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On a Multiobjective Transportation Problem.

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Introduction:

For some practical problems, the conventional transportation problem which usually minimizes only the total cost of transportation, became of no benifit to the real situation. For example, high way motor carriers, although considerably cheaper than airfreight, require much longer shipment times, and consequently the freshness and perishability of many grocery products are influenced. Also, for a certain finished goods inventory, the shorter the shipment time the faster is the response of a logistic system to surges in demand. Furthermore, pipeline inventories and associated opportunity costs are directly related to shipment times. Thus, for some problems of real situation, an average shipment time may be used as an objective function beside the total costs. These two objectives, i.e., minimizing total transportation costs and average shipment times are in general conflicting. It is not possible in general to minimze both (or all in case of more than two objective functions) simultaneously. However, there are several different criteria which could be considered and employed to provide alternate solutions to multiobjective problems. One can determine the "efficient" solutions, and consequently the efficient curve, which denote the minimum attainable value of each of the objective functions for different values of the other ones. This idea of determining the efficient curve aids the decision making of the manager in eliminating the inefficient solutions. The decision maker can then subjectively choose that efficient point which mostly suits his company. Another approach to the problem is to find the feasible solution which can be considered the best regarding the optimization of all objective functions. That can be the solution for which the summation of the relative deviations of the different objective functions from their optima is minimum. This turns to--

be the optimal solution of an overall objective function equal to the summation of the different objective function each one weighted by the reciprocal of its optimal value (see, reference (6)). On the other hand if it is possible for the manager to order the objective functions according to their relative periorities, then the problem can be handled as follows: a subset of the overall feasible region which contains optimal solutions to the most important objective function is to be determined. Out of this subset a set of solutions which optimize the second important objective function may be located and so on the process continues for all functions. The most inner subset is considered to be the optimal region for the problem (see, reference (5)). Another alternative is to transform the problem into a program of a single objective function if one of the functions could be considered (by the manager for example) as the most important one over all others. In this cases that function is taken to be the main objective function and all other ones are to be transformed into constraints with lower bounds. The lower bounds represent the required ratios of the optimal values of the objective functions, e.g, it is required at least 90 % of the optimal value of the second objective function, 85% of the fourth one etc. (see, reference (5)).

In this paper, we suggest a method for finding the efficient points of a multiobjective transportation problem when the conflicting objectives of minimizing total costs and shipment times are considered. Corresponding to each efficient point, the method provides the optimal routes, modes of transportation, and the appropriate shipping amounts. The method has been based one the idea of Srinivasan & Thompsons method(see reference 4). In section I we.

present some theoretical results of the multiobjective transportation problem. In section II, a method for finding all optimal basic solution to the standard transportation problem is provided; we sometimes need to identify all optimal solutions (if there exists more than one) to the first or second objective during the process of locating the efficient points. Section III presents an algorithm and a worked example.

I. Theoretical Results For a Multiobjective Transportation Problem

Let

$I = \{1,2, \ldots, m\}$, be the set of origins O (rows),

$J = \{1,2, \ldots, n\}$, " " " " destinations D (columns)

and $K = \{1,2,\ldots, P\}$, " " " " modes of transportations.

Let

x_{ijk} be the amount shipped from the origin O_i to destination D_j via mode k,

C_{ijk} be the unit shipping cost from O_i to D_j via k,

t_{ijk} be the time of shipping any amount from O_i to D_j via k,

a_i be the available amount at O_i ($a_i \geq 0$),

and b_j be the requirement at D_j ($b_j \geq 0$).

The problem is to generate the efficient solutions to the following multiobjective transportation problem:

Minimize the toal cost

$$C(X) = \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} c_{ijk} \quad x_{ijk}$$

and the total time

$$T(X) = \sum_{i} \sum_{j} \sum_{k} t_{ijk} \quad x_{ijk}$$

Subject to

$$\sum_{j} \sum_{k} x_{ijk} = a_i \qquad i \in I \qquad \ldots. (1)$$

$$\sum_{i} \sum_{k} x_{ijk} = b_j \qquad j \in J \qquad \ldots. (2)$$

$$x_{ijk} \geqslant o, \quad \forall \; i,j \text{ and } k \qquad \ldots. (3)$$

--- (I)

The following conditions are assumed:

(1) The p modes of transportation are available for all routes. If this is not the case, the unit costs c_{ijk} and times t_{ijk} for the modes not available for a route from O_i to D_j can be set equal to a large positive number M, so that, the optimal solution will not utilize such modes.

