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Formulation of Some Optimization Problems in Queueing Theory

By

T.L. Saaty.

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CAIRO

3, MOHAMED MAZHAR - ZAMALEK

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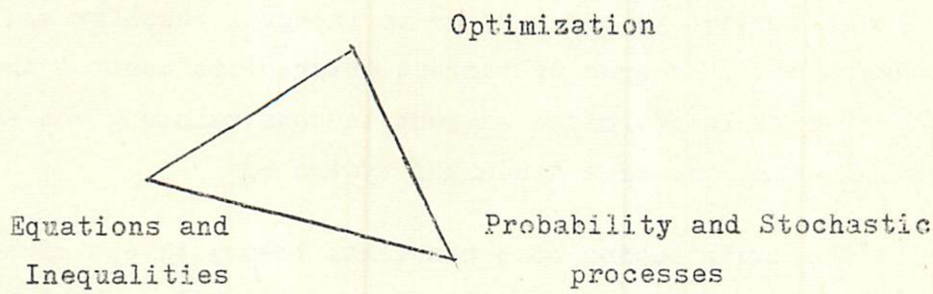
1. Introduction : Here we shall be concerned with the formulation of optimization problems for queueing theory-essentially a new and difficult aspect of the subject . The motivation and justification of the usefulness of these problems may be found in the following framework .

Virtually all of applied mathematics may be divided into optimization models , deterministic models and probabilistic models with numerical analysis as a general technique for solution . Schematically we have a triangle



By optimization we understand not only the theory of maxima and minima and the calculus of variations but also the theory of games (minimax) and generalizations of these ideas . In referring to deterministic models one often has in mind equations or inequalities . These can be ordinary (algebraic and transcendental) , differential , difference , integral , functional or combinations of them (e.g., recurrent integro-differential , and differential-difference) . The three fields are related by the triangle in the sense that many stochastic processes can be described by equations e.g., differential , integral , integro-differential and differential-difference . Also the contribution of probability theory to equations may be exemplified by those differential equations in which the right side is a stochastic forcing function or by differential equations in random variables . Again necessary and sufficient conditions of optimization theory are expressed in terms of equations and inequalities e.g., the Euler equations and the Weierstrass inequality condition in the calculus of variations etc. Conversely , optimization theory imposes a richer structure on systems of equations and inequalities by introducing functions given in any of the forms listed above e.g., integral , functional , etc. to be maximized or minimized subject to constrains , given a mixture of the different types of equations (or inequalities) . For example dynamic programming optimizes a functional expression subject to ordinary constraints ; ordinary programming optimizes an ordinary expression subject to ordinary inequalities whereas in the calculus of variations the function to be optimized is an integral or a functional subject to ordinary or to integral equality and inequality constraints . An area of current interest is control theory in which an integral is optimized subject to constraints given as the differential equations of a dynamical system .

The contribution of probability theory to optimization is exemplified by applying probability to maximization problems . There are areas in which optimization is applied to a probability problem in a broad optimization sense . A well known application of optimization to probability theory is information theory , where it is shown that the Gaussian distribution has maximum chaos (entrop) i.e.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{x^2}{2\sigma^2}\right] \text{ minimizes}$$

$$-\int_{-\infty}^{\infty} f(x) \log f(x)\,dx.$$ In fact its information content is

$$\log 1/\sqrt{2\pi e \sigma^2}$$

Finally , stochastic linear programming is an example of a union of all three fields .

Queueing theory is a field of probability widely explored in operations research . However , in its present form it is a descriptive field i.e., it does not have the "normative" framework of optimization applied to it . Of course there are queueing problems involving minimization of costs and other economic(therefore extrinsic) measures. Intrinsically we are interested in optimization problems in queueing theory whereby existing results are further exploited. For example , under various assumptions on the input distribution to a single service-channel queue one often derives expressions for the waiting time distribution as a function of the service distribution. In many cases it is desirable to find the service distribution (appropriately constrained and qualified) which yeilds the minimum expected waiting time. Then in practice one may attempt to set up the corresponding type of service which minimizes the expected waiting time . Another example utilizes the distribution of a busy period : how does one minimize the expected idle time of a service channel by fixing the service distribution and varying the input distribution? To find the combination of input and service distributions which minimizes the expected waiting time is yet another problem . To our knowledge no such problems have yet been attacked .

The formal approach of the next section given rise to variational problems . We will be happy if the reader is sufficiently interested to contribute to this new chapter in operations research .

