$$

whence $x < 7.9$.

2.2 Limits for complex roots by Westerfield and Parodi

Consider the polynomial

$$x^n+a_1x^{n-1}+\dots+a_n \qquad (4)$$

with real and complex coefficients .

We shall denote by q_r the quantities[x)]

$$\sqrt[r]{|a_r|} , r = 1,2,\dots,n \qquad (5)$$

arranged in order of decreasing magnitude

$$q_1 \geqslant q_2 \geqslant \dots \geqslant q_n \qquad (6)$$

It has been showed by Westerfield that all roots (real and complex) of the polynomial (4) satisfy the conditions:

$$|x| \leqslant q_1 + q_2 \qquad (7)$$

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x) The real positive value of the root is taken.

and

$$|x| \leqslant q_1 + 0.6180\ q_2 + 0.2213\ q_2 + 0.0883\ q_4 +$$
$$+ 0.0375\ q_5 + 0.0185\ q_6 + 0.0074\ q_7 + 0.0081\ q_8 \qquad (8)$$

In the case of the coefficient a_1 of the polynomial (4) being much larger than the other coefficients, we can apply a simple and effective estimate found by M. Parodi:

Let $|a_1| > 2\sqrt{S}$

where

$$S = |a_2| + |a_3| + \ldots + |a_n| \qquad (9)$$

and

$$S > 1. \qquad (10)$$

The polynomial (4) has one, and only one, root within the circle

$$|x + a_1| \leqslant \sqrt{S} \qquad (11)$$

<u>Example</u> : Find the limits for the roots of the polynomial

$$x^4 - 48x^3 + 797x^2 - 5350x + 12297 = o$$

$\sqrt[1]{|-48|} = 48$, $\sqrt[2]{|797|} \approx 28.2$,

$\sqrt[3]{|-5350|} \approx 17.5$, $\sqrt[4]{|12297|} \approx 10.5$

Thus $q_1 = 48$, $q_2 = 28.2$, $q_3 = 17.5$, $q_4 = 10.5$.

According to formula (7) we find :

$$|x| \leqslant 76.2 \ .$$

If we apply formula (8) we get the following value for the limits:

$$|x| \leqslant 48 + 17.4 + 3.9 + 0.9 = 70.2$$

By Maclaurin's method, we find from (2) that $x < 49$; however this gave a limit only for real positive roots; while the value 70.2 is a limit for the moduli of all roots (real and complex).

3. Approximate Determination of the Roots of a polynomial by means of Vieta's Formula

The method provides the possibility of the easy determination of roots which are larger or smaller in magnitude than most of the other. Its advantage over other method lies in the fact that it requires a minimal quantity of calculation. However, the accuracy with which roots are determined is often very small : usually one succeeds indetermining only the order of magnitude of the largest and the smallest roots.

Vieta's formulae

These formulae connect the roots $x_1 , x_2 , \ldots , x_n$ of the polynomial

$$P(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$$

with its coefficients :

$$\left.\begin{aligned}
-\frac{a_1}{a_0} &= x_1 + x_2 + \ldots + x_n \\
\frac{a_2}{a_0} &= x_1 x_2 + x_1 x_3 + \ldots + x_{n-1} x_n \\
-\frac{a_3}{a_0} &= x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 + \ldots + x_{n-2} x_{n-1} x_n \\
&\vdots \\
(-1)^{n-1} \frac{a_{n-1}}{a_0} &= x_1 x_2 \ldots x_{n-2} x_{n-1} + x_1 x_2 \ldots x_{n-2} x_n + \ldots + \\
&\qquad \ldots + x_2 x_3 \ldots x_{n-1} x_n \\
(-1)^n \frac{a_n}{a_0} &= x_1 x_2 \ldots\ldots x_n
\end{aligned}\right\} \quad (A_1)$$

3.1 Calculation of the larger roots

Let

$$|x_1| \geqslant |x_2| \; \dots \geqslant |x_n| . \qquad (A_2)$$

If $|x_1|$ is appreciably larger than the moduli of all the other roots, then it is possible to ignore the numbers x_2, x_3, ..., x_n

$$-\frac{a_1}{a_0} \approx x_1 \qquad (A_3)$$

Thus the largest roots approximately satisfies the equation

$$a_0 x + a_1 = 0 \qquad (A_4)$$

If the moduli of the first two roots are appreciably larger than the moduli of the remaining roots, we get from the first two of Vieta's formulae :

$$\left.\begin{aligned} -\frac{a_1}{a_0} &\approx x_1 + x_2 \\ \frac{a_2}{a_0} &\approx x_1 \; x_2 \end{aligned}\right\} \qquad (A_5)$$

