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FOREIGN LOANS AND ECONOMIC DEVELOPMENT

PART II

THE FIXED ANNUITY PRINCIPLE

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CONTENTS

I. Introduction:

In Part I of this memo.(1) we dealt with the problem of assessing the advantages of foreign loans, using a model suggested by Qayum.(2) Three main problems have been considered:

1. Discussion of the advantage criteria, with special emphasis on long-term development.
2. Correction of Qayum's formula.
3. Introduction of the replacement period as a parameter in evaluating foreign loans.

The main outcome of that analysis was that the decisive factor in determining whether a loan has a negative or a positive effect on the debtor country's capital, and hence income, at the end of the repayment period, is the difference between the ratio of the marginal savings rate to the capital coefficient, and the rate of interest on the loan. Whenever that ratio exceeds the interest rate, a positive effect can be realized, and vice versa. On the other hand, the elongation of the repayment period was not always a blessing. Its role is to magnify the effect of the loan, whether positive or negative. Grace periods of an equal length were found to be more beneficial in introducing extra gains which might, in some cases, outweigh any initial losses.

Taking account of the orders of magnitude of the parameters(3) σ and γ, normally prevailing in developing countries, one should expect relatively low rates of interest if resort to foreign loans would contribute at all to the process of long-term development. Moreover, if the interest rate is very close to the ratio σ/γ, the repayment period should be extended considerably over time in order to recover the whole benefit of the loan at some given date. In some cases, the repayment period has to be several times as long as the period required for replacing the assets financed through the loan.

We have concluded our previous paper by asserting the need for:

1. Reconsidering the behaviour of the parameters, and
2. Studying the possibility of adopting the fixed instalment principle.

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(1) Memo. 779, Part I. I.N.P., June 1967
(2) Memo. 570, I.N.P., May, 1965
(3) For the system of notation, see Memo. 779, Part I, pp. 4-5

We shall start by considering this latter problem in order to see how far our previous conclusions would be affected by a change of the repayment conditions. Later on we shall consider the policy implications of alternative advantage criteria, and their effects on the values of the parameters

II. The Fixed Annuity Formula:

In Part I, we have given the formula for a fixed annuity based on compound interest. Thus, if the Annuity is A, its ratio to the loan is b:

$$b = A/L, \qquad \text{or,} \qquad A = bL \tag{1}$$

The value of b under the compound interest rule, b_c is (4)

$$b_t = b_c = \frac{r.(1+r)^e}{(1+r)^e - 1} \qquad (t = 1.....e) \tag{2}$$

where r is the interest rate and e is the repayment period. A more plausible assumption is to assume that repayment follows the simple interest rule:

Suppose that a loan L received in year 0 is to be repaid over e years at a simple rate of interest r. Repayment is made by means of a fixed annuity A, paid at the end of each year. To determine A, we start by considering all flows envolved at a given point of time, e.g., at the end of the repayment period. Thus the inflow will be:

$$L\,(1 + er)$$

The repayment made in the t-th year will reach the value

$$A\left[1 + r\,(e - t)\right]$$

at the same point of time. Hence the total outflow evaluated at the end of year e will be

$$\sum_{t=1}^{e} A\left[1 + r\,(e-t)\right] = A\left[e + r\sum_{t=1}^{e}(e-t)\right]$$
$$= eA + rA\sum_{j=0}^{e-1} j = eA\left[1 + \frac{r\,(e-1)}{2}\right]$$

(4) There is a printing error in this formula as given in memo. 779 Part I, p. 7 The factor r is missing from the numerator.

Equating the two flows, and substituting (1), we can easily find that:

$$b = \frac{2\ (1 + er)}{e\left[2 + r\ (e - 1)\right]} \qquad (3)$$

It has been shown that if the principal is to be repaid at equal instalments while interest is charged on the remainder, the annuity will be decreasing. Its value was found to be:(5)

$$b_t = A_t/L = \frac{1}{e}\left[1 + (e - t + 1)\ r\right] \qquad (I/3)$$

For year t = 1, this value is greater than b since:

$$b_1/b = \frac{1}{e}\ (1 + er)\ \frac{e\left[2 + r(e - 1)\right]}{2\ (1 + er)} = 1 + r\left(\frac{e - 1}{2}\right) \geqslant 1$$

The equality sign holds only when e = 1, while the inequality holds for larger values of e, so long as r is positive. On the other hand, the annuity b_t will be decreasing at a rate equal to r/e. This means that its ratio to a growing income will be falling more rapidly. In other words, the fixed annuity principle is less unfavourable than the decreasing annuity approach, to the debtor country.