2. Statement of three problems :

(a) Given the conditions in the steady state :

$$W(t) = \int_0^\infty W(x)u(t-x)dx \qquad \text{(Lindley's equation)}$$

$$u(s) = \int_0^\infty b(y)a(y-s)dy \qquad s \leq 0$$

$$u(s) = \int_0^\infty b(y+s)a(y)dy \qquad s \geq 0$$

$$\int_0^\infty t^k\, a(t)dt = \alpha_k$$

$$\int_0^\infty t^k\, b(t)dt = \beta_k \ , \ k=0 \ , \ 1,2$$

where $\alpha_0 = \beta_0 = 1$ and where $W(t)$ is the probability of waiting for a time of length t in a single channel first come first served queue with input times density function a(x) and service time density function b(y) , find a(x) and b(y) which minimize the expect waiting time .

$$I = \int_0^\infty t \frac{dW}{dt} dt$$

We may simplify the above problem by assuming that the distribution of input times is known to be poisson i.e., $a(x) = \alpha e^{-\alpha x}$. Note that it is essential to prescribe some of the moments of these distributions to avoid results which cannot always be easily implemented in practice such as obtaining a Dirac delta function for an answer.

(b) If we assume $a(x) = \alpha e^{-\alpha x}$ then fortunalety we can obtain an expression for $W^*(s)$ the Laplace transform of W(t) , using the integro-differential equation of Takacs . In that case we have

$$W^*(s) = \frac{1-\rho}{s\left\{1-\alpha\left[\frac{1-b^*(s)}{s}\right]\right\}}$$

where ρ is the utilization factor and $W(0) = 1 - \rho$ and $b^*(s)$ is the Laplace transform of the service time density function $b(y)$.

The problem is to find $b(y)$ subject to

$$\int_0^\infty t^k \; b(t)dt = \beta_k \; ; k = 0, 1, 2$$

which using the inverse transform

$$W(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \, W^*(s) \, ds ,$$

minimizes

$$\int_0^\infty t \frac{dW}{dt} dt = \frac{1}{2\pi i} \int_0^\infty \int_{c-i\infty}^{c+i\infty} \frac{t(1-\rho) \, e^{st} \, ds \, dt}{1-\alpha \frac{\left[1-b^*(s)\right]}{s}}$$

(c) Another example uses the generating function $P(z)$ of p_n the probability that there are n customers in a single channel first come first served system in equilibrium. If $\beta(s)$ is the Laplace Stieltjes transform of $B(t)$ the cumulative service time distribution i.e.,

$$\beta(s) = \int_0^\infty e^{-st} \, dB(t)$$

and if $a(t) = \alpha e^{-\alpha t}$ then it is known that

$$P(z) = \frac{(1-\rho) \, \beta\left[\alpha (1-z)\right]}{1 - \left\{\frac{1-\beta\left[\alpha (1-z)\right]}{1-z}\right\}}$$

Since $P(z) = \sum_{n=0}^{\infty} p_n \, z^n$

the average number waiting in the system is given by

$$L = \left. \frac{dP(z)}{dz} \right|_{z=1}$$

The problem here is to find B(t) which minimizes L subject to

$$\int_0^{\infty} t^k \; dB(t) = \beta_k \; , \; k = 0,1, \ldots .$$

3. Remark about the solution

The solution of such problems is often derived from variational methods applied to integrals subject to contraints . Here however the task is by no means easy i.e., the conditions cannot be written down in a next explicit form and even if this were possible , the result is likely to be a system of integral equation which is usually solved by approximation techniques , all indicating that this is a nontrivial field for investigation .

Necessary conditions for the second problem with Lindley's formulation using the variational approach may be reduced to one of finding b(t) which renders the following integral stationary .

$$\int_0^{\infty} \Big\{ t \frac{dW}{dt} + \sum_{k=0}^{2} \lambda_k t^k b(t) + \lambda(t) \Big[W(t) - \int_0^t W(x) \int_0^{\infty} b(y-x+t) a(y) \, dx - \int_t^{\infty} W(x) \int_0^{\infty} b(y) \, a(y+x-t) dy dx \Big] \Big\} dt - \sum_{k=0}^{2} \lambda_k \beta_k$$

where $\alpha e^{-\alpha t}$ with the appropriate argument is used for a(t). Here λ_k, k = 0,1,2 and $\lambda(t)$ are the usual Lagrange multipliers used in variational problems .