Thus the two larger roots of the given polynomial approximately satisfy the equation

$$a_0 x^2 + a_1 x + a_2 = 0 \qquad (A_6)$$

Analogously, if the moduli of three roots are appreciably larger than the moduli of the remaining ones, these roots are approximately determined by the equation :

$$a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 0 \qquad (A_7)$$

The truth of this statement follows from the relation :

$$\left.\begin{aligned} -\frac{a_1}{a_0} &\approx x_1 + x_2 + x_3 \\ \frac{a_2}{a_0} &\approx x_1 x_2 + x_1 x_3 + x_2 x_3 \\ -\frac{a_3}{a_0} &\approx x_1 x_2 x_3 \end{aligned}\right\} \quad (A_8)$$

obtained from (A_1) and being Viet's formulae for equation (A_7).

3.2 Calculation of the smaller roots

If we substitute into (A_3) a new argument $y = \frac{1}{x}$ and apply the results we have got for large roots, and then change back from y to the argument $x = \frac{1}{y}$, we get the following results .

If $|x_n|$ is appredicably smaller than the moduli of the other roots of the given polynomial, $|x_1|$ may be approximately determined by the equation

$$a_{n-1} x + a_n = 0 \qquad (A_8)$$

If the moduli of x_{n-1} and x_n are appreciably smaller than the moduli of the remaining roots, the three roots are approximately determined by the equation :

$$a_{n-3} x^3 + a_{n-2} x^2 + a_{n-1} x + a_n = 0 \qquad (A_9)$$

Analogous theorems hold also for any number of roots with larger or smaller moduli.

<u>Example</u> Determine the roots of the polynomials

$$P(x) = x^4 + 39x^3 + 958x^2 - 1080x - 2000$$

we try to determine the largest root by means of the equation

$$x + 39 = 0$$

Then $x_1 = -39$. However a trial convinces us that $x_1 = -39$ is not even approximately a root.

We form the second equation :

$$x^2 + 39x + 958 = 0$$

From which

$$x_1 = -19.5 + 24.04\, i$$

$$x_2 = -19.5 - 24.04\, i$$

The exact roots are $x_1 = -20 \pm 24.48\, i$,

$$x_2 = -20 - 24.48\, i$$

For determining the smallest root we take the equation

$$-1080x - 2000 = 0,$$

from which $x_4 \approx -1.85$. A trial shows that the number found is not a root.

We take the equation $958x^2 - 1080x - 2000 = 0$

Then $x_4 = -0.99$, $x_3 = 2.12$ (exact values are $x_4 = -1$, $x_3 = 2$)

4. Iteration in the complex plane

A study closely analogous to that in [1] for the iteration methods may be applied to the solution of equations involving functions of a complex variable. For example, Newton's method may be applied readily if a suitable starting value is available.

4.1 Example

Using the starting value $x_0=i, i=\sqrt{-1}$, and applying Newton's formula to the equation :

$$f(x)=x^4+x^3+5x^2+4x+4=0 \tag{12}$$

we obtain

$$x_1=i - \frac{f(i)}{f'(i)} = i - \frac{3i}{1+6i} = 0.486+0.919\, i \;,$$

$$x_2= 0.486+0.919\, i - \frac{-0.292+0.174i}{1.780+6.005i} = -0.499+0.866i$$

as two approximations to the solution

$$x = \frac{-1+i\sqrt{3}}{2}$$

4.2 The square root of a real number

If we write $x = \sqrt{a}$, then $f(x) = x^2-a$ where $a \gtrless 0$. Newton's iteration method here assumes the form

$$x_{i+1} = x_i - (x_i^2 - a)/2x_i \tag{13}$$

or, more simply

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{a}{x_i}\right) \tag{14}$$

If recursion (14) is to be coded for a computer, it will be desirable to have the starting value x_0 chosen to exceed the first iterate x_1. Since the code should be applicable to finding the square root of any positive, number a, large or small, and since it is convenient to start with a preasigned value, say $x_0 = 1$, we introduce a change of variables to meet these requirements. If the program is based on decimal arithmetic, we introduce a new quantity b which provides that $a = 10^{2k}b$, b an integer, and $\frac{1}{100} < b \leq 1$. We find $\sqrt{b}$ using (14) and convert to $\sqrt{a}$ through the relation $\sqrt{a} = 10^k \sqrt{b}$.

If the computations indicated in (14) are done in the base 2, a natural choice of range for b is normally $\frac{1}{4} < b \leq 1$, Such that $a = 2^{2k}$ b. The starting value $x_0 = 1$ again yields a decreasing sequence of iterates converging to $\sqrt{b}$ and hence $\sqrt{a} = 2^k \sqrt{b}$. The sequence of calculations is indicated in the following flow cahrt.

4.3 Flow-chart for $\sqrt{a}$, with relative error

(a real number)