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ITERATIVE PRICE AND QUANTITY DETERMINATION FOR SHORT-RUN PRODUCTION AND FOREIGN TRADE PLANNING

by

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Today, more than ever before in the history of science, theoretical formulation goes hand in hand with computational feasibility.

R. Bellman and S. Dreyfus

Iterative Price and Quantity Determination for Short-run Production and Foreign Trade Planning

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Economic interrelations of importance for short-run production and foreign trade planning may, as a first approximation [1], be described by a very large linear programming model. The great size of this model necessitates, if it should become managable for practical use, special analysis of its equational structure and exploitation of its special features. This study will be centered on the utilization of the structural properties of the equations describing

1) The author wishes to express his sincere gratitude to Dr. Alfred Zauberman, London School of Economics and Political Science, for stimulating discussions and proposals in regard to the elaboration of this mathematical appendix and to Dr. Salah Hamid, Director of the Operations Research Center of the Institute of National Planning, Cairo, for enabling the undertaking of this study as part of the center's research activities for the preparation of the Egyptian Five Year Plan.

the relationships of foreign trade and production variables.

This paper may be seen as a continuation of the discussion by W. Trzeciakowski, J. Mycielski, K. Rey, J. Głowacki, W. Piaszczyński in Poland, by A. Nagy, T. Liptak, A. Marton, M. Tardos in Hungary, by V. Pugachev, V. Volkonskij, Yu. Chernyak, A. Modin in the Soviet Union and by P. Pigot in France as well as of earlier contributions by the author. (Cf. par. 15 and literature references at the end.)

Unfortunately, an older generation of economists, both in the East and in the West, seems to have difficulty in understanding the importance of the challenges of these problems, and that mathematical and computational analyses will become as important to the economist in the future as it is today to the mechanist or the physicist.

A Survey of the Model Analyzed

The table at the end of this mathematical appendix may be unfolded during the reading.

1. Variables

Quantity variables are denoted as follows:

Production levels of various industrial branches (i) with internal procerses (j) are denoted by the vectors x_i (with elements x_{ij}).	x_i (i=0,1, ,m)

Export and import variables are denoted by the vectors y_i, each of which embraces certain commodity numbers and the relevant markets for the commodities in question (thus similarly with elements y_{ij}) y_i $(i=0,1, ,m)$

Price variables are denoted as follows:

The feasible prices of the various foreign currency resources are denoted by the vector v (with elements j) v

The feasible prices of the various commodities used or produced by the industrial branches (i) are denoted by the vectors u_i (with elements u_{ij}) u_i $(i=0,1, ,m)$

The feasible prices of the production capacity vectors (cf. $-\bar{x}_i$ in par. 3 below) are denoted by k (with elements k_{ij}) k_i $(i=0,1, ,m)$

The feasible prices of the export and import constraint vectors (cf. $-\bar{y}_i$ in par. 3 below) are denoted by h_i (with elements h_{ij}) h_i $(i=0,1, ,m)$

The iteration is denoted by the variables r, s, t. Subiterations (s) within an iteration (r) by rs, etc.

Auxiliary variables in master (i.e. coordinating) problems are denoted by z. Though the same name (z) is used in various masters they are not identical. z_r, z_{irs}, z_t etc.

2. Equations or Inequality Constraints

The balance of payments constraints

$$(1) \quad C_o y_o + \ldots + C_i y_i + \ldots + C_m y_m = bp$$

where $C_i (i=0,1, \ ,m)$ are matrices of the foreign prices obtained or paid for export or import quantities to various markets (and possibly including certain conditions for the commodity structure as determined by trade agreements). bp is the vector of net requirement of foreign currency holdings (and possible of the trade composition). Its elements are bp_j.

The commodity balances which state that import - export + production - use in production should equal the requirement vectors b_i with elements b_{ij}:

$$(2) \quad B_{oo} y_o \qquad + A_{o1} x_1 + \ldots + A_{oi} x_i + \ldots + A_{om} x_m + A_{oo} x_o = b_o$$

$$B_{11} y_1 \qquad + A_{11} x_1 \qquad + A_{1o} x_o = b_1$$

$$
\begin{array}{llll}
B_{ii}y_i & +A_{ii}x_i & & +A_{io}x_o=b_i \\
B_{mm}y_m & & +A_{mm}x_m & +A_{mo}x_o=b_m
\end{array}
$$

The important assumption has been made here that the production structure may be characterized by the following four features:

i) Certain commodities, e.g. labour, electricity and water, are inputs to or outputs from all branches of production, as depicted by the first row in (2) of the o-group of equations.

ii) Certain production processes require inputs from almost all branches of production, as may be the case for the chemical industry. Such production processes are grouped together in the penultimate column of the x_o activities.

iii) Except for the common input or output commodities defined above and those processes which use inputs from almost all industrial branches, the industries are supposed to be groupable in branches which only use or produce mutually exclusive groups of commodities, e.g. the textile industries producing only commodities belonging to a "textile" commodity group, the mechanical industry only those belonging to a "mechanical" group as depicted by the matrices A_{ii}.

iv) Special constraints on the export and import variables such as balance of payments (and possibly on the balancing of certain commodities as determined by trade agreements) are supposed to be included in the matrices C_i.

3. Bounds

The production level vectors have to be within the capacity bounds

(1) $0 \leq x_i \leq \bar{\bar{x}}_i$ (i=0,1, ,m)

Similarly, the export and import vectors have to be within their corresponding marketing bounds

(2) $0 \leq y_i \leq \bar{\bar{y}}_i$ (i=0,1, ,m)

We wish to state all the conditions of the original problem in the form of $\geq$ or = as we will then obtain all feasible price solutions to the corresponding dual as non-negative magnitudes. We therefore multiply the right hand part of the above conditions by (-1) and get

(3) $-x_i \geq -\bar{\bar{x}}_i$ (i=0,1, ,m)

and

(4) $-y_i \geq -\bar{\bar{y}}_i$ (i=0,1, ,m)

4. Preference Function

The preference function is formally defined by the expression

(1) Min $g_o y_o + \dots + g_i y_i + \dots + g_m y_m + f_1 x_1 + \dots + f_i x_i + \dots f_m x_m + f_o x_o$

No discussion is made on the actual coefficients.

5. Summary of the Model

$$g_o y_o + g_1 y_1 + \ldots + g_i y_i + \ldots + g_m y_m + f_1 x_1 + \ldots + f_i x_i + \ldots + f_m x_m + f_o x_o = \text{Min}$$

$$C_o y_o + C_1 y_1 + \ldots + C_i y_i + \ldots + C_m y_m = bp$$

$$B_{oo} y_o + A_{o1} x_1 + \ldots + A_{oi} x_i + \ldots + A_{om} x_m + A_{oo} x_o = b_o$$

$$B_{11} y_1 + A_{11} x_1 + A_{1o} x_o = b_1$$

$$B_{ii} y_i + A_{ii} x_i + A_{io} x_o = b_i$$

$$B_{mm} y_m + A_{mm} x_m + A_{mo} x_o = b_m$$

$$-y_o \geqq -\bar{y}_o$$

$$-y_1 \geqq -\bar{y}_1$$

$$-y_i \geqq -\bar{y}_i$$

$$-y_m \geqq -\bar{y}_m$$

$$-x_1 \geqq -\bar{x}_1$$

$$-x_i \geqq -\bar{x}_i$$

$$-x_m \geqq -\bar{x}_m$$

$$-x_o \geqq -\bar{x}_o$$

$$x_i \geqq 0, \; y_i \geqq 0 \qquad (i = 0,1, \; ,m)$$

6. The Dual Formulation

Instead of considering the original formulation, it will be useful at various calculation stages to deal with the dual:

$$+ b_o'u_o + b_1'u_1 + \ldots + b_i'u_i + \ldots + b_m'u_m - \bar{y}_o'h_o - \bar{y}_1'h_1 - \ldots - \bar{y}_i'h_i - \ldots - \bar{y}_m'h_m - \bar{x}_1'k_1 - \ldots - \bar{x}_i'k_i - \ldots - \bar{x}_m'k_m - \bar{x}_o'k_o = \text{Max}$$

$$A_{oo}'u_o + A_{1o}'u_1 + \ldots + A_{io}'u_i + \ldots + A_{mo}'u_m - k_o \leq f_o'$$

$$A_{om}'u_o + A_{mm}'u_m - k_m \leq f_m'$$

$$A_{oi}'u_o + A_{ii}'u_i - k_i \leq f_i'$$

$$A_{o1}'u_o + A_{11}'u_1 - k_1 \leq f_1'$$

$$v + B_{mm}'u_m - h_m \leq g_m'$$

$$v + B_{ii}'u_i - h_i \leq g_i'$$

$$v + B_{11}'u_1 - h_1 \leq g_1'$$

$$v + B_{oo}'u_o - h_o \leq g_o'$$

$$v \geq 0$$

$$u_i \geq 0\,,\; h_i \geq 0,\; k_i \geq 0 \qquad (i=0, 1, \;\; , m)$$

(' denotes transposition of a vector or a matrix)

7. Parameters of Action

will in the following be both the quantity variables x_i, y_i (i=0,1, ,m) and the price variables v , u_i, h_i and k_i (i=0,1, ,m), as well as the auxiliary variables z_r, z_{irs}, etc.