(2) For any route from O_i to D_j there are no two modes k and l with $c_{ijk} \leqslant c_{ijl} < M$ and $t_{ijk} \leqslant t_{ijl} < M$. If this is not so, c_{ijl} is set equal to $t_{ijl} = M$.

(3) A single homogeneous good is considered in the system. If there were multiple goods, the problem may be solved separately for each of the goods.

(4) $\sum_i a_i = \sum_j b_j$. A dummy origin or destination can be added in order this condition to hold.

A feasible solution X to (I), i.e., satisfying (1) - (3), is called efficient if there does not exist other feasible solution $\hat{X}$ such that

$$C(\hat{X}) \leqslant C(X) \quad \text{and} \quad T(\hat{X}) < T(X)$$

or

$$C(\hat{X}) < C(X) \quad \text{and} \quad T(\hat{X}) \leqslant T(X) .$$

We now consider the parametric problem $P(\lambda)$:

$$\left.\begin{array}{l} \text{Min} \quad (1 - \lambda)C(X) + \lambda T(X) \quad \text{, under constraints} \\ \text{(1)-(3), where} \quad 0 \leqslant \lambda \leqslant 1. \quad \ldots\ldots\ldots\ldots \end{array}\right\} \text{(II)}$$

Problem (II) is equivalent to problem (I) in the sense that: every optimal solution to $P(\lambda)$ is an efficient solution to (I) and conversely, if X is efficient then there exists a scalar λ in the unit interval (o,1) such that X is an optimal solution to $P(\lambda)$. The sufficient condition is Lemma (2) in reference (1).

The necessary condition may be proved as follows:

Let $o < \lambda < 1$ and assume that an optimal solution X to $P(\lambda)$ is not efficient. Then there exists a feasible solution $\hat{X}$ such that

$$C(\hat{X}) \leqslant C(X) \quad \text{and} \quad T(\hat{X}) < T(X),$$

or

$$C(\hat{X}) < C(X) \quad \text{and} \quad T(X) \leqslant T(X).$$

Hence

$$(1- \lambda) \; C(\hat{X}) + \lambda T(\hat{X}) < (1- \lambda) \; C(X) + \lambda T(X),$$

which contradicts the assumption that X was optimal to $P(\lambda)$. The previous argument does not hold when $\lambda = 0$ or $\lambda = 1$. It holds only when there exists a unique optimal solution to P(λ) for both $\lambda = 0$ and $\lambda = 1$. Hence, it is assumed that there exists a unique optimal solution to each of P(o) and P(1).

Let (CXT) denote the two dimensional space with cost C as the abscissa and time T as the ordinate. Let W(λ) be the convex set of all optimal solutions to P(λ). Thus for each optimal solution $X \in W(\lambda)$, there exists an image $Q(X) = \{ C(X), \quad T(X) \}$ in (CXT).

Let $Q\left[W(\lambda) \right]$ be the set of images corresponding to all solutions in W(λ) under this mapping. The following theorem proivds a characterization of the set of efficient points.

<u>Theorem 1</u>

(For the proof see Lemma 3 in reference (1)). For each fixed value of λ satisfying $o \leq \lambda \leq 1$, $Q\left[W(\lambda) \right]$ is either a singleton or a compact line segment in (CXT). In the second case if $Q(X_1)$ and $Q(X_2)$ are the end points of the line segment corresponding to the optimal solutions X_1 and X_2, then

$$Q\left(x_1 \delta + (1 - \delta) X_2 \right) = Q(x_1 \delta) + (1 - \delta)\, Q(x_2),$$

for all δ satisfying $o \leq \delta \leq 1$.