Besides, the total repayments made on the basis of the fixed annuity principle is smaller than that following the decreasing annuity rule:

$$\sum_{t=1}^{e} b_t = \sum_{t=1}^{e} \frac{1}{e}\left[1 + (e + 1)\ r - tr\right] = 1 + \frac{r\ (e + 1)}{2}$$

$$\sum_{t=1}^{e} b = e \cdot \frac{2\ (1 + er)}{e\left[2 + r(e - 1)\right]} = 1 + \frac{r\ (e + 1)}{2 + r\ (e - 1)}$$

Thus:

$$\sum b_t > eb, \qquad \text{for any } e > 1$$

which means that the outflow of capital will be larger for the variable annuity approach.

It would be also expected that the capitalized value of the variable annuities is larger than that of the fixed annuity, which was found to be (1 + er). Now,

(5) Equations quoted from 779, Part I, will be given their numbers preceded with I.

$$\sum_{t=1}^{e} b_t \left[1 + (e - t) r\right] = \frac{1}{e} \sum_{j=0}^{e-1} (1 + r + jr)(1 + jr)$$
$$= (1 + r) + \frac{r(2 + r)(e - 1)}{2} + \frac{r^2 (e - 1)(2e - 1)}{6}$$
$$= 1 + er + \frac{(e - 1)(e + 1)}{3} r^2 > (1 + er)$$

Thus the fixed annuity approach seems to be more favourable to the debtor country, on either of the three aspects considered:

1. The value of the annuity is smaller at the beginning, hence its burden on initial lower incomes is smaller.
2. It leads to a smaller capital outflow.
3. The capitalized value of the outflow is also smaller.

It might be argued, therefore, that the stringent conditions obtained before could be somewhat relaxed if we accept the fixed annuity approach. To the answer of this question, we shall address our-selves in the following sections.

III. The Impact of a Single Loan:

Let us start by assuming the set of assumptions (1) - (5), section V, memo 779, Part I (pp. 14-15). Assumption (6) is to be replaced by the fixed annuity assumption. The base year values are given:

$$Y_0, \quad S_0 = sY_0, \quad I_0 = (s + \lambda) Y_0$$

For year 1, the values of income and savings are the same as before:

$$Y_1 = \left(1 + \frac{s + \lambda}{\gamma}\right) Y_0 \qquad \text{(I/10)}$$

$$S_1 = \left(d + \lambda\frac{\sigma}{\gamma}\right)Y_0 = c\, Y_0 \qquad \text{(I/12)}$$

where

$$d = s\left(1 + \frac{\sigma}{\gamma}\right) \qquad \text{(I/32)}$$

But instead of (I/13), we have:

$$I_1 = S_1 - A = \left(1 + \frac{\sigma}{\gamma}\right) S_0 + \left(\frac{\sigma}{\gamma} - b\right) L$$

or,
$$I_1 = \left[d + \lambda \left(\frac{\sigma}{\gamma} - b\right)\right] Y_0 \qquad (4)$$

For the following years, $t = 2, \ldots, e$, we have the following system of equations:

$$Y_t = Y_{t-1} + \frac{1}{\gamma} I_{t-1} \qquad (5)$$

$$S_t = S_1 + \sigma (Y_t - Y_1) \qquad (6)$$

$$I_t = S_t - A = \sigma(Y_t - Y_1) + I_1 \qquad (7)$$

Equations (5) and (6) are the same as (I/15) and (I/16), while (7) differs from (I/17) in so far as the value of annuity is concerned. Substituting (7) in (5) we obtain:

$$Y_t = (1 + \frac{\sigma}{\gamma}) Y_{t-1} + \frac{1}{\gamma} (I_1 - \sigma Y_1) \qquad (8)$$

where I_1 and Y_1 are given by (4) and (I/10). The solution of this first-order difference equation is:

$$Y_t = (1 + \frac{\sigma}{\gamma})^{t-1} \cdot (\frac{1}{\sigma} I_1) + (Y_1 - \frac{1}{\sigma} I_t)$$

or,

$$Y_t \; (1 + \frac{\sigma}{\gamma})^{t-1}\left[\frac{s}{\sigma} (1 + \frac{\sigma}{\gamma}) + \frac{\lambda}{\sigma} (\frac{\sigma}{\gamma} - b)\right] Y_o + \frac{1}{\sigma} (\sigma - s + \lambda b) Y_o \qquad (9)$$