8. A Graphic Picture

of the equation system, the variables, constants, prices and preference coefficients is given in Table 4 at page 56.

The pluses and minuses denote +1 and −1, respectively. The reformulated bounds (cf. 3.3, 3.4) are found in the lower half of the table.

9. Main Principles of Solution

As the foreign trade and production variables are subject to very different structural constraints, they should be treated as being qualitatively different, and different methods of calculation should consequently be employed in solving them.

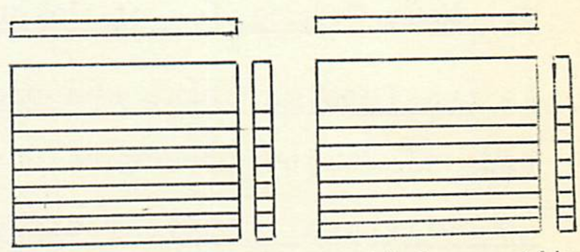
This will be an important theme of the following exposition.

Another, will be that of breaking the problem into smaller more rapidly solved subproblems, the solution of which are coordinated at various levels.

10. A General Survey of the Iterative Procedure

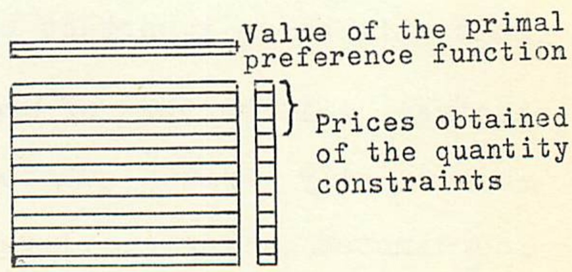
An attempt will now be made to give a birds-eye view of the general course of the solution process. The linear programme will be symbolized by a large rectangle together with two narrow ones. If some quantity solution is inserted and leads to the fulfilment of some equations the satisfied equations are indicated by horizontal lines. If some price solution to the dual problem is inserted, the satisfied price equations (columns) of the dual are indicated by vertical lines.

1. Various known quantity solutions satisfying part of the equations (shown by lines in the figures) are combined

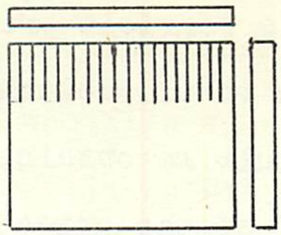


Partial quantity solution 1   Partial quantity ...solution r

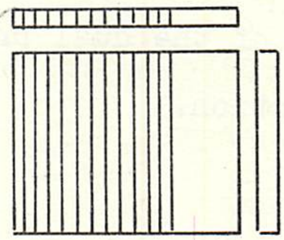
2. to a quantity solution satisfying all quantity equation (rows). As a result prices of the earlier unfulfilled equations are determined as is also an estimate of the minimum of the primal preference function.



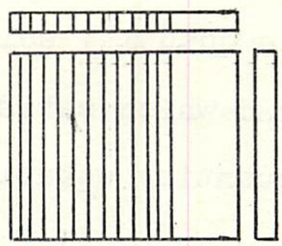
3. The feasible prices obtained are inserted in the dual problem



4. and a feasible price solution of the left columns is obtained. (The price equations of the right columns have been temporarily disregarded as they are difficult to satisfy).



5. This feasible price solution to the left columns is combined with earlier known price solutions which likewise satisfy only the left columns



Partial price solution 1

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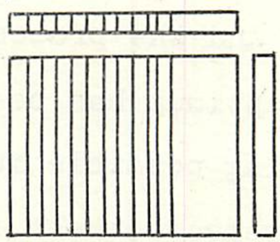
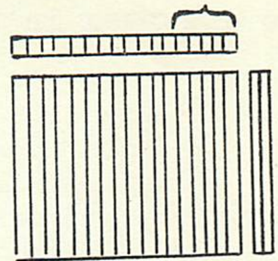


Partial price solution r

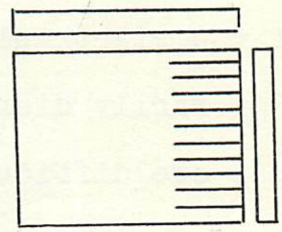
6. to give a feasible price solution to all columns. As a result we obtain the "prices" of the price equations, i.e., quantities as well as an estimate of the maximum of the dual preference function.

Quantities of price constraints obtained

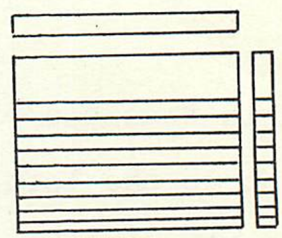


Value of the dual preference function

7. These quantity solutions are inserted in the right part of the quantity equations and



8. feasible quantity solutions are found which satisfy the lower quantity equations.



Partial quantity solution r+1

RETURN TO 1. The partial quantity solution is combined with the earlier known ones and the process repeated until the optimum has been found or the remaining possible improvement i.e. the difference between the minimum (cf. point 2) and the maximum estimates (cf. point 6) is less than a certain tolerance.

11. The Favourable Properties of Some Subproblems Involving B_{ii} Matrices

The way in which we will solve the general problem will partly be based upon exploiting the favourable structure of the B_{ii} matrices. We will mainly have to deal with two different types of subproblems involving these matrices.

The first of these we may name

11.A. The Export and Import Quantity Problem

which will be of the type

(1)
$$\begin{aligned} \text{Min} \quad & gy \\ Cy &= bp \\ By &= b \\ -y &\geq -\bar{y} \\ y &\geq 0 \end{aligned}$$

In dealing with a problem of this type, we introduce the indexes c for commodity, d for incompletely convertible currency block or district and a for trade activity, i.e. either export (E) or import (I).

The price of commodity c in incompletely inconvertible currency block or district d is defined as p_{cdE} (≥ 0) for exports and p_{cdI} (≤ 0) for imports.[1)]

The problem may now be restated as

$$(2) \quad \text{Min} \sum_{c} \sum_{d} \sum_{a} g_{cda} \; y_{cda}$$

$$\sum_{c} \sum_{a} p_{cda} \; y_{cda} = bp_{d} \qquad (d=\text{all})$$

$$\sum_{d} \sum_{a} \text{sign}(a) \; y_{cda} = b_{c} \qquad (c=\text{all})$$

$$-y_{cda} \geq -\bar{y}_{cda} \qquad (c,d,a=\text{all})$$

$$y_{cda} \geq 0 \qquad (c,d,a=\text{all})$$

$$\text{sign}(a) = \begin{vmatrix} - \\ + \end{vmatrix} \text{if } a = \begin{vmatrix} E \\ I \end{vmatrix}$$

An illustration of this type of problem is given in table 1.

---

1) Individual constraints tying the export of one commodity with the import of another commodity agreed upon in bilateral trade agreements may be accounted for by a slight revision of the problem 11:2, the master 11:3 and the preference function of the subproblem 11:4, but may probably be better handled by being included in the subproblem 11:4 and devising a special algorithm for its solution, which would differ from the one presented in par. 11.A.

Table 1. Typical structure of the export and import quantity problem.