Now, let us introduce the modes of the transportation into the picture. Let $d_{ij}(\lambda)$ be defined as

$$d_{ij}(\lambda) = \min_{k \in K} \left[(1 - \lambda)\, c_{ijk} + \lambda t_{ijk} \right] . \quad \ldots . \ (4)$$

Let $K_{ij}(\lambda) \subseteq K$ be the set of indices over which (4) attains its minimum for a particular route (i,j) and for a given value of λ , and the variables y_{ij} be $y_{ij} = \sum_{k \in K_{ij}(\lambda)} x_{ij}$, for a given λ and solution X.

Let us consider the following standard transportation problem $P'(\lambda)$:

$$\min \sum_{i \in I} \sum_{j \in J} d_{ij}(\lambda)\, y_{ij} \quad \ldots. \ (5)$$

subject to

$$\sum_{j} Y_{ij} = a_i \quad , i \in I , \ \ldots. \quad (6)$$

$$\sum_{i} Y_{ij} = b_j \quad , j \in J , \ \ldots. \quad (7)$$

and

$$y_{ij} \geqslant 0 \quad , \quad \forall \, i,j . \quad \ldots. \quad (8)$$

Let

$$c^1_{ij}(\lambda) = \min_{k \in K_{ij}(\lambda)} c_{ijk} , \quad \ldots\ldots \quad (9)$$

and $k^1_{ij}(\lambda)$ be the index in $K_{ij}(\lambda)$ at which this minimum is attained. Let $t^1_{ij}(\lambda)$ denote the unit time corresponding to the mode $k^1_{ij}(\lambda)$.

Similarly, let

$$t^2_{ij}(\lambda) = \min_{k \in K_{ij}(\lambda)} t_{ij} , \quad \ldots\ldots \quad (10)$$

and $k^2_{ij}(\lambda)$ denote the unique mode in $K_{ij}(\lambda)$ at which this minimum is attained and $C^2_{ij}(\lambda)$ the unit cost corresponding to $k^2_{ij}(\lambda)$. Theorem (2) below charactrizes the optimal solutions to $P(\lambda)$ and Theorem (3) relates the optimum solution of (5) - (9) to that of $P(\lambda)$.

Theorem 2

(i) For every optimal solution X to P(λ), $x_{ijk} = 0$ for $k \in K_{ij}(\lambda)$.

(ii) Given an optimal solution X to P(λ), any other solution X' satisfying:

- $x'_{ijk} \geq 0$ for $k \in K_{ij}(\lambda)$, and

- $\sum_{k \in K_{ij}(\lambda)} x'_{ijk} = \sum_{k} x_{ijk} = y_{ij}$ is also optimal to P(λ).

Theorem 3

Corresponding to any optimal solution Y to P'(λ), (for some given λ), the solution $X_1 = \{x^1_{ijk}\}$ that has the minimum total cost is given by:

$$x^1_{ijk} = \begin{cases} y_{ij} & k = k^1_{ij}(\lambda) \\ 0 & k \neq k^1_{ij}(\lambda) \end{cases} \quad , \forall i,j \quad \ldots. (11)$$

The corresponding total cost and time are:

$$C(X^1) = \sum_i \sum_j c^1_{ij}(\lambda)\, y_{ij} \quad \ldots. (12)$$

and

$$T(X^1) = \sum_i \sum_j t^1_{ij}(\lambda)\, y_{ij} \quad \ldots. (13)$$

The solution X^2 that minimizes the total time is given by

$$x^2_{ijk} = \begin{cases} y_{ij} & k = k^2_{ij}(\lambda) \\ 0 & k \neq k^2_{ij}(\lambda) \end{cases} \quad , \forall i,j \quad \ldots. (14)$$

$$C(X^2) = \sum_i \sum_j c^2_{ij}(\lambda)\, y_{ij} \quad \ldots. (15)$$

and

$$T(X^2) = \sum_i \sum_j t^2_{ij}(\lambda)\, y_{ij} \,. \quad \ldots. (16)$$

From the above theorems, it follows that the set of efficient points (C X T) of problem (T) can be obtained by applying the following basic stages:

1) For each chosen value of λ we optimize the problem p' (λ).

2) Using theorem (3), the optimal solution of p(λ) corres ponding to that of p'(λ) is constructed.