If no loan was taken, the GDP. can be calculated from this last formula by putting $\lambda = 0$:

$$Y'_t = (1 + \frac{\sigma}{\gamma})^t \frac{s}{\sigma} Y_o - (1 - \frac{s}{\sigma}) Y_o \qquad (10)$$

Thus for any year $t = 2, \ldots, e, e + 1$, we have:

$$Y_t - Y'_t = (1 + \frac{\sigma}{\gamma})^{t-1} \frac{\lambda}{\sigma} (\frac{\sigma}{\gamma} - b) Y_o + \frac{\lambda}{\sigma} b Y_o \qquad (11)$$

In particular, $V = Y_{e+1} - Y'_{e+1}$ defined by (I/29) is equal to:

$$V = \left[E (\frac{\sigma}{\gamma} - b) + b \right] \frac{\lambda}{\sigma} Y_o \qquad (12)$$

where E is defined according to (I/32)

$$E = (1 + \frac{\sigma}{\gamma})^e$$

The criteria ε and δ defined by (I/28) are:

$$\varepsilon = \frac{\gamma}{\lambda} \frac{V}{Y_o} = \left[E (\frac{\sigma}{\gamma} - b) + b\right] \frac{\gamma}{\sigma}$$

Hence:

$$\varepsilon = \frac{\gamma}{\sigma} (E - 1) (\frac{\sigma}{\gamma} - b) + 1 \qquad (13)$$

and, $$\delta = \varepsilon - 1 = \frac{\gamma}{\sigma} (E - 1) (\frac{\sigma}{\gamma} - b) \qquad (14)$$

For given values of $\frac{\sigma}{\gamma}$ and r, these expressions are increasing functions of e. As e increases the value of b diminishes. It might be possible that for small values of e the factor $(\frac{\sigma}{\gamma} - b)$ is negative. As e increases it would change signs, and its value will be increasing. At the same time the value of the positive factor multiplying it, (E - 1) will be increasing. This result is different from our earlier finding, for the case of a decreasing annuity, but it is in line with the findings of Qayum and of I.B.R.D. experts.(6)

Again, since b is an increasing function of r, both ϵ and δ will be decreasing as r increases. On the other hand, we can write:

$$\frac{\gamma}{\sigma}(E - 1) = \sum_{0}^{e-1} (1 + \frac{\sigma}{\gamma})^t$$

which is an increasing function of $\frac{\sigma}{\gamma}$. The same is also true for the factor $(\frac{\sigma}{\gamma} - b)$, which means that both criteria are increasing functions of σ, and decreasing functions of γ. Let us now turn to the other parameters envolved, before discussing the implications of the alternative criteria.

IV. The Formulae with Lags and Replacement:

Suppose that projects financed by the loan have a gestation lag of f years, which means that they start production in year f + 1. Further, a grace period of a length h is accorded. As had been suggested in Part I, we consider that the life time of the projects financed by the loan is k, which might be equal to or different from e. To calculate the values of the alternative criteria, we subdivide investment in any year t into two parts, one due to incomes derived from the loans, j", and the remainder J':

$$I_t = J'_t + J''_t \qquad (15)$$

The part J' changes its composition starting year h + 1. For the first h years we have:

$$J'_t = s\,(1 + \frac{\sigma}{\gamma})^t\ Y_o, \qquad (t = 1,\ldots,h) \qquad (16)$$

In year h + 1 it becomes equal to:

$$J'_{h=1} = \left[s\,(1 + \frac{\sigma}{\gamma})^{h+1} - \lambda b\right] Y_o = (dH - \lambda b)\, Y_o$$

For the repayment period we have:

(6) See, e.g., G.M. Alter:" The Servicing of Foreign Capital Inflow by under developed countries". in, H.S. Ellis & H.C. Wallich (ed): Economic Development for Latin Amercia; Micmillan, N. Y., 1961

$$J'_t = J'_{h+1}\,(1 + \frac{\sigma}{\gamma})^{t-h-1} = (dH - \lambda b)(1 + \frac{\sigma}{\gamma})^{t-h-1}\;Y_o$$

which is true for $t = h + 1, \ldots, \tau \leqslant h + e$. The cumulative investments of type J' can be found by summation:

$$\sum_1^{\tau} J'_t = \sum_1^{h} J'_t + \sum_{h+1}^{\tau} J'_t$$
$$= \frac{\gamma}{\sigma}\Big[d\,(H - 1) + (dH - \lambda b)(\frac{T}{H} - 1)\Big]\,Y_o$$

where H is defined by (I/32), and T is defined in a similar manner:

$$T = (1 + \frac{\sigma}{\gamma})^{\tau} \qquad (17)$$

In other words:

$$\sum_1^{\tau} J'_t = \frac{\gamma}{\sigma}\Big[d\,(T - 1) - \lambda b\,(\frac{T}{H} - 1)\Big]\,Y_o \qquad (18)$$