Actually we begin by finding an expression for the integral I in terms of the function b . Inserting the expression for a(y) in the expression for u(s) , changing variable , etc. we obtain*

$$u(s) = \alpha e^{\alpha s} \quad \bar{B}(s), \qquad \bar{B}(s) = \begin{cases} \int_s^\infty b(y)e^{-\alpha y} \, dy, & s \geq 0, \\ \int_0^\infty b(y)e^{-\alpha y} \, dy, & s \leq 0. \end{cases} \tag{1}$$

Thus W satisfies

$$W(t) = \int_0^\infty W(x) \, \alpha e^{\alpha t - \alpha x} \quad \bar{B}(t-x) \, dx \text{ , or}$$

$$R(t) = \alpha \int_0^\infty R(x) \, \bar{B}(t-x)dx, \quad R(t) = e^{-\alpha t} \quad W(t). \tag{2}$$

We note that

$$R(0) = \alpha . \bar{B}(0) \quad \int_0^\infty R(x)dx \text{ (since } \bar{B}(s) = \bar{B}(0) \text{ for } s \leq 0 \text{)}. \tag{3}$$

Setting

$$V(t) = R(t) - R(0) \tag{4}$$

We see that (using (3))

$$R(t) - R(0) = \alpha \int_0^\infty R(x) \left[\bar{B}(t-x) - \bar{B}(0) \right] dx. \tag{5}$$

Since $\bar{B}(s) = \bar{B}(0)$ for $s \leq 0$, we see that V satisfies

$$V(t) = \alpha R(0) \int_0^t C(t-x)dx + \alpha \int_0^t V(x)C(t-x)dx; C(s) = \bar{B}(s) - \bar{B}(0). \tag{6}$$

Solving this for V by successive approximations , we obtain

$$V(t) = R(0) \left\{ \alpha \int_0^t C(t-x)dx + \alpha^2 \int_0^t C(t-x) \left[\int_0^x C(x-y)dy \right] dx \right.$$

$$\left. + \alpha^3 \int_0^t C(t-x) \left[\int_0^x C(x-y) \left\{ \int_0^y C(y-z)dz \right\} dy \right] dx + \ldots \right\} \tag{7}$$

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* I am grateful to professor C.B. Morrey for suggesting this approach

so that

$$R(t) = R(0)\left\{1 + \alpha\int_0^t C(t-x)\,dx + \alpha^2\int_0^t C(t-x)\left[\int_0^x C(x-y)dy\right]dx + \ldots\right\}$$

(8)

$$C(s) = -\int_0^s b(y)e^{-\alpha y}\,dy\ ,\quad C'(s) = -b(s)e^{-\alpha s}$$

$$W(t) = e^{\alpha t}\ R(t)$$

Clearly if the function b is uniformly bounded, so is C .
In this case it is easy to show the convergence of the series for R(t) for each $t \geq 0$. However we need to know more about the behaviour of R(t) for large t.

By making the changes of variable (for each fixed $t > 0$)

$$x' = t-x\ ;\quad x' = t-x\ ,\ y' = x-y$$
$$x' = t-x\ ;\quad y' = x-y\ ,\ z' = y-z$$

in the various multiple integrals , we may obtain

$$R(t) = R(0)\left\{1+\alpha\int_0^t C(x')dx' + \alpha^2\iint_{\Delta_2(t)} C(x')C(y')dA_{x'y'}\right.$$

$$\left. + \alpha^3\iiint_{\Delta_3(t)} C(x')C(y')C(z')dV_{x'y'z'} \quad + \ldots\ldots\right\} \qquad (9)$$

$\Delta_2(t)$: $x \geq 0$, $y \geq 0$, $0 \leq x + y \leq t$;

$\Delta_3(t)$; $x \geq 0$, $y \geq 0$, $z \geq 0$, $0 \leq x + y + z \leq t$; ...

This can be put in the form

$$R(t) = R(0)\left\{1+\alpha\int_0^t C(x)dx + \alpha^2\int_0^t\int_0^{t-x} C(x)C(y)dydx\right.$$

$$\left. + \alpha^3\int_0^t\int_0^{t-x}\int_0^{t-x-y} C(x)C(y)C(z)\,dzdydx + \ldots\right\}. \qquad (10)$$

If we now suppose that a given function b(y) minimizes I as desired , we can find the conditions on b by replacing

$$b(y) \text{ by } b(y,\lambda) = b(y) + \lambda\bar{\beta}(y) \; (\bar{\beta}(y) = \delta b(y))$$