3) If for any λ the optimal solution is not unique, we choose among the alternate optimal solutions a limit - cost solution X^c and a limit-time solution X^t such that $C(X^c) \leq C(X)$ and $T(X^t) \leq T(X)$, where X is any optimal solution to p (λ).

4) By theorem 1, the set of efficient points is the line segment connecting the limit-cost point $[C(X^c), T(X^c)]$ and the limit-time point $[C(X^t), T(X^t)]$.

(If the optimal solution is unique for any λ then the limit-cost and limit-time points coincide).

It is well known that any optimal solution to a linear programming problem is a convex combination of its basic optimal solutions. The following theorem uses this result to identify the limit-cost and limit-time solutions for any λ. Using this theorem we can implement the previous stage 3.

Theorem 4*

Let $Y_1, Y_2, \ldots, Y_g$ be the basic optimal solutions to p'(λ). Let $X_1^1, X_2^1, \ldots, X_g^1$ and $X_1^2, X_2^2, \ldots, X_g^2$ be the solutions constructed

* For the proofs of theorems (2)-(4) see reference (4).

corresponding to $Y_1, \dots Y_g$ using theorem (3). Then the limit-cost solution X_1 to $p(\lambda)$ is given by that solution which minimizes $C(X_i^1)$, $i=1,\dots,g$, and the limit-time solution X^2 is given by that solution which minimizes $T(X_i^2)$, $i = 1, \dots, g$.

II- A Method for Generating All the Alternate Basic Optimal Solutions to the Transportation Problem

Let us assume that the transportation problem; min.

$Z = \sum_i \sum_j c_{ij} x_{ij}$ subject to $\sum_i x_{ij} = b_i$, $\sum_j x_{ij} = a_i$,

$i= 1,\dots, m$, $j= 1, \dots, n$, and $x_{ij} \geqslant 0$, has more than one optimal solution. This can be recognized from the final optimal transportation tableau, when the coeffients of some of the nonbasic variables have the value zero. An indicator α of a basic optimal solution Y to the previous transportation problem is the set $(i_1, i_2, \dots, i_{m+n-1}) \subset IUJ$, where $i_1, i_2, \dots, i_{m+n-1}$ are the indices of the basic variables. Two indicators α_1, α_2 are said to be neighbouring indicators if all except one of the elements of α_1 and α_2 are the same. The neighbouring set of an indicator α consists of all neighbouring indicators to α and will be denoted by $G(\alpha)$. Let P denote the set of all indicators of the optimal basic solutions. The method is based on the use of two sets. The first is $S^r \subset P$, we call it the served set, and the second is $W^r \subset P$, we call it the waiting set. We start the process with an optimal basic solution Y_o. We store the indicator α_o associated with Y_o in the set $S^r, r=1$, i.e., $S^1=0$. We analyze the optimal transportation tableau corresponding to α_o to specify all the nonbasic variables of zero coefficients. If more than one neighbouring indicator of α_o will be created from the current tableau, we store all of them except α_o in the set $W^{r}, r=1$, i.e., $W^1 = G(\alpha_o) - \alpha_o$. If only one indicator is specified, say $/\alpha'/$ we calculate its corresponding solution and put α' in $S^{r}, r = 2$, i.e., $S^2 = \alpha_o \cup \alpha'$. We pick the last element stored in W^1, say/ α_1/ if more than one indicator have been stored in W^1, and calculate the associated basic optimal solution. We examine the optimal transportation tableau corresponding to α_1 in order to create the new indicators (if there is any) neighbouring to α_1. We update S^2 and W^2 by adding the new neighbouring indicators of α_1 to W^2 and removing α_1 from W^2 and adding it to S^2 so we get

$$S^2 = \alpha_o \cup \alpha_1 \quad \text{and} \quad W^2 = W^1 \cup G(\alpha_1) - S^2.$$

We pick the last indicator stored in S^2 and repeat the same process. At the g-th stage the waiting set W^g and served set S^g will be:

$$S^g = S^{g-1} \cup \alpha_{g-1} \quad \text{and} \quad W^g = W^{g-1} \cup G(\alpha_{g-1}) - S^g.$$

The process will terminate finitely when $W^r = \emptyset$. We now prove that all optimal basic solutions will have been found finitly when $W^r = \emptyset$. The proof runs as follows:

From the construction of S^g and W^g we have $S^g \cap W^g = \phi$. Since the number of basic optimal solutions is finite($\frac{(nm + n + m)!}{nm!\ (n + m)!}$ is an upper bound) and any indicator in W^g will sonner or later leave W^g and enter S^g then after a finite number of iterations W^r will be empty. The only statement to be proved is that $W^r = \phi$ implies $S^r = P$.