The investments financed by savings generated from loan incomes start in year f + 1,

$$J''_{f+1} = \lambda \frac{\sigma}{\gamma}\,Y_o$$

For the following years,

$$J''_t = \lambda\frac{\sigma}{\gamma}(1 + \frac{\sigma}{\gamma})^{t-f-1}\,Y_o \qquad (t \geqslant f+1)$$

Hence,

$$\sum_1^{\tau} J''_t = \sum_{f+1}^{\tau} J''_t = \lambda\,(\frac{T}{F} - 1)\,Y_o \qquad (19)$$

It follows that the sum-total of investments during the τ years is:

$$\sum_1^{\tau} I_t = \frac{\gamma}{\sigma}\Big[d\,(T - 1) + \lambda\frac{\sigma}{\gamma}\,(\frac{T}{F} - 1) - \lambda b\,(\frac{T}{H} - 1)\Big]\,Y_o \qquad (20)$$

Starting year h + e + 1, repayments disappear which means that:

$$I_t = I_{h+e+1}\,(1 + \frac{\sigma}{\gamma})^{t-h-e-1}$$

It can be seen that:

$$I_{h+e+1} = I_{h+e}\,(1 + \frac{\sigma}{\gamma}) + \lambda b\,Y_o$$

To calculate I_{h+e}, we calculate its two components:

$$J'_{h+e} = (dH - \lambda b)\,(1 + \frac{\sigma}{\gamma})^{e-1}\;Y_o$$

$$J''_{h+e} = \lambda \frac{\sigma}{\gamma} (1 + \frac{\sigma}{\gamma})^{h+e-f-1} Y_o$$

Substituting in the above expressions we obtain:

$$I_{h+e+1} = (dH + \lambda \frac{\sigma}{\gamma} \frac{H}{F} - \lambda b)(1 + \frac{\sigma}{\gamma})^e Y_o + \lambda b Y_o$$

Hence for $t \geqslant h + e + 1$

$$I_t = (dH + \lambda \frac{\sigma}{\gamma} \frac{H}{F} - \lambda b)(1 + \frac{\sigma}{\gamma})^{t-h-1} Y_o + \lambda b(1 + \frac{\sigma}{\gamma})^{t-h-e-1} Y_o \quad (21)$$

and,

$$\sum_{e+h+1}^{\tau} I_t = \frac{\gamma}{\sigma} \left[d(T - HE) + \lambda \frac{\sigma}{\gamma} (\frac{T}{F} - E^*) + \lambda b (\frac{T}{HE} - \frac{T}{H} + E - 1) \right] Y_o$$

where, as before, $E^* = HE/F$. In order to obtain the cumulation of investment starting year 1, we have to add to this last sum the cumulation (20) up to $\tau = h+e$. Hence

$$\sum_{1}^{h+e} I_t = \frac{\gamma}{\sigma} \left[d(HE - 1) + \lambda \frac{\sigma}{\gamma} (E^* - 1) - \lambda b (E - 1) \right] Y_o$$

It follows that, for $\tau > h + e$:

$$\sum_{1}^{\tau} I_t = \frac{\gamma}{\sigma} \left[d(T - 1) + \lambda \frac{\sigma}{\gamma} (\frac{T}{F} - 1) - \lambda b \frac{T}{HE} (E - 1) \right] Y_o \quad (22)$$

Finally, in the case of no loans we have for any t:

$$I'_t = d(1 + \frac{\sigma}{\gamma})^{t-1} \quad (23)$$

Hence, for any τ,

$$\sum_{1}^{\tau} I'_t = d \frac{\gamma}{\sigma} (T - 1) \quad (24)$$

It follows that:

$$\Delta_\tau = \sum_{1}^{\tau} (I_t - I'_t) \quad (25)$$

has the following values for the different values of τ:

(1) $\underline{\tau = f + k < h + e}$: Using (20) and (24),

$$\Delta_{f+k} = \lambda \frac{\gamma}{\sigma} \left[\frac{\sigma}{\gamma} (K - 1) - b (K^* - 1) \right] Y_o \quad (26)$$

where, as in (I/61), $K^* = FK/H$,