Then C(s) , R(t) , and W(t) are replaced by

$$C(s;\lambda) = C(s) + \lambda\Gamma(s) , \; \Gamma(s) = - \int_0^s \bar{\beta}(y)e^{-\alpha y}\, dy$$

$$R(t;\lambda) = R(0;\lambda)\left\{1+\alpha\int_0^t C(x,\lambda)\, dx + \alpha^2\iint_{\Delta_2(t)} C(x,\lambda)C(y,\lambda)dA_{xy} + \dots\right\} \quad (11)$$

$$W(t;\lambda) = e^{\alpha t} R(t;\lambda), \; W_t(t;\lambda) = e^{\alpha t}\left[R_t(t;\lambda) + \alpha R(t;\lambda)\right]$$

$$I(\lambda) = \int_0^\infty te^{\alpha t}\left[(R_t(t;\lambda) + \alpha R(t;\lambda)\right]dt$$

The derivative R_t can be found by using the form (10).

Now,since b(y;0) =b(y) minimizes the integral I, it follows that we must have

$I'(0) = 0$ for every admissible function $\bar{\beta}(y)$.

Thus the condition on b is found by differentiating (11) with respect to λ and setting $\lambda = 0$.

Now,from (3),(8), etc. it seems that R, and hence W is determined only up to a factor R(0) = W(0) so that the integral I has that factor also , so perhaps we must require

$W(0;\lambda) = R(0,\lambda) = 1$ for all λ near 0 .

Then $R(t;\lambda)$ will be given by the series in the brace in (11).

But then, from(3) , the function $b(y;\lambda)$ must be such that

$$1 = \alpha.\bar{B}(0;\lambda) \int_0^{\infty} R(x;\lambda)dx \text{ , where}$$

$$\bar{B}(0;\lambda) = \bar{B}(0) + \lambda \int_0^{\infty} \bar{\beta}(y)e^{-\alpha y}\, dy \tag{12}$$

As an aid in carrying out the differentiation, suppose

$$f_2(t;\lambda) = \iint_{\Delta_2(t)} C(x;\lambda)C(y;\lambda)dA_{xy} = \int_0^t \int_0^{t-x} C(x;\lambda)C(y;\lambda)dydx$$

$$= \int_0^t C(x;\lambda)\, \mathcal{C}(t-x;\lambda)dx,\ \mathcal{C}(s;\lambda) = \int_0^s C(y;\lambda)\, dy \tag{13}$$

$$f_{2t}(t;\lambda) = \int_0^t C(x;\lambda)C(t-x;\lambda)dx$$

$$f_{2\lambda}(t;\lambda) = \iint_{\Delta_2(t)} \left[C(x;\lambda)C_\lambda(y;\lambda) + C_\lambda(x;\lambda)C(y;\lambda)\right] dA_{xy}$$

$$= 2\iint_{\Delta_2(t)} C(x;\lambda)C_\lambda(y;\lambda)dA_{xy} = 2 \int_0^t \int_0^{t-x} \Gamma(x)C(y;\lambda)dydx$$

using the symmetry of $\Delta_2(t)$ in x and y . The higher integrals in (11) can be differentiated in like manner .

Thus R_t , $R_{t\lambda}$, and R_λ may be found as infinite series.

Differentiating (12) , we get

$$0 = \alpha . B(0;\lambda) . \int_0^\infty R_\lambda(x;\lambda)dx + \alpha \bar{B}_\lambda (0;\lambda) . \int_0^\infty R(x;\lambda)dx$$

$$= \frac{\bar{B}_\lambda (0;\lambda)}{\bar{B} \ (0;\lambda)} + \alpha \bar{B}(0;\lambda) . \int_0^\infty R_\lambda(x;\lambda)\,dx \qquad (14)$$

Setting $\lambda = 0$ will lead to a condition of the form

$$\int_0^\infty A(x)\,\bar{\beta}(x)\,dx = 0. \qquad (15)$$

The other conditions on $\bar{\beta}$ will be

$$\int_0^\infty t^k\,\bar{\beta}(t)dt = 0 , \qquad (16)$$

All admissible $\bar{\beta}$ must satisfy (15) and (16). Then the Lagrange multiplier rule will yield some condition on b(y) which appears to be expressible only in infinite series form.

Thus we have a procedure for the numerical calculation of b(t) .

The foregoing example serves to illustrate a possible method of attack on the suggested problems . It seems clear that in most cases the use of a computer would be required using numerical methods for solving integral equations [23] .

These problems in which greater opportunity for controlling the queue is seen , bring queueing theory closer to one of the basic interests of operations research-optimization .

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