Let α_o be an element of P. Since we start with an optimal basic solution then $S^r \neq \phi$. Let α_1 be an element of $S^r \subset P$. Then we have a finite sequence of neighbouring indicators

$$\alpha_1 , \quad \alpha_2 , \quad \ldots , \quad \alpha_k = \alpha_o ,$$

such that

$$\alpha_{j+1} \in G(\alpha_j) \quad , \qquad j = 1,2, \ldots, k-1.$$

The existence of this sequence follows from the fact that any optimal basic solution to the transportation problem can be obtained from any other optimal basic solution by a finite number of iterations; this follows from the fact that the graph of the feasible region of any linear programming problem is connected. If some $\alpha_j \in S^r$, then it had entered S^r at some iteration $g_i \leq r$, say. Thus either α_{j+1} is an element of S^{g_i}, and hence $\alpha_{j+1} \in S^r$, or α_{j+1} is not an element of S^{g_i} and $\alpha_{j+1} \in G(\alpha_j)$ which implies $\alpha_{j+1} \in W^{g_i+1}$. Since $W^r = \Phi$, then α_{j+1} must have left W^r at some iteration between the $(g_i + 1)$-th and the r-th stage and entered the set S^{g_i+1} so that $\alpha_{j+1} \in S^r$. Thus in any case $\alpha_j \in S^r$ implies $\alpha_{j+1} \in S^r$. Since $\alpha_1 \in S^r$ we have by induction $\alpha_k = \alpha_o \in S^r$. Therefore $P \subseteq S^r$. But $S^r \subseteq P$. whence $P = S^r$.

The method presented in this section will be used to find the alternate basic optimal solutions required by theorem (4).

III Statement of The Algorithm And a Worked Example.

We combine the results of the previous two sections to provide the algorithm below. Applying the following steps, the set of efficient points of problem (I) can be located, and the optimal transportation routes, modes and shipping amounts can be determined.

Step 1. Let $\lambda = o$. Determine d_{ij} (o) and K_{ij} (o) by using equation (4).*
Using equation (9) - (10), find

$$k^1_{ij}\ (o),\ k^2_{ij}\ (o),\ c^1_{ij}\ (o),\ t^1_{ij}\ (o),\ c^2_{ij}\ (o),\ \text{and}\ t^2_{ij}\ (o),$$

for all $i \in I$ & $j \in J$. Solve the transportation problem $P'(o)$, i.e., (5) - (8), and find the basic optimal solution Y' .

Corresponding to Y' find the limit - cost solution X^1 by equation (11).

Find the limit-cost point $[C(X^1), T(X^1)]$ by (12) - (13). The efficient point in (CXT) space corresponding to $\lambda=0$ is given by $[C(X^1), T(X^1)]$ (Note that Y' is unique by assumption, thus the limit-cost and time points coincide for $\lambda=0$).

Step 2. Choose different values for λ in the interval (o,1). For each chosen value of λ, apply the following:

(i) Determine K_{ij} (λ) using equation (4) and evaluate k^1_{ij} (λ), k^2_{ij} (λ), and the corresponding c^1_{ij} (λ), t^1_{ij} (λ), c^2_{ij} (λ) and t^2_{ij} (λ) from from equations (9)-(10) for all i & j.

(ii) Solve the problem $P'(\lambda)$. There are two different cases:

a) If the basic optimal solution Y' to $P'(\lambda)$ is unique, then find the limit-cost solution X^1 using (11) and the limit-cost point $[C(X^1), T(X^1)]$ using (12)-(13). Similarly determine the limit-time solution X^2 using (14) and the limit-time point $[C(X^2), T(X^2)]$ using (14)-(15).

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* Since it is assumed that there are no two modes with the same C_{ijk}, then the set $K_{ij}(o)$ is a singleton.