Comment		g_{11E}	g_{12E}	g_{14E}	g_{21I}	g_{22I}	g_{23I}	g_{25I}	g_{51I}	g_{53E}	g_{53I}	g_{55E}	g_{64E}	g_{64I}		
Incompletely	v_1	p_{11E}			p_{21I}				p_{51I}						=	bp_1
convertible	v_2		p_{12E}			p_{22I}										bp_2
currency	v_3						p_{23I}			p_{53E}	p_{53I}					bp_3
blocks.	v_4			p_{14E}									p_{64E}	p_{64I}		bp_4
	v_5							p_{25I}				p_{55E}				bp_5
Commodity																
only exported	u_1	-1	-1	-1											=	b_1
only imported	u_2				+1	+1	+1	+1								b_2
not subject to	u_3															0
foreign trade	u_4															0
both imported	u_5								+1	-1	+1	-1				b_5
and exported	u_6												-1	+1		b_6
	h_{11E}	-1													≥	$-\bar{y}_{11E}$
	h_{12E}		-1													$-\bar{y}_{12E}$
	h_{14E}			-1												$-\bar{y}_{14E}$
	h_{21I}				-1											$-\bar{y}_{21I}$
	h_{22I}					-1										$-\bar{y}_{22I}$
	h_{23I}						-1									$-\bar{y}_{23I}$
	h_{25I}							-1								$-\bar{y}_{25I}$
	h_{51I}								-1							$-\bar{y}_{51I}$
	h_{53E}									-1						$-\bar{y}_{53E}$
	h_{53I}										-1					$-\bar{y}_{53I}$
	h_{55E}											-1				$-\bar{y}_{55E}$
	h_{64E}												-1			$-\bar{y}_{64E}$
	h_{64I}													-1		$-\bar{y}_{64I}$
		y_{11E}	y_{12E}	y_{14E}	y_{21I}	y_{22I}	y_{23I}	y_{25I}	y_{51I}	y_{53E}	y_{53I}	y_{55E}	y_{64E}	y_{64I}		

The problem (11:2) may be solved by making the preference function and the currency equations into a master of the type [1])

(3) $$\text{Min } \left(\sum_{c} \sum_{d} \sum_{a} g_{cda} \; y^{r}_{cda} \right) z_r$$

$$\left(\sum_{c} \sum_{a} p_{cda} \; y^{r}_{cda} \right) z_r = bp_d \qquad (d=all)$$

$$\sum_{r} z_r = 1$$

$$z_r \geqq 0 \qquad (r=all)$$

and the subproblem

(4) $$\text{Min } \sum_{c} \sum_{d} \sum_{a} (g_{cda} - p_{cda} \, v_d) \; y_{cda}$$

$$\sum_{d} \sum_{a} \text{sign}(a) \; y_{cda} = b_c \qquad (c=all)$$

$$-y_{cda} \geqq -\bar{y}_{cda} \qquad (c,d,a=all)$$

$$y_{cda} \geqq 0 \qquad (c,d,a=all)$$

$$\text{sign}(a) = \begin{vmatrix} - \\ + \end{vmatrix} \quad \text{if} \quad a = \begin{vmatrix} E \\ I \end{vmatrix}$$

v_d being the iterative or shadow price of the dth resource of the master 11:3.

1) A more effective formulation of this master problem will be considered in paragraphs 12 and 14.

The last minimization problem (4) may be separated into as many independent minimization problems as there are commodities. Every one of these problems will contain only one constraining equation and be of the type:

$$(5)\quad \text{Min} \sum_d \sum_a g^*_{cda}\, y_{cda} \qquad (g^*_{cda} = g_{cda} - p_{cda} v_d)$$

$$\sum_d \sum_a \text{sign}(a)\, y_{cda} = b_c$$

$$-\,y_{cda} \geq -\bar{y}_{cda} \qquad (d, a = \text{all})$$

$$y_{cda} \geq 0 \qquad (d, a = \text{all})$$

$$\text{sign}(a) = \left| \begin{matrix} - \\ + \end{matrix} \right| \text{ if } a = \left| \begin{matrix} E \\ I \end{matrix} \right|$$

This subproblem embracing only one commodity may be readily solved by means of the following algorithm, in the description of which partial use of the international algorithmic language ALGOL[1] has been made.

For brevity of exposition we have omitted reference at most places to the commodity index c.

Step 0. As we want to minimize the operations necessary to carry the y^r_{cda} solution into the master (11:3), we once for all calculate the value of the following terms for all commodities (the index c is not indicated) and for all markets (the index variable d):

---

1) For readers not familiar with ALGOL we point out, that a variable index appears within a parenthesis (e.g. by $e_E(d)$ we simply mean e_{dE}), that := means substitution i.e. ←, that begin and end work much like very large parentheses and that : is used to indicate a place (label). For particulars cf. the primer by Daniel D. McCracken (12).

$e_E(d) := g_E(d) \times \bar{y}_E(d);$ $\quad e_I(d) := g_I(d) \times \bar{y}_I(d);$

$q_E(d) := p_E(d) \times \bar{y}_E(d);$ $\quad q_I(d) := p_I(d) \times \bar{y}_I(d);$

and define $e_E(E(0)) := e_I(I(0)) := q_E(E(0)) := q_I(I(0)) := 0;$

$$g_E^*(E(0)) := g_I^*(I(0)) := \text{large negative number};$$

The effect in the master problem (11:3) of the y^r_{cda} solution of subproblem (11:4) will be defined by the vector [1)]

$$(6) \qquad V^r = \begin{bmatrix} \sum_c \sum_d \sum_a g_{cda}\, y^r_{cda} \\ \sum_c \sum_a p_{cda}\, y^r_{cda} \\ 1 \end{bmatrix}$$

<u>Step 1.</u> Before every solution (r) of the subproblem (11:4) the vector V^r has to be set equal to zero except for its unit element.

<u>Step 2.</u> Thereafter we do the following procedure for all the commodities of the subproblem (11:4):

---

1) It will probably to suitable to let this vector have one redundant element as has been done at page 20,cf. the footnote at the same page.

Step 2A. For the commodity c

the names of the export markets or districts are arranged according to rising $g^{*}_{E}(d)$. As a result we get the export order vector E. The consequtive terms of E indicate, which export market is the best, next best, and so on to the worst. The names of the import markets or districts are similarly arranged in rising $g^{*}_{I}(d)$ order, and as a result we get the import order vector I.

The consequtive terms of I will also give the name of the best, the next best and so on to the worst import market.

Step 2B. Thereafter the following few operations are done which aim at pairing favourable export opportunities or domestic requirements with import opportunities or available supplies.

The local[1] integer variables i and j will be used for indexing. The cumulative sum of export and import bounds will be denoted by the real variables $S\bar{y}_E$ and $S\bar{y}_I$.

---

1) A local variable is one which is only defined for a certain part of the algorithm and though it may have the same name as some variable used in other parts (e.g. i, j in par. 2) is not identical with the latter.

```
begin
   comment: neither net export nor net import (b=0);
   i:=j:=1; Sȳ_E := ȳ_E(E(1)); Sȳ_I := ȳ_I(I(1));
   comment: net export;
   if b < 0 then begin j := 0; Sȳ_I := -b; goto L end
   else
   comment: net import;
   if b > 0 then begin i := 0; Sȳ_E := b; goto L end;
   comment: no remaining profitable re-export activities;
   if g*_E(E(i)) ≥ - g*_I(I(j)) then goto READY;
L: | if Sȳ_I ≥ Sȳ_E then
   | begin comment: cf. footnote;
   |    V(0):=V(0)+e_E(E(i));V(E(i)):=V(E(i))+q_E(E(i));i:=i+1;
   |    if g*_E(E(i)) < -g*_I(I(j)) then
   |    begin Sȳ_E := Sȳ_E + ȳ_E(E(i)); goto L end else
        begin y := Sȳ_E - Sȳ_I + ȳ_I(I(j)); V(0) :=V(0)+y×g_I(I(j));
        V(I(j)) :=V(I(j)) + y×p_I(I(j)) end
   end
```

1) The element of V corresponding to V(E(0)) and V(I(0)) is redundant for the solution of the master (11:3) and only used to permit the same algorithm being used for various values of the commodity constraint b.

```
else if Sȳ_I < Sȳ_E then
begin
   V(0):=V(0)+e_I(I(j));V(I(j)):=V(I(j))+q_I(I(j)); j:=j+1;
   if g*_E(E(i)) < -g*_I(I(j)) then
   begin Sȳ_I :=Sȳ_I + ȳ_I(I(j)); goto L end else
   begin y:=Sȳ_I-Sȳ_E +ȳ_E(E(i)); V(0):=V(0)+y×g_E(E(i));
   V(E(i)) := V(E(i)) + y × p_E(E(i)) end
end
else comment: Sȳ_I = Sȳ_E ;
begin
   V(0):=V(0)+e_E(E(i))+e_I(I(j));V(E(i)):=V(E(i))+q_E(E(i));
   V(I(j)) := V(I(j)) + q_I(I(j)); i := i+1; j :=j+1;
   if g*_E(E(i)) < -g*_I(I(j)) then
   begin Sȳ_E := Sȳ_E+ȳ_E(E(i)); Sȳ_I :=Sȳ_I+ȳ_I(I(j));goto L end
end;

READY:end
```

In general, either of the two parts of the algorithm indicated by vertical lines has to be run through, though <u>only</u> as many times as there are export and import bounds that will become active. Very few operations, all of which are simple additions and testings have to be made during these runs.

The total number of export and import bounds will <u>only</u> affect the necessary number of multiplications to obtain the g^{*}_{cda} terms and the sorting required to get the export and import order vectors in step 2A. This work may be brought to a minimum by using our knowledge of the approximative (i.e. the previous) order of the g^{*}_{da} terms.

11.B The Export and Import Price Problem

The second problem involving B_{ii} matrices which we will have to deal with is of the type

$$(7) \qquad \text{Max } b'u - \bar{y}'h$$

$$B'u - h \leq g$$

$$u, h \geq 0$$

in which u is the vector of prices of the commodities and h of the export and import bounds in the various foreign markets. The detailed structure of this problem is illustrated in the table below.

Table 2. Typical structure of the export and import price problem.

(Commodities 3 to 6 not subject to foreign trade.)

Max	b_1	b_2					b_7	$-\bar{y}_{11I}$	$-\bar{y}_{12E}$	$-\bar{y}_{13I}$	$-\bar{y}_{14I}$	$-\bar{y}_{21E}$	$-\bar{y}_{22E}$	$-\bar{y}_{71I}$	$-\bar{y}_{74I}$		
y_{11I}	+1							-1								$\leq$	g_{11I}
y_{12E}	-1								-1								g_{12E}
y_{13I}	+1									-1							g_{13I}
y_{14I}	+1										-1						g_{14I}
y_{21E}		-1										-1					g_{21E}
y_{22E}		-1											-1				g_{22E}
y_{71I}							+1							-1			g_{71I}
y_{74I}							+1								-1		g_{74I}
	u_1	u_2					u_7	h_{11I}	h_{12E}	h_{13I}	h_{14I}	h_{21E}	h_{22E}	h_{71I}	h_{74I}		

It is evident that this maximization problem may be separated into one for every u_c (c being the commodity index). There will thus be as many independent maximization problems as there are commodities. Every one of these will be of the type (omitting the commodity index c):

$$(8) \quad \text{Max} \quad bu - \sum_d \sum_a \bar{y}_{da} \, h_{da}$$

$$\text{sign}(a) \; u \quad - h_{da} \leq g_{da} \qquad (d, a = \text{all})$$

$$u, \; h_{da} \geq 0 \qquad (d, a = \text{all})$$

$$\text{sign}(a) = \begin{pmatrix} - \\ + \end{pmatrix} \text{ if } a = \begin{pmatrix} E \\ I \end{pmatrix}$$

u = a single variable, corresponding to the price of one commodity, h_{da} = the price of the export or import bound on this commodity in market d.

If we transfer the u variable to the right side and multiply by -1 we get the identical problem

$$(9) \quad \text{Max} - \sum_d \sum_a \bar{y}_{da} \, h_{da} + bu$$

$$h_{da} \geq - g_{da} + \text{sign}(a) \; u \qquad (d, a = \text{all})$$

$$h_{da} \geq 0 \qquad (d, a = \text{all})$$

$$u \geq 0$$

For any fixed value of u it will always be best to select the lowest possible value of h_{da} which fulfills the two inequalities above, which is

$$(10) \qquad h_{da} = \text{Max}\,(-g_{da} + \text{sign}(a)\,u\,;\,0) \qquad (d,a=\text{all})$$

Thus we may insert these expressions (11:10) for h_{da} in the preference function and get an expression in the sole variable u

$$(11) \quad z(u) = \text{Max} \; - \sum_{d} \sum_{a} \bar{y}_{da} \,(\text{Max}\,(-g_{da} + \text{sign}(a)\,u;\,0)) + bu$$

$$\text{sign}(a) = \begin{pmatrix} - \\ + \end{pmatrix} \text{ if } a = \begin{pmatrix} E \\ I \end{pmatrix}$$

$$u \geq 0$$

In maximizing this function it will be of importance to know for which ranges of u the expressions $\text{Max}(-g_{da} + \text{sign}(a)\,u;\,0)$; when $u \geq 0$; begin to become greater than 0. These ranges of u are given in the table below.

Table 3. Values of u for which the expression $\text{Max}\,(-g_{da} + \text{sign}(a)\,u;0)$ begins to differ from zero.

Sign of $(-g_{da})$	Sign(a)	Name of the combination of signs	Range of u
+	+	K_1	$u \geq 0$
−	+	K_2	$u \geq g_{dI}$
+	−	K_3	$0 \leq u \leq -g_{dE}$
−	−	K_4	the expression equals zero for all values of u

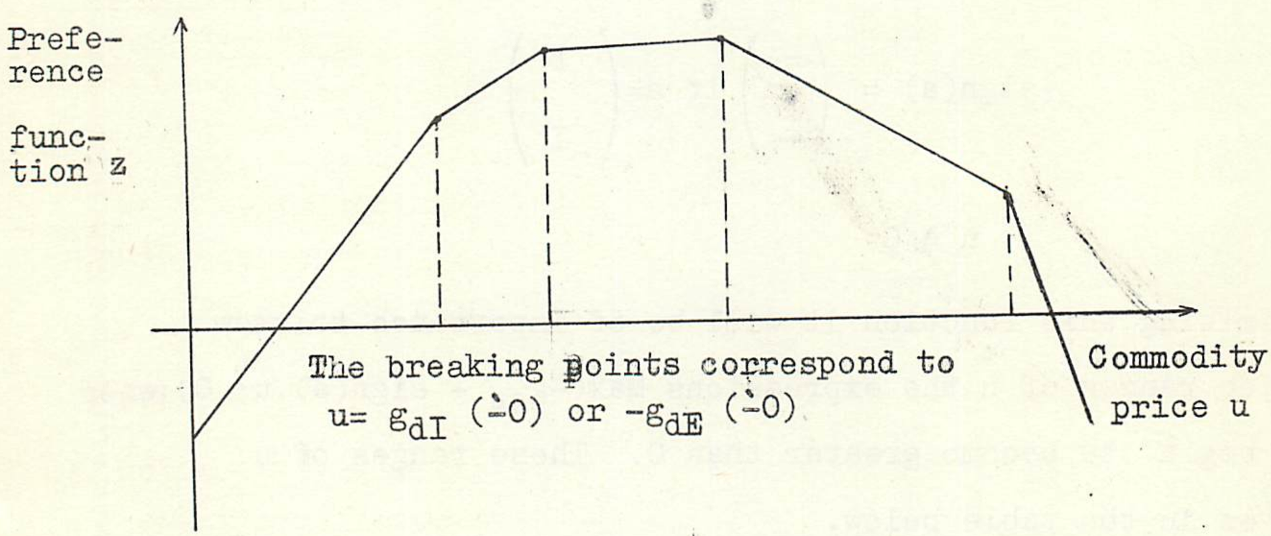
The g_{da} values which are critical are thus

$$(12) \qquad g_{dI} \doteq 0$$

$$-g_{dE} \doteq 0$$

If we order these values, we will get the consequtive points at which the preference function will change its direction.

Fig. 1 The typical outlook of the preference function as a function of u



An important question is whether this function has one or several maxima, as it will influence the number of points u for which we have to calculate its value.

We therefore separate the terms of (11:11) into the four groups of (11:13) defined in table 3 by the indexes K_1, K_2, K_3, K_4

and get

(13) $z(u) =$

	Comments
$= - \sum_{K_1} \bar{y}_{K_1}(-g_{K_1}) - \underline{\sum_{K_1} \bar{y}_{K_1} u} -$	the terms always included and the variable terms always decreasing for increasing u;
$- \underline{\sum_{K_2 \in (u \geq g_{K_2})}} \bar{y}_{K_2}(u - g_{K_2}) -$	the terms are introduced for $u \geq g_{K_2}$ and are <u>always decreasing</u> for increasing u;
$- \sum_{K_3 \in (u \leq -g_{K_3})} \bar{y}_{K_3}(-g_{K_3}) + \underline{\sum_{K_3 \in (u \leq g_{K_3})} \bar{y}_{K_3} u} -$	the variable terms <u>are increasing when u changes from 0 to</u> $-g_{K_3}$, the sum of each corresponding constant and variable term approaches 0 as u approaches the corresponding $-g_{K_3}$ value and assumes the value 0 for $u \geq -g_{K_3}$;
$- \sum_{K_4} \bar{y}_{K_4} \cdot 0 +$	the terms are always zero for the permitted non-negative range of u ;
$+ \underline{b\,u}$	the term is <u>always increasing or decreasing</u> depending on the sign of b

We note that no new increasing term is ever introduced after u has become greater than zero, the increasing terms are switched off gradually (for u assuming the values $-g_{K_3}$) and new decreasing terms are switched on gradually (for u assuming the values g_{K_2}). This mean that once z has begun to decrease it may <u>never</u> begin to rise again. To find the maximum of z we have therefore only to determine for which g_{da} value the derivative in regard to u changes from positive to negative, or between which g_{da} values the derivative equals zero. Thus for some value $u = g_{da}$ we have only to form the sum

$$(14)\quad \frac{dz}{du} = - \sum_{K_1} \bar{y}_{K_1} - \sum_{K_2 \epsilon (u \geq g_{K_2})} \bar{y}_{K_2} - \sum_{K_3 \epsilon (u \leq -g_{K_3})} \bar{y}_{K_3} + b$$

and note for which value of u it changes sign or becomes zero. We may start with the correct expression for some arbitrary value of u and increase or decrease the value of u to the nearest g_{da} if the expression is positive (or negative).

When the optimal value of u (or range of values) has been determined the other iterative prices h_{da} are easily obtained from the expressions of (11:10):

$$(15)\quad h_{da} = \text{Max}\,(-g_{da} + \text{sign}(a)\, u \;;\; 0) \qquad (d,a = \text{all})$$

Thus the determination of the iterative prices of the sub-problem dealt with may be done in a very swift way.

12. The Decomposition of the Short-run Production and Foreign Trade System.

12.1. Combination of partial solutions (y^r, x^r) to enable the fulfillment of all quantity equations.

Knowing various sets (r) of feasible x, y solutions to all bounds and all equation groups except the balance of payment and the o-group, we attempt to find a combination of solution vectors which satisfies all constraints. In principle we are interested in solving the master:

(1) $$\text{Min } (gy^r + fx^r)\, z_r$$

$$(Cy^r + Ox^r)\, z_r = bp \qquad \text{(O=a zero matrix)}$$

$$(B_o y^r + A_o x^r) z^r = b_o \qquad (B_o = B_{oo} + \text{ a zero matrix})$$

$$\sum_r z_r = 1$$

$$z_r \geq 0 \qquad \text{(r=all)}$$

In practice we will use a slightly more sophisticated formula, which may be expected to be more efficient as it solves the problem (12:1) i) in several stages; and ii) for a given number of auxiliary variables z increases the range of possible combinations though at the cost of increasing the number of equations.

12.1A. The $(x_i, y_i)^r$ sets are first combined to satisfy the o-group of equations

$$(2)\ \mathrm{Min}((g_o - vC_o)y_o^t)z_t + \sum_{i=1}^{m}((g_i - vC_i)y_i^{rs} + f_i x_i^{rs})z_{irs} + (f_o x_o^r)z_r$$

$$(B_{oo}y_o^t)z_t + \sum_{i=1}^{m}(A_{oi}x_i^{rs})z_{irs} + (A_{oo}x_o^r)z_r$$

$$\sum_t z_t = 1$$

$$\sum_r z_r = 1$$

$$\sum_s z_{irs} = z_r \qquad (i=1,2, \ ,m;\ r = \text{all})$$

$$z_t, z_r, z_{irs} \geq 0 \qquad (i=1,2, \ ,m; t,r,s = \text{all})$$

This formulation will permit us to use one solution x_o together with many different solutions of each x_i group and with various y_o solutions. This is of great importance as the number of equations of the o-group which have to be satisfied may be supposed to be fairly large(for instance, of the order of 200). If we have one x_o solution and, for instance, 5 solutions of each x_i group and 10 to the y_o group, we will have obtained 1+50x 5+10 = 261 vectors fairly easily. These may be combined to satisfy the 200+1+50+1 = 252 equations.

It should be noted that this introduction of additional equations is not necessary from a formal point of view. The same over all solution may be obtained by making all possible extreme combinations of group solutions and including the corresponding column in the master:

$$(3) \qquad \text{Min}\,((g - vC)y^r + fx^r)z_r$$

$$(B_o y^r + A_o x^r)z_r = b_o$$

$$\sum_r z_r = 1$$

$$z_r \geq 0$$

An equivalent programme would then consist of $5^{50} \times 10 \approx 10^{36}$ variables and 200 + 1 equations.

If the computational work required to solve a linear programme increases approximately according to the formula

(4) $T_{lp} = m^2 \times n \times \text{constant}$

in which m equals the number of equational constraints and n the number of variables, we may expect the following total computational time for solving (12:2)

(5) $252^2 \times 261 \times \text{constant} \approx$ the order of 10^7

while for the equivalent formulation of (12:3) we might expect

(6) $201^2 \times 10^{36} \times \text{constant} \approx$ the order of 10^{41}

The formulation (12:2) may therefore be expected to be more effective than (12:3).

12.1B The resulting (y,x) sets are then combined to satisfy all quantity equations.

Having found a solution y,x ($= y_o^t\, z_t,\ y_i^{rs}\, z_{irs},\ x_i^{rs}\, z_{rs}, x_o^r\, z_r$) which satisfies all constraints except the balance of payment ones, we wish to find one which will also satisfy the latter. In principle we do it by solving a still higher master

(7) $$\mathrm{Min}(gy^r + fx^r)\, z_r$$

$$(Cy^r + 0x^r) z_r = b_p$$

$$\sum_r z_r = 1$$

$$z_r \geq 0$$

In solving the previous master (12:2) as many of the z_r, z_{irs}, z_t will differ from 0 as there were equations. The various possible y_i vectors would in turn be a combination of the original y_i^r vectors multiplied by the non-negative z_{irs} terms and added together. If the master (12:2) embraced a great number of equations, it may perhaps be more effective to establish the net bounds for the $B_{ii} y_i$ problems, and solve them anew for given quantities, i.e. solve the m + 1 problems:

(8) $$\mathrm{Min}\ (g_i - vC_i) y_i$$

$$B_{ii}\, y_i = b_i - A_i\, x$$

$$-y_i \geq -\bar{y}_i$$

$$y_i \geq 0$$

(x being a solution satisfying the (12:2) master) using the procedure dealt with in (11:4) and sequel.

In analogy with (12:2) we solve the master (12:7) by formulating it as

$$\text{(9)} \qquad \text{Min} \sum_{i=o}^{m} (C_i y_i^{rs}) z_{irs} + (fx^r) z_r$$

$$\sum_{i=o}^{m} (C_i y_i^{rs}) z_{irs} = b_p$$

$$\sum_{r} z_r = 1$$

$$\sum_{s} z_{irs} = z_r \qquad (i,r=\text{all})$$

$$z_r,\ z_{irs} \geq 0 \qquad (i,r,s=\text{all})$$

(x^r being a feasible solution to the (12:2) master)

It should be noted that <u>the partition into subproblems (i) need not be identical with the earlier partition used in respect to production and may vary from iteration to iteration.</u> We may also iterate between 12:9 and 12:8 as many times we like, and in doing that we are free to choose a finer or coarser subproblem division in order to obtain the optimal <u>structure</u> and <u>size</u> of subproblems of export and import quantities as well as the <u>swiftest routing</u> of their solution (cf. par. 14). The purpose of the earlier detailed investigation into the properties of a simple export and import quantity problem for <u>fixed</u> export or import quantities, should now be evident.

12.2 Feasible prices v, u_o obtained from the highest quantity masters.

As a result of solving (12:9) we have obtained the value of the original preference function, feasible prices v of the rows bp, from the renewed solution of (12:2) feasible prices u_o of the rows b_o .

12.3-4 Calculation of u_i, h_i and k_i on the basis of given v, u_o

Our attention is now turned to the dual problem.

The selected v , u_o are feasible price solutions to part of the price equations. In principle we are now interested in complementing them by feasible $u_1, \ldots, u_m$; $h_o, \ldots, h_m$ and $k_1, \ldots, k_o$ in such a way that the dual preference function (cf. 6:1) is maximized.

In other words, we wish to obtain an improved solution to the dual problem (6:1), using our knowledge of earlier partly feasible price solutions and the newly attained price solutions v, u_o .

As the number of equations is supposed to be extremely large, we will gain by solving the u_i, h_i and k_i in a two or three stage process, which will mainly depend upon the form of the matrix A_{ii}.

In principle, we attempt to solve the dual problem by dividing into a price master of the type:

(10) $$\text{Max}\left(bp'v^r + \sum_{i=o}^{m} b_i' u_i^r - \sum_{i=o}^{m} \bar{y}_i' h_i^r - \sum_{i=1}^{m} \bar{x}_i' k_i^r\right) z_r - \bar{x}_o' k_o$$

$$\left(\sum_{i=o}^{m} A_{io}' u_i^r\right) z_r - k_o \leq f_o$$

$$\sum_r z_r = 1$$

$$z_r \geq 0, \quad k_o \geq 0$$

(a more effective formulation will be considered in (12:27))

and the subproblem of finding such u_i vectors which will maximize a modified preference function,

(11) $$\text{Max}\, bp_o' v + (b_o - A_{oo} x_o)' u_o + \sum_{i=1}^{m} (b_i - A_{io} x_o)' u_i - \bar{y}_o' h_o - \sum_{i=1}^{m} \bar{y}_i' h_i - \sum_{i=1}^{m} \bar{x}_i' k_i$$

$$A_{oi}' u_o + A_{ii}' u_i \qquad - k_i \leq f_i' \qquad (i=1, \ ,m)$$

$$C_i' v + B_{ii} u_i \quad - h_i \qquad \leq g_i' \qquad (i=1, \ ,m)$$

$$C_o' v + B_{oo}' u_o \quad - h_o \qquad \leq g_o'$$

$$v \ , u_i \ , h_i, k_i \geq 0 \qquad (i=0, 1, \ ,m)$$

This subproblem may be made separable, by inserting the last feasible v , u_o price solution obtained from the quantity masters (12:9 and 12:2) and this will give:

one subproblem of the type

(12) $\quad \text{Max} \quad -\bar{y}_o' h_o$

$$-h_o \leq g_o' - C_o' v - B_{oo}' u_o$$

$$h_o \geq 0$$

and m subproblems of the type

(13) $\quad \text{Max } (b_i - A_{io} x_o)' u_i - \bar{y}_i' h_i - \bar{x}_i' k_i$

$$A_{ii}' u_i \qquad -k_i \leq f_i' - A_{oi}' u_o$$

$$B_{ii}' u_i \qquad -h_i \qquad \leq g_i' - C_i' v$$

$$u_i, h_i, k_i \geq 0$$

The problem (12:12) is equivalent to the following problem (obtained by multiplying by minus ones):

(14) $\quad \text{Min } \bar{y}_o' h_o$

$$h_o \geq -g_o' + C_o' v + B_{oo}' u_o = g_o^{*'}$$

$$h_o \geq 0$$

The minimum of this expression is readily found as

(15)
$$h_{oj} = \begin{pmatrix} g_{oj}^{*} \\ 0 \end{pmatrix} \text{ if } g_{oj}^{*} \begin{pmatrix} \geq 0 \\ \not\geq 0 \end{pmatrix}$$

The subproblem (12:13) will be solved in various ways depending on the dimensions of the A_{ii} and the B_{ii} matrices.

A_{ii} of the standing rectangle type

If A'_{ii} is of the narrow lying rectangle form it may be most effective to solve the problem (12:13) by formulating it as a master of the type:

$$\text{(16)} \qquad \text{Max}((b_i - A_{io}x_o)u_i^r - \bar{y}'_i h_i^r)z_r - \bar{x}'_i k_i$$

$$(A'_{ii}u_i^r)z_r \qquad - k_i = f'_i - A'_{oi}u_o$$

$$\sum_r z_r = 1$$

$$z_r \geqq 0 \qquad k_i \geqq 0$$

(which will give x_i values as "shadow" quantities) and the subproblem

$$\text{(17)} \qquad \text{Max}(b_i - A_{io}\, x_o - A_{ii}\, x_i)u_i - \bar{y}'_i h_i$$

$$B'_{ii}u_i \qquad -h_i \leqq g'_i + C'_i v$$

$$u_i \geqq 0 \qquad h_i \geqq 0$$

the detailed solution of which was the subject of par.11.B.

As the problem (12:17) will not determine u_i values for commodities not subject to foreign trade (cf. note in par.11B), the master above (12:16) will have to be formulated on the following lines

$$(18) \quad \mathrm{Max}((b_{i)} - A_{io)} x_o)' u^r_{i)} - \bar{y}'_{i)} h^r_{i)}) z_r + (b_{i(} - A_{io(} x_o)' u_{i(} - \bar{x}'_i k_i$$

$$(A'_{ii)} u^r_{i)}) z_r + A'_{ii(} u_{i(} \qquad - k_i = f'_i - A'_{oi} u_o$$

$$\sum_r z_r = 1$$

$$z_r \geq 0, \quad u_{i(} \geq 0 \quad k_i \geq 0$$

By the vector $u^r_{i)}$ is meant the solution to (12:17) but excluding all the $u_{i(}$ prices which correspond to commodities not subject to foreign trade. The closing parenthesis may be memorized as "already committed to", the opening as "open for determination". The matrices $A_{ii)}$ and $A_{io)}$ have been obtained from the A_{ii} and A_{io} matries by depriving them of the rows $A_{ii(}$ and $A_{io(}$ corresponding to the commodities not subject to foreign trade. The same is the case with the $b_{i)}$ and $b_{i(}$ vectors.

<u>A_{ii} of the lying rectangle type</u>

If A'_{ii} is of the narrow standing rectangle form it may be more effective to solve (12:13) by solving its dual

$$(19) \quad \mathrm{Min}\ (g_i - vC_i) y_i + (f_i - u_o A_{io}) x_i$$

$$B_{ii} y_i + A_{ii} x_i \geq b_i - A_{io} x_o$$

$$-y_i \geq -\bar{y}_i$$

$$-x_i \geq -\bar{x}_i$$

$$y_i \geq 0, \; x_i \geq 0$$

When the number of y_i variables is very great it may probably be solved with advantage as a master programme of the type:

(20) $$\text{Min}((g_i - vC_i)y_i^r)z_r + (f_i - u_o A_{io})x_i$$

$$(B_{ii}y_i^r)z_r + A_{ii}\ x_i \geq b_i - A_{io}\ x_o$$

$$-\ x_i \geq -\bar{x}_i$$

$$\sum_r z_r = 1$$

$$z_r \geq 0 \qquad x_i \geq 0$$

(this problem gives the feasible prices u_i and k_i

and the subproblem

(21) $$\text{Min}(g_i - vC_i - u_i\ B_{ii})y_i$$

$$-y_i \geq -\bar{y}_i$$

$$y_i \geq 0$$

By reformulating (12:21) as

(22) $$\text{Min}(g_i - vC_i - u_i\ B_{ii})y_i = g_i^* y_i$$

$$y_i \leq \bar{y}_i$$

$$y_i \geq 0$$

it is readily seen that its solution is extremely simple

(23) $$y_{ij} = \begin{pmatrix} 0 \\ \bar{y}_{ij} \end{pmatrix} \text{ if } g_{ij}^* \qquad \begin{pmatrix} \geq 0 \\ < 0 \end{pmatrix}$$

but we note that in addition to sending a y_i vector into (12:20) we have after having found an acceptable u_i, k_i dual solution to (12:20) to send the latters together with the appropriate h_i prices to the overall price master (12:10). These dual prices h_i will be easily found by formulating (12:21) as

(24) $$\text{Max } -\bar{y}_i' h_i$$

$$-h_i \leq g_i' - C_i'v - B_{ii}' u_i$$

$$h_i \geq 0$$

which is equivalent to

(25) $$\text{Min } \bar{y}_i' h_i$$

$$h_i \geq -(g_i' - C_i'v - B_{ii}'u_i) = g_i^*$$

$$h_i \geq 0$$

thus

(26) $$h_{ij} = \begin{pmatrix} 0 \\ \\ g_{ij}^* \end{pmatrix} \text{ if } g_{ij}^* \begin{pmatrix} \leq 0 \\ \\ > 0 \end{pmatrix}$$

12:5-6 Obtaining feasible prices v, u_i, h_i, k_i, an estimate of the dual preference function and an x_o solution.

As the result of the above calculations we have found sets of prices v, u_i, h_i $(i=0,1, \ ,m)$ and $k_i (i=1, \ ,m)$, every one of which only fulfils the equations of (12:11). In analogy to what was done with the partial quantity solutions (12:2), we may now attempt to solve the overall price master (12:10) by formulating it as

$$(27) \quad \mathrm{Max}(b_p' v^r + b_o' u_o^r - \bar{y}_o' h_o^r) z_r + \sum_{i=1}^{m} (b_i' u_i^{rs} - \bar{y}_i' h_i^{rs} - \bar{x}_i' k_i^{rs}) z_{irs} - \bar{x}_o' k_o$$

$$(A_{oo} u_o^r) z_r + \sum_{i=1}^{m} (A_{io} u_i^{rs}) z_{irs} - k_o = f_o$$

$$\sum_r z_r = 1$$

$$\sum_s z_{irs} = z_r \qquad (i=1, \ ,m; \ r = \text{all})$$

$$z_r, \ z_{irs}, \ k_o \geq 0 \qquad (i=1, \ ,m; r,s = \text{all})$$

As a result we obtain the "shadow" quantities x_o of the above price problem and a value of the dual preference function.

12:7-8 Determination of y_i, x_i for fixed x_o, v, u_o

If we insert the last x_o values obtained from the master (12:27) into our primal equation system (5:1) we may readily obtain a new partial quantity solution (y,x) if we disregard the balance of payments and the 0-group of commodity constraints and use a modified preference function.

$$(28) \quad \mathrm{Min}(g_o - vC_o - u_o B_{oo}) y_o + \sum_{i=1}^{m} (g_i - vC_i) y_i + \sum_{i=1}^{m} (f_i - u_o A_{oi}) x_i$$

$$B_{ii} y_i + A_{ii} x_i = b_i - A_{io} x_o \quad (i=1, ,m)$$

$$-y_o \geq -\bar{y}_o$$

$$-y_i \geq -\bar{y}_i$$

$$-x_i \geq -\bar{x}_i \quad (i=1, ,m)$$

$$y_i \geq 0 \quad (i=0, ,m)$$

$$x_i \geq 0 \quad (i=1, ,m)$$

This problem is separable into

one of the type

$$(29) \quad \mathrm{Min}(g_o - vC_o - u_o B_{oo})\, y_o = g_o^* \, y_o$$

$$-y_o \geq -\bar{y}_o$$

$$y_o \geq 0$$

with the obvious solution

$$(30) \qquad y_{oj} = \begin{pmatrix} 0 \\ \bar{y}_{oj} \end{pmatrix} \quad \text{if} \quad g^{*}_{oj} \begin{pmatrix} \geq 0 \\ < 0 \end{pmatrix}$$

and m of the type

$$(31) \qquad \text{Min}(g_i - vC_i)y_i + (f_i - u_o A_{oi})x_i$$

$$\begin{aligned} B_{ii}y_i + A_{ii}x_i &= b_i - A_{io}x_o \\ -y_i &\geq -\bar{y}_i \\ -x_i &\geq -\bar{x}_i \\ y_i \geq 0 \quad x_i &\geq 0 \end{aligned}$$

As was the case in dealing with the corresponding price problems in (12:13), we may also choose to solve these quantity problems in two ways mainly depending upon the dimensions of the matrice A_{ii}.

A_{ii} of the standing rectangle type

We may then consider the dual of problem (12:31) which is just (12:13) and decompose it into the master (12:18) and the subproblem (12:17). The x_i quantities will then be obtained as "shadow quantities" from the master (12:18) and the y_i quantities from solving the dual of (12:17) which is

$$(32)\quad \text{Min } (g_i - vC_i)y_i$$

$$B_{ii}y_i = b_i - A_{io}x_o - A_{ii}x_i$$

$$-y_i \geqq -\bar{y}_i$$

$$y_i \geqq 0$$

the solution of which was dealt with in par. 11.A.

$\underline{A_{ii}}$ <u>of the lying rectangle type</u>

It may again be preferable to use a master identical with (12:20) and a subproblem (12:22).

In solving the subproblems involving the y_i and u_i variables, it should be noted that these subproblems may in turn be partioned into smaller ones and so on until we have as many subproblems as there are commodities. Appropriate changes will then have to be made in the masters above. This possibility may be used to speed up calculations. This will be briefly dealt with in par. 14.

As a result of these calculations we will have obtained a new partial (y,x) solution and may again combine it with the other known ones in(12:2),which was the point of departure of this chapter.

13. The successive Contraction of the possible Range of the Optimum Value of the Preference Function.

The successive solutions of the highest quantity master (12:9) will give a falling sequence of possible values of the preference function. The successive solutions of the highest price master(12:27) will give rising sequence of values of the dual preference function. In the optimum the values of these two functions will be equal. A useful estimate of the possibilities of still further decreasing the preference function will be obtained as the greatest possible improvement cannot lead to a value that is lower than the last and highest value of the dual preference function.

14. The Optimal Structure, Size and Number of Subproblems and the Routing of Iterations.

The most important factors for the swift solution of the linear programming problem dealt with will probably be the selection of the most effective structure, size, number of subproblems and the routing of the iterations.

Some of the most interesting possibilities which seem to appear here will be mentioned below.

Sensitivity of a subproblem

In dealing with for instance, an export and import quantity problem we may notice that for certain commodities the comparable prices (g^{*}_{da}) in various markets only slightly differ; very large unused import or export possibilities are existant and the value of the export or import is relatively large in relation to that of other commodities.

The solution of the subproblem will then be extremely dependent upon small changes in the currency exchange rates (iterative prices v). The balance of payment vector which enters the corresponding master problem, will in turn strongly change its character, which will influence the new currency prices.

In solving the whole problem it may then be most efficient to partition, for instance, the export and import quantity subproblem into one subproblem embracing very many insensitive and unimportant commodities and one embracing very few but sensitive and in value important commodities.

To obtain a feasible solution to the whole problem we repeatedly solve the relatively small but most important subproblem and only some very few times the less important though very large in number of commodities.

The advantages of this principle are readily seen.

Suppose we have a master of 100 equations. If we introduce only one formal subproblem we will usually have to solve the subproblem at least 101 times to get a feasible solution of the master. If we assume that we have one largesized subproblem embracing 99% of the commodity numbers and one very small but sensitive and important embracing 1% of the commodity numbers, we may solve the small at least 98 times and the large 4 times to obtain a feasible solution to the master of 100+2 equations. The effort spent in solving the foreign trade quantity subproblem, will then be equal to 98% + 4×99% ≈ 5 solutions of the entire subproblem. This would lead to a 95% reduction of computational work in regard to the straight forward approach which used only one subproblem.

<u>The sensitivity of a subproblem in respect to particular price changes</u>.

If, for instance, the foreign trade subproblems may be so constructed that they include only those commodities which are traded with <u>some particular currency regions, then the</u> resulting subproblem will only be sensitive in regard to iterative price changes of the corresponding currencies. If in an iteration no appreciable iterative price changes of some currencies but fairly large changes of certain other currencies have taken place it may then be most effective to solve those subproblems which embrace the regions for which large iterative price changes have taken place.

<u>The extremality of a subproblem</u>

In dividing a subproblem into for instance, two subproblems (each of which are assumed to be equally sensitive to price changes) it seems to be probable that a division into one with <u>predominant positive effects</u> (e.g. export commodities) on the master problem and another with predominant negative effects (e.g. import commodities) will be more effective than two with more mixed effects (e.g. each one including both export and import commodities in equal proportions). In the first case we are likely to obtain more extreme vectors in the master programme, which may permit the formation of more advantageous solutions.

The size of a subproblem

Even though we may have succeeded in partitioning the problem into some subproblems embracing approximately equally sensitive commodities the problem of whether the size of the subproblem is the most appropriate one remains. If we, for instance, would divide one of them into two and employ the policy of immediately revising the master after the solution of each, we may make use of the improvement of the iterative prices gained from solving the first half of the original subproblem, for solving the second half. This will lead to an increase in the relative number of times which we will solve the master, but may lead to a decrease in the total computational work required to reach the optimal solution. Even if we will not revise the master after the solution of each of the two new subproblems, we may still have a decrease in the number of times we will have to solve the master as we have a greater number of partial solutions that may be combined in the master.

The routing of the iteration process.

If we have several subproblems to one master problem, a considerable saving of computational work may often be made by immediately revising the master after the solution of a subproblem and then selecting that subproblem for solution, which may be expected to have the greatest influence on the general solution so far obtained. This will mean that we will repeatedly solve the most sensitive subproblem, then at one time or another swift over to a less sensitive one, and again work repeated by with the more sensitive ones, etc. We will then have to introduce a special mathematical programme a Policy Problem , the solution of which gives the subproblem which has to be solved in the current iteration.

The topics mentioned in this paragraph would for their detailed analysis require much the same space as this mathematical appendix. Their discussion will therefore have to wait for another opportunity.

15. Relations to Certain East-European Investigations

One of the aims of this paper has been the further development of some formulations for both production and foreign trade planning made by J. Mycielski, K. Rey and W. Trzeciakowski, (1,2). Certain disadvantages are associated with their conception in that:

i) the proposed procedure will not necessarily give a series of solutions converging to the optimal solution[1]);

ii) the restricted use of the knowledge gained about other possible solutions, but have the advantage of

iii) implicitly raising the question of whether a swifter road to the optimal prices and quantities may be found than that which is derived by combinations of known solutions[2]). To implement such an idea various approaches are possible. We may make an attempt by introducing the principle of over- and underrelaxation or in economic interpretation of "speculation" and "inertia", as has been done in certain studies in the Soviet Union.

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1) The thought expressed in "Decomposition and Optimization of Short-Run Planning", (1), p.35, that:

"On the given step of iteration the inequality $\sum_{j=1}^{v} A_{k_i j} > B_{k_i}$ suggests that l_{k_i} must be lowered, and the opposite one that it must be raised"

will by itself not guarantee convergence. The simplest way of showing this seems to be by inspecting table 3, rows 3 and 4 in the author's study "Iterative Pricing for Planning Foreign Trade" (9). Using the terminology of Mycielski-Rey-Trzeciakowski and adding superscripts to indicate iteration we find that

Continuation of footnote from p. 50.

$l_1^3 = M_1^3 = 1.300$ contributes to $\sum_{j=1}^{v} A_{1j} z_j = -0.85 < B_1 = 0$,
thus, according to the principle suggested it may be raised to
$l_1^4 = M_1^4 = 2.900$;
$l_2^3 = M_2^3 = 0.210$ contributes to $\sum_{j=1}^{v} A_{2j} z_j = -22.85 < B_2 = +1$,
thus in accordance with the said principle it may be raised to
$l_2^4 = M_2^4 = 1.000$;
$l_3^3 = M_3^3 = 0.160$ contributes to $\sum_{j=1}^{v} A_{3j} z_j = 60.00 > -16 = B_3$,
thus still in accordance with the said principle that it may be decreased to $l_3^4 = M_4^4 = 0.095$;

If we study the balances of trade which correspond to the M_1^4; M_2^4 and M_3^4 thus chosen we see that while still adhering to the above principle we may set $M_1^5 = M_1^3$, $M_2^5 = M_2^3$ and $M_3^5 = M_3^3$ and continue in this way, always putting the iterative prices of an <u>even</u> iteration equal to those of the fourth iteration, and those of an <u>uneven</u> iteration equal to those of the third, <u>without ever reaching the optimal solution</u>.

2) This possibility may be felt by inspecting Table 4 in the author's study "Iterative Pricing for Planning Foreign Trade" (9) , where certain price sets as in iteration 8 seem to deviate strongly from what in the end will turn out to be the optimal prices.

The production and foreign trade model of this paper attempts to use the method of D. Pigot (4) and theoretically related concepts of Kornai and Liptak as described in A. Nagy-T. Liptak (3) for solving systems with some filled rows and columns in an otherwise separable problem, and analyzes how the foreign trade matrices should be treated.

An interesting conclusion is that the problem of optimal allocations of export and import quantitives on incompletely convertible currency territories will become a subproblem in the overall system of economic planning. This will enhance the importance of empirical studies of this subproblem which are being undertaken by A. Marton and M. Tardos (6) in Hungary and by W. Trzeciakowski in Poland.

This study deviates also from some Russian concepts of employing one auxiliary constraining inequality at each level of a pyramidal economic planning system and considers that it in general will be more effective to employ several in-equalities (cf. the discussion in par. 12 and 14).

Complementary views on how blocks of different levels of planning models may be integrated to form an all embracing planning system are contained in Yu. I Chernyak (7) and A. Modin (8).

16. Literature

A production and foreign trade planning model is given in

1. Jerzy Mycielski, Krzysztow Rey and Witold Trzeciakowski, "Decomposition and Optimization of Short-Run Planning", in Tibor Barna (Editor) Structural Interdependence and Economic Development, London, 1963,

as well as in an extended Polish version

2. "Optimum całościowe a optima cząstkowe w planowaniu handlu zagranicznego" in Przegląd Statystyczny, No. 1, 1963, pp. 119-137.

Important questions of decomposition of linear programmes in regard to foreign trade planning are considered in

3. A. Nagy and T. Liptak, "Short-Run Optimization Model of Hungarian Cotton Fabric Exports", Economics of Planning, Oslo, No. 2, Sept., 1963, pp. 89-113.

The decomposition of a linear programming problem with filled rows and columns is treated in a compact presentation by

4. D. Pigot, "Double décomposition d'un programme linéaire", in the forthcoming proceedings of the Third Conference of the International Federation of Operational Research Societies held in Oslo 1st to 5th July, 1963.

Important experiences of the efficiency of various solution policies in decomposed linear programming problems are

rendered in

5. J.-M. Gauthier and F. Genuys, "Expériences sur le principe de décomposition des programmes linéares", 1er Congrés de l'AFCALTI, 1960.

Some discussion on the preference function is given in

6. Adam Marton and Marton Tardos, "On optimizing the commodity pattern on foreign trade markets", Kőzgazdaságí Szemle, Budapest, August 1963, pp. 932-944.

Concepts of a pyramidal system of planning models simulating the planning process of the Soviet Union are evolved in

7. Yu. I. Chernyak, "The Electronic Simulation of Information Systems for Central Planning", Economics of Planning, Oslo, No. 1, April, 1963, pp. 23-40;

8. A. Modin, "Developing Interbranch Balances for Economic Simulation", Economics of Planning, Oslo, No. 2, Sept., 1963.

A numerically illustrated account of the optimization procedure for distribution of given exports and imports (re-exports excluded), on incompletely convertible currency territories is rendered in

9. Tom Kronsjö, "Iterative Pricing for Planning Foreign Trade", Economics of Planning, Oslo, No. 1, April, 1963, pp. 1-22; as well as in an extended Russian version

10. ———, "Postroenie optimalnych planov vneshne-torgovych raspredelenij po metodu iterativnogo cenobrazovaniya",

Institute for International Economic Studies, Stockholm, 1963, 23 pp.

A more formal mathematical exposition is made in

11. ______, "Decomposition of Large Linear Programmes, illustrated with an example from the foreign trade theory of a planned economy", in the forthcoming proceedings of the <u>Third Nord SAM (Nordic Symposium on the Application of Computing Machinery)</u>, Helsinki, 15th-20th August, 1963.

An excellent introduction to Algol is given in

12. Daniel D. Mc Cracken, <u>A guide to Algol programming</u>, New York, 1962, 106 pp.

The importance at present attached to mathematical education of economists in the Soviet Union is apparent from the university programmes, accounted for in

13. Tom Kronsjö, "Tendencies in Soviet Economic Scientific Education", Economics of Planning (then <u>Øst-Økonomi</u>), Oslo, No. 1, March, 1962, pp. 2-20.

14. A. Ya. Boyarskij, "University Programme: Mathematics in Economics", Economics of Planning (then <u>Øst-Økonomi</u>), Oslo, No. 2, July, 1962, pp. 105-115.

15. Tom Kronsjö, "Soviet Engineering-Economic Education", Economics of Planning (then <u>Øst-Økonomi</u>), Oslo, No. 3, December, 1962, pp. 184-194.

16. B.I. Michalevskij, "University Course: Economic-Mathematical Methods", <u>Economics of Planning</u>, Oslo, No. 3, Dec., 